Mathematical Logic - 2016

Exercises: First Order Logics (FOL)

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DPLL

DPLL Procedure: Main Steps

1. It identifies all literal in the input proposition P

$B \wedge \neg C \wedge (B \vee \neg A \vee C) \wedge (\neg B \vee D)$

2. It assigns a truth-value to each variable to satisfy them

$B \wedge \neg C \wedge (B \vee \neg A \vee C) \wedge (\neg B \vee D)$ $\nu(B) = T$; $\nu(C) = F$

3. It simplifies P by removing all clauses in P which become true under the truth-assignments at step 2 and all literals in P that become false from the remaining clauses (this may generate <u>empty clauses</u>)

D

4. It recursively checks if the simplified proposition obtained in step 3 is satisfiable; if this is the case then P is satisfiable, otherwise the same recursive checking is done assuming the opposite truth value (*).

D YES, it is satisfiable for $\nu(D) = T$. NOTE: $\nu(A)$ can be T/F

DPLL Procedure: Example 1

$P = A \wedge (A \vee \neg A) \wedge B$

- There are still variables and clauses to analyze, go ahead
- P does not contain empty clauses, go ahead
- It assigns the right truth-value to A and B: $\nu(A) = T$, $\nu(B) = T$
- It simplifies P by removing all clauses in P which become true under $\nu(A) = T$ and $\nu(B) = T$

  This causes the removal of all the clauses in P
- It simplifies P by removing all literals in the clauses of P that become false from the remaining clauses: nothing to remove
- It assigns values to pure literals. nothing to assign
- All variables are assigned: it returns true

DPLL Procedure: Example 2

$P = C \wedge (A \vee \neg A) \wedge B$

- There are still variables and clauses to analyze, go ahead
- P does not contain empty clauses, go ahead
- It assigns the right truth-value to C and B: $\nu(C) = T, \nu(B) = T$
- It simplifies P by removing all clauses in P which become true under $\nu(C) = T$ and $\nu(B) = T$.

  P is then simplified to $(A \vee \neg A)$
- It simplifies P by removing all literals in the clauses of P that become false from the remaining clauses: nothing to remove
- It assigns values to pure literals: nothing to assign
- It selects A and applies the splitting rule by calling DPLL on
  - $A \wedge (A \vee \neg A)$ AND $\neg A \wedge (A \vee \neg A)$

  which are both true (the first call is enough). It returns true

DPLL Procedure: Example 3

$P = A \wedge \neg B \wedge (\neg A \vee B)$

- There are still variables and clauses to analyze, go ahead
- P does not contain empty clauses, go ahead
- It assigns the right truth-value to A and B

  $\nu(A) = T, \nu(B) = F$
- It simplifies P by removing all clauses in P which become true under $\nu(A) = T$ and $\nu(B) = F$.

  P is simplified to $(\neg A \vee B)$
- It simplifies P by removing all literals in the clauses of P that become false from the remaining clauses: the last clause becomes empty
- It assigns values to pure literals: nothing to assign
- All variables are assigned but there is an empty clause: it returns false

FOL

The need for greater expressive power

- We need FOL for a greater expressive power. In FOL we have:
  - constants/individuals (e.g. 2)
  - variables (e.g. x)
  - Unary predicates (e.g. Man)
  - N-ary predicates (eg. Near)
  - functions (e.g. Sum, Exp)
  - quantifiers (∀, ∃)
  - equality symbol = (optional)

Alphabet of symbols in FOL

Variables	$x_1, x_2, \ldots, y, z$
Constants	$a_1, a_2, \ldots, b, c$
Predicate symbols	$A^1_1, A^1_2, \ldots, A^n_m$
Function symbols	$f^1_1, f^1_2, \ldots, f^n_m$
Logical symbols	$\wedge, \vee, \neg, \supset, \forall, \exists$
Auxiliary symbols	( )

- Indexes on top are used to denote the number of arguments, called arity, in predicates and functions.
- Indexes on the bottom are used to disambiguate between symbols having the same name.
- Predicates of arity =1 correspond to properties or concepts

Write in FOL the following NL sentences

- "Einstein is a scientist"

  Scientist(einstein)

- "There is a monkey"

  $\exists x$ Monkey(x)

- "There exists a dog which is black"

  $\exists x$ (Dog(x) $\wedge$ Black(x))

- "All persons have a name"

  $\forall x$ (Person(x) $\supset$ $\exists y$ Name(x, y))

Write in FOL the following NL sentences

- "The sum of two odd numbers is even"

  $\forall x\ \forall y\ (\ Odd(x) \wedge Odd(y) \supset Even(Sum(x,y))\ )$

- "A father is a male person having at least one child"

  $\forall x\ (\ Father(x) \supset Person(x) \wedge Male(x) \wedge\ \exists y\ hasChilden(x, y)\ )$

- "There is exactly one dog"

  $\exists x\ Dog(x) \wedge\ \forall x\ \forall y\ (\ Dog(x) \wedge Dog(y) \supset x = y\ )$

- "There are at least two dogs"

  $\exists x\ \exists y\ (\ Dog(x) \wedge Dog(y) \wedge \neg(x = y)\ )$

The use of FOL in mathematics

- Express in FOL the fact that every natural number x multiplied by 1 returns x (identity):

$$\forall x \, (\, Natural(x) \supset (Mult(x, 1) = x) \,)$$

- Express in FOL the fact that the multiplication of two natural numbers is commutative:

$$\forall x \, \forall y \, (\, Natural(x) \wedge Natural(y) \supset (Mult(x, y) = Mult(y, x)) \,)$$

The use of FOL in mathematics

- FOL has being introduced to express mathematical properties

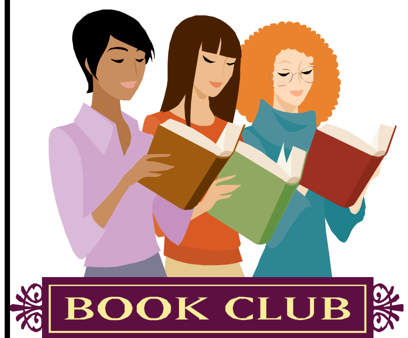
- The set of axioms describing the properties of equality between natural numbers (by Peano):

<u>Axioms about equality</u>

1. $\forall x_1 \, (x_1 = x_1)$ reflexivity
2. $\forall x_1 \, \forall x_2 \, (x_1 = x_2 \supset x_2 = x_1)$ symmetricity
3. $\forall x_1 \, \forall x_2 \, \forall x_3 \, (x_1 = x_2 \wedge x_2 = x_3 \supset x_1 = x_3)$ transitivity
4. $\forall x_1 \, \forall x_2 \, (x_1 = x_2 \supset S(x_1) = S(x_2))$ successor

NOTE: Other axioms can be given for the properties of the <u>successor</u>, the <u>addition</u> (+) and the <u>multiplication</u> (x).

Modeling the club of married problem



There are exactly three people in the club, Tom, Sue and Mary. Tom and Sue are married. If a member of the club is married, their spouse is also in the club. Mary is not married.

L = {tom, sue, mary, Club, Married}

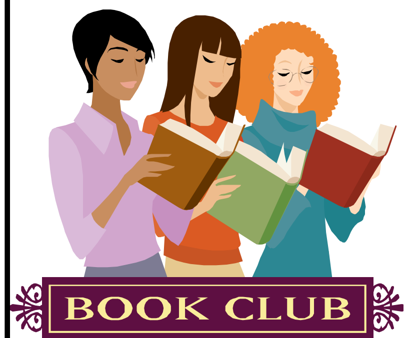
$\mathrm{Club}(tom) \wedge \mathrm{Club}(sue) \wedge \mathrm{Club}(mary) \wedge \forall x\, (\mathrm{Club}(x) \supset (x = tom \vee x = sue \vee x = mary))$

$\mathrm{Married}(tom, sue)$

$\forall x\, \forall y\, ((\mathrm{Club}(x) \wedge \mathrm{Married}(x, y)) \supset \mathrm{Club}(y))$

$\neg\, \exists x\, \mathrm{Married}(mary, x)$

Modeling the club of married problem (II)



There are exactly three people in the club, Tom, Sue and Mary. Tom and Sue are married. If a member of the club is married, their spouse is also in the club.

Add enough common sense FOL statements (e.g. everyone has at most one spouse, nobody can be married to himself or herself, Tom, Sue and Mary are different people) to make it entail that Mary is not married in FOL.

L = {tom, sue, mary, Club, Married}

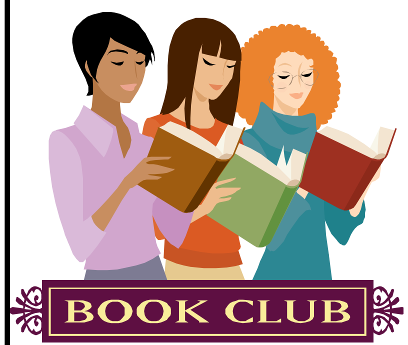
S1: $Club(tom) \wedge Club(sue) \wedge Club(mary) \wedge \forall x\, (Club(x) \supset (x = tom \vee x = sue \vee x = mary))$

S2: $Married(tom, sue)$

S3: $\forall x\ \forall y\ ((Club(x) \wedge Married(x, y)) \supset Club(y))$

S4: $\neg\ \exists x\ Married(mary, x)$

Modeling the club of married problem (III)



There are exactly three people in the club, Tom, Sue and Mary. Tom and Sue are married. If a member of the club is married, their spouse is also in the club.

Add enough common sense FOL statements (e.g. everyone has at most one spouse, nobody can be married to himself or herself, Tom, Sue and Mary are different people) to make it entail that Mary is not married in FOL.

We need to add the following:

S5: $\forall x\ \forall y\ \forall z\ ((Married(x, y) \land Married(x, z) \supset y = z)$ — at most one wife

S6: $\neg\ \exists x\ Married(x, x)$ — nobody is married with himself/herself

S7: $\neg\ (tom{=}sue) \land \neg\ (tom{=}mary) \land \neg\ (mary{=}sue)$ — unique name assumption

Interpretation

FOL interpretation for a language L

A first order interpretation for the language $L = \langle c_1, c_2, \ldots, f_1, f_2, \ldots, R_1, R_2, \ldots \rangle$ is a pair $\langle \Delta, \mathcal{I} \rangle$ where

- Δ is a non empty set called interpretation domain
- $\mathcal{I}$ is is a function, called interpretation function
  - $\mathcal{I}(c_i) \in \Delta$ (elements of the domain)
  - $\mathcal{I}(f_i) : \Delta^n \longrightarrow \Delta$ (n-ary function on the domain)
  - $\mathcal{I}(P_i) \subseteq \Delta^n$ (n-ary relation on the domain)

where n is the arity of f_i and P_i.

Example of interpretation

Example (Of interpretation)

Symbols Constants: *alice*, *bob*, *carol*, *robert*
Function: *mother-of* (with arity equal to 1)
Predicate: *friends* (with arity equal to 2)

Domain $\Delta = \{1, 2, 3, 4, \dots\}$

Interpretation $\mathcal{I}(alice) = 1$, $\mathcal{I}(bob) = 2$, $\mathcal{I}(carol) = 3$,
$\mathcal{I}(robert) = 2$

$$\mathcal{I}(\textit{mother-of}) = M \qquad \begin{array}{l} M(1) = 3 \\ M(2) = 1 \\ M(3) = 4 \\ M(n) = n + 1 \text{ for } n \geq 4 \end{array}$$

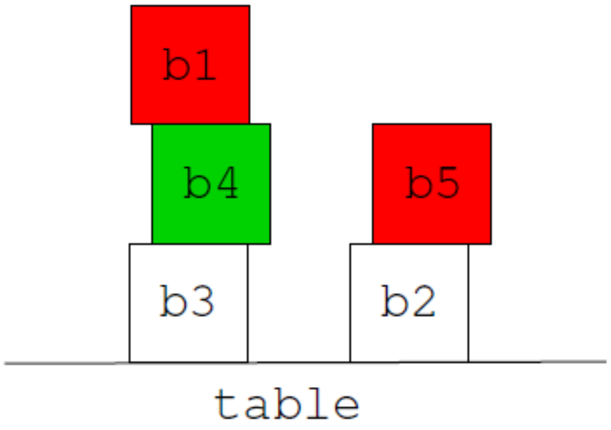
$$\mathcal{I}(friends) = F = \left\{ \begin{array}{lll} \langle 1, 2 \rangle, & \langle 2, 1 \rangle, & \langle 3, 4 \rangle, \\ \langle 4, 3 \rangle, & \langle 4, 2 \rangle, & \langle 2, 4 \rangle, \\ \langle 4, 1 \rangle, & \langle 1, 4 \rangle, & \langle 4, 4 \rangle \end{array} \right\}$$

Modeling “blocks world”

Non Logical symbols

Constants: A, B, C, D, E, F;

Predicates: On^2, $Above^2$, $Free^1$, Red^1, $Green^1$.



Interpretation $\mathcal{I}_1$

- $\mathcal{I}_1(A) = b_1$, $\mathcal{I}_1(B) = b_2$, $\mathcal{I}_1(C) = b_3$, $\mathcal{I}_1(D) = b_4$, $\mathcal{I}_1(E) = b_5$, $\mathcal{I}_1(F) = table$
- $\mathcal{I}_1(On) = \{\langle b_1, b_4\rangle, \langle b_4, b_3\rangle, \langle b_3, table\rangle, \langle b_5, b_2\rangle, \langle b_2, table\rangle\}$
- $\mathcal{I}_1(Above) = \{\langle b_1, b_4\rangle, \langle b_1, b_3\rangle, \langle b_1, table\rangle, \langle b_4, b_3\rangle, \langle b_4, table\rangle, \langle b_3, table\rangle, \langle b_5, b_2\rangle, \langle b_5, table\rangle, \langle b_2, table\rangle\}$
- $\mathcal{I}_1(Free) = \{\langle b_1\rangle, \langle b_5\rangle\}$, $\mathcal{I}_1(Green) = \{\langle b_4\rangle\}$, $\mathcal{I}_1(Red) = \{\langle b_1\rangle, \langle b_5\rangle\}$

Interpretation of terms

Definition (Assignment)

An assignment a is a function from the set of variables to Δ.

$a[x/d]$ denotes the assignment that conincides with a on all the variables but x, which is associated to d.

Interpretation of terms

The interpretation of a term t w.r.t. the assignment a, in symbols $\mathcal{I}(t)[a]$ is recursively defined as follows:

$$
\begin{aligned}
\mathcal{I}(x_i)[a] &= a(x_i) \\
\mathcal{I}(c_i)[a] &= \mathcal{I}(c_i) \\
\mathcal{I}(f(t_1, \dots, t_n))[a] &= \mathcal{I}(f)(\mathcal{I}(t_1)[a], \dots, \mathcal{I}(t_n)[a])
\end{aligned}
$$

Satisfiability of a formula

Definition (Satisfiability of a formula w.r.t. an assignment)

An interpretation $\mathcal{I}$ satisfies a formula ϕ w.r.t. the assignment a according to the following rules:

$$
\begin{array}{rcl}
\mathcal{I} \models t_1 = t_2[a] & \text{iff} & \mathcal{I}(t_1)[a] = \mathcal{I}(t_2)[a] \\
\mathcal{I} \models P(t_1, \dots, t_n)[a] & \text{iff} & \langle \mathcal{I}(t_1)[a], \dots, \mathcal{I}(t_n)[a] \rangle \in \mathcal{I}(P) \\
\mathcal{I} \models \phi \wedge \psi[a] & \text{iff} & \mathcal{I} \models \phi[a] \text{ and } \mathcal{I} \models \psi[a] \\
\mathcal{I} \models \phi \vee \psi[a] & \text{iff} & \mathcal{I} \models \phi[a] \text{ or } \mathcal{I} \models \psi[a] \\
\mathcal{I} \models \phi \supset \psi[a] & \text{iff} & \mathcal{I} \not\models \phi[a] \text{ or } \mathcal{I} \models \psi[a] \\
\mathcal{I} \models \neg\phi[a] & \text{iff} & \mathcal{I} \not\models \phi[a] \\
\mathcal{I} \models \phi \equiv \psi[a] & \text{iff} & \mathcal{I} \models \phi[a] \text{ iff } \mathcal{I} \models \psi[a] \\
\mathcal{I} \models \exists x \phi[a] & \text{iff} & \text{there is a } d \in \Delta \text{ such that} \mathcal{I} \models \phi[a[x/d]] \\
\mathcal{I} \models \forall x \phi[a] & \text{iff} & \text{for all } d \in \Delta, \mathcal{I} \models \phi[a[x/d]]
\end{array}
$$

Decide whether the following formula are satisfied by $\mathcal{I}_1$

1. A is above C, D is above F and on E
   $Above(A, C) \wedge Above(D, F) \wedge On(D, E)$ (NO, "On")
2. A is green while C is not
   $Green(A) \wedge \neg Green(C)$ (NO)
3. Everything is on something
   $\forall x \exists y On(x, y)$ (NO, "table")
4. Everything that is free has nothing on it
   $\forall x (Free(x) \supset \neg \exists y On(y, x))$ (YES)
5. Everything that is green is free
   $\forall x (Green(x) \supset Free(x))$ (NO)
6. There is something that is red and is not free
   $\exists x (Red(x) \wedge \neg Free(x))$ (NO)
7. Everything that is not green and is above B, is red
   $\forall x (\neg Green(x) \wedge Above(x, B) \supset Red(x))$ (NO)

FOL Tableaux

. . . for propositional connectives

α rules $\quad \dfrac{\phi \wedge \psi}{\begin{array}{c}\phi \\ \psi\end{array}} \qquad \dfrac{\neg(\phi \vee \psi)}{\begin{array}{c}\neg\phi \\ \neg\psi\end{array}} \qquad \dfrac{\neg\neg\phi}{\phi} \qquad \dfrac{\neg(\phi \supset \psi)}{\begin{array}{c}\phi \\ \neg\psi\end{array}}$

β rules $\quad \dfrac{\phi \vee \psi}{\phi \mid \psi} \qquad \dfrac{\phi \supset \psi}{\neg\phi \mid \psi} \qquad \dfrac{\neg(\phi \wedge \psi)}{\neg\phi \mid \neg\psi} \qquad \dfrac{\phi \equiv \psi}{\begin{array}{c|c}\phi & \neg\phi \\ \psi & \neg\psi\end{array}} \qquad \dfrac{\neg(\phi \equiv \psi)}{\begin{array}{c|c}\phi & \neg\phi \\ \neg\psi & \psi\end{array}}$

$$\frac{\forall x \phi(x)}{\phi(t)} \qquad\qquad \frac{\exists x \phi(x)}{\phi(c)} \text{ for a new constant } c$$

FOL Tableaux

Check via tableaux the validity/satisability of the formula:

- $\phi = \forall xy(P(x) \supset Q(y)) \supset (\exists xP(x) \supset \forall yQ(y))$

$$\neg(\forall xy(P(x) \supset Q(y)) \supset (\exists xP(x) \supset \forall yQ(y)))$$

$$(\forall xy(P(x) \supset Q(y))$$
$$\neg(\exists xP(x) \supset \forall yQ(y))$$

$$\exists xP(x)$$
$$\neg \forall yQ(y)$$

$$P(a)$$
$$\neg Q(b)$$

$$P(a) \supset Q(b)$$

$\neg P(a)$ $Q(b)$

X   X

MODAL LOGIC

Introduction

- We want to model situations like this one:
  1. "Fausto is always happy" circumstances"
  2. "Fausto is happy under certain
- In PL/ClassL we could have: HappyFausto
- In modal logic we have:
  1. $\Box$ HappyFausto
  2. $\Diamond$ HappyFausto

As we will see, this is captured through the notion of "possible words" and of "accessibility relation"

Syntax

- We extend PL with two logical modal operators:
  □ (box) and ◊ (diamond)

  □P : “Box P” or “necessarily P” or “P is necessary true”
  ◊P : “Diamond P” or “possibly P” or “P is possible”

  Note that we define □P = ¬◊¬P, i.e. □ is a primitive symbol

- The grammar is extended as follows:

  <Atomic Formula> ::= A | B | ... | P | Q | ... | ⊥ | ⊤ |
  <wff> ::= <Atomic Formula> | ¬<wff> | <wff>∧ <wff> | <wff>∨ <wff> |
  <wff> → <wff> | <wff> ↔ <wff> | □ <wff> | ◊ <wff>

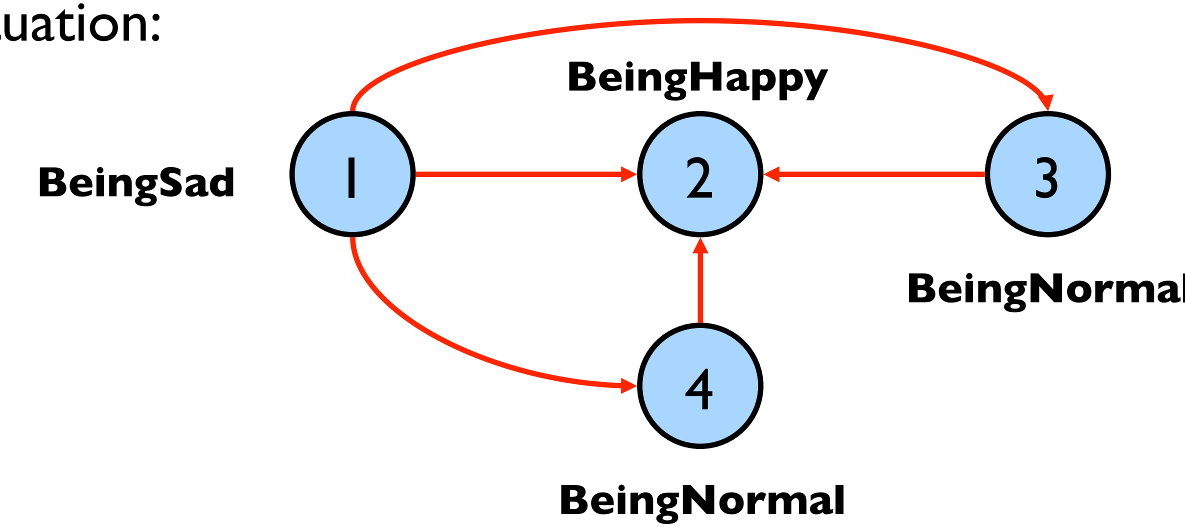
Semantics: Kripke Model

- A Kripke Model is a triple M = <W, R, I> where:
  - W is a non empty set of worlds
  - $R \subseteq W \times W$ is a binary relation called the accessibility relation
  - I is an interpretation function $I: L \rightarrow pow(W)$ such that to each proposition P we associate a set of possible worlds I(P) in which P holds

- Each $w \in W$ is said to be a world, point, state, event, situation, class … according to the problem we model
- In a Kripke model, <W, R> is called <u>frame</u> and is a relational structure.

Semantics: Kripke Model

- Consider the following situation:



- M = <W, R, I>

  W = {1, 2, 3, 4}

  R = {<1, 2>, <1, 3>, <1, 4>, <3, 2>, <4, 2>}

  I(BeingHappy) = {2} I(BeingSad) = {1} I(BeingNormal) = {3, 4}

Truth relation (true in a world)

Truth is relative to a world, so we define that relation of $\models$ between a world in a model and a formula

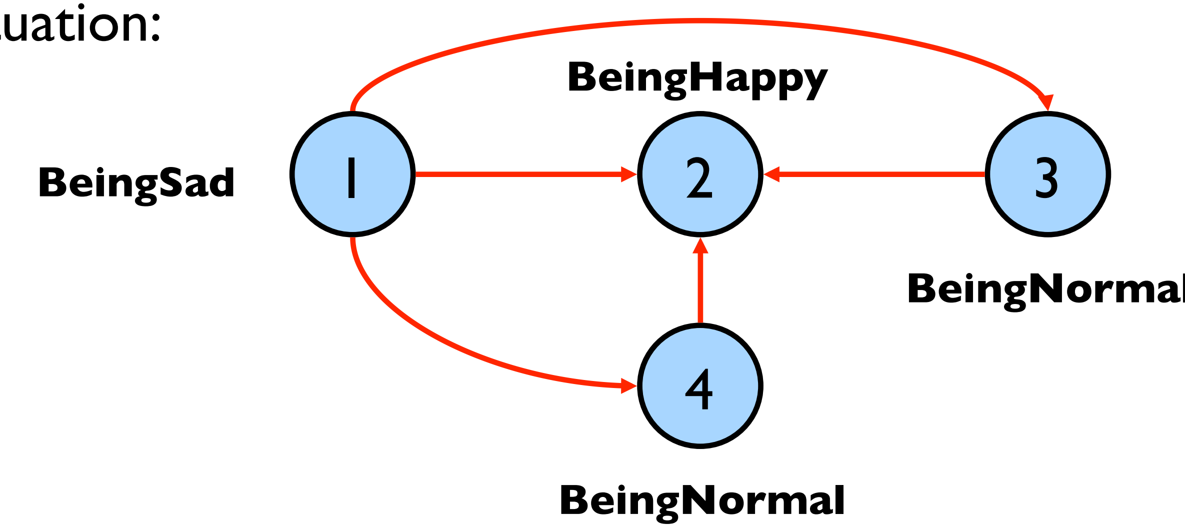
$$\mathcal{M}, w \models p \text{ iff } w \in \mathcal{I}(p)$$
$$\mathcal{M}, w \models \phi \wedge \psi \text{ iff } \mathcal{M}, w \models \phi \text{ and } \mathcal{M}, w \models \psi$$
$$\mathcal{M}, w \models \phi \vee \psi \text{ iff } \mathcal{M}, w \models \phi \text{ or } \mathcal{M}, w \models \psi$$
$$\mathcal{M}, w \models \phi \supset \psi \text{ iff } \mathcal{M}, w \models \phi \implies implies \mathcal{M}, w \models \psi$$
$$\mathcal{M}, w \models \phi \equiv \psi \text{ iff } \mathcal{M}, w \models \phi \text{ iff } \mathcal{M}, w \models \psi$$
$$\mathcal{M}, w \models \neg\phi \text{ iff not } \mathcal{M}, w \models \phi$$
$$\mathcal{M}, w \models \Box\phi \text{ iff for all } w' \text{ s.t. } wRw', \mathcal{M}, w' \models \phi$$
$$\mathcal{M}, w \models \Diamond\phi \text{ iff there is a } w' \text{ s.t. } wRw' \text{ and } \mathcal{M}, w' \models \phi$$

ϕ is globally satisfied in a model $\mathcal{M}$, in symbols, $\mathcal{M} \models \phi$ if

$$\mathcal{M}, w \models \phi \quad \text{for all } w \in W$$

Semantics: Kripke Model

- Consider the following situation:



- M = <W, R, I>

  W = {1, 2, 3, 4}

  R = {<1, 2>, <1, 3>, <1, 4>, <3, 2>, <4, 2>}

  I(BeingHappy) = {2}   I(BeingSad) = {1}   I(BeingNeutral) = {3, 4}

  M, 2 ⊨ BeingHappy   M, 2 ⊨ ¬BeingSad

  M, 4 ⊨ □BeingHappy   M, 1 ⊨ ◊BeingHappy   M, 1 ⊨ ¬◊BeingSad

Satisfiability and Validity

- **Satisfiability**

  A proposition $P \in L^{ML}$ is satisfiable in a Kripke model $M = <W, R, I>$ if $M, w \vDash P$ for all worlds $w \in W$.

  We can then write $M \vDash P$

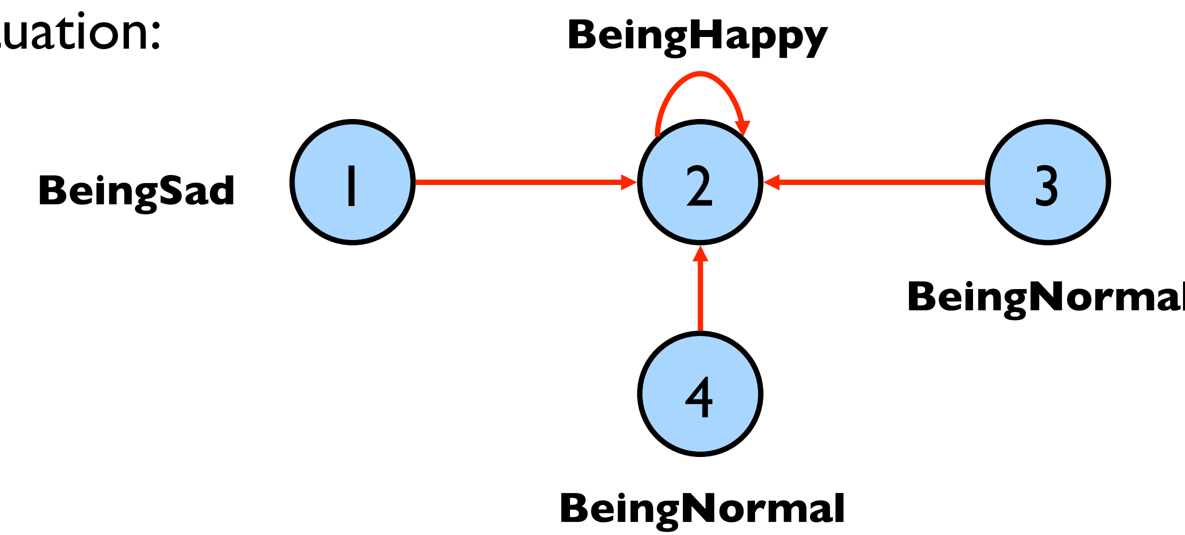
- **Validity**

  A proposition $P \in L^{ML}$ is valid if P is satisfiable for all models M (and by varying the frame $<W, R>$).

  We can write $\vDash P$

Satisfiability

- Consider the following situation:



- M = <W, R, I>

  W = {1, 2, 3, 4}

  R = {<1, 2>, <2, 2>, <3, 2>, <4, 2>}

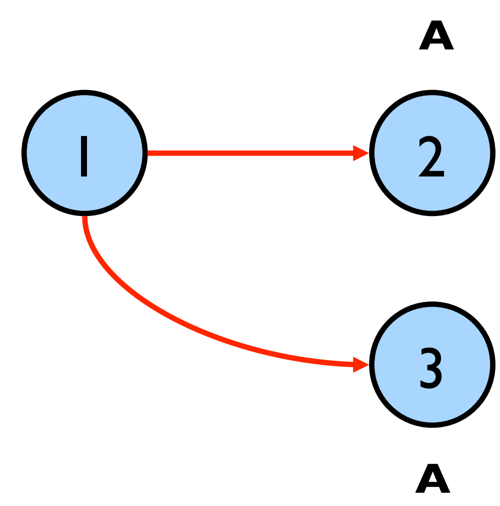
  I(BeingHappy) = {2} I(BeingSad) = {1} I(BeingNormal) = {3, 4}

  Does $\Box$BeingHappy is satisifable?

  M, w $\models$ $\Box$BeingHappy for all w $\in$ W, therefore $\Box$BeingHappy is satisifable in M.

Validity

- Prove that P: $\Box A \rightarrow \Diamond A$ is valid



- In all models M = <W, R, I>,

  (1) $\Box A$ means that for every $w \in W$ such that wRw' then $M, w' \vDash A$

  (2) $\Diamond A$ means that for some $w \in W$ such that wRw' then $M, w' \vDash A$

  It is clear that if (1) then (2) in the example

  (this is valid in <u>serial frames</u>)

Kinds of frames

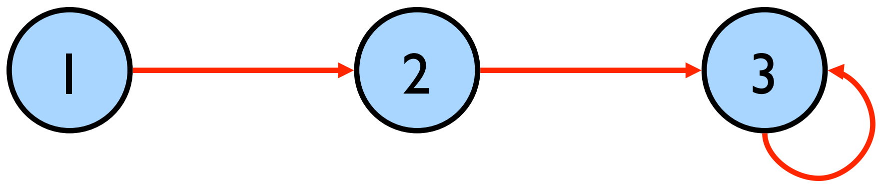
- Given the frame F = <W, R>, the relation R is said to be:

  - **Serial** iff for every $w \in W$, there exists $w' \in W$ s.t. wRw'
  - **Reflexive** iff for every $w \in W$, wRw
  - **Symmetric** iff for every $w, w' \in W$, if wRw' then $w'Rw$
  - **Transitive** iff for every $w, w', w'' \in W$, if wRw' and $w'Rw''$ then wRw''

- We call a frame <W, R> serial, reflexive, symmetric or transitive according to the properties of the relation R

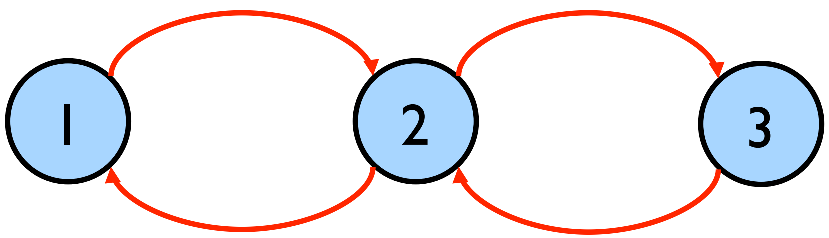
Kinds of frames

- Serial: for every w ∈ W, there exists w' ∈ W s.t. wRw'

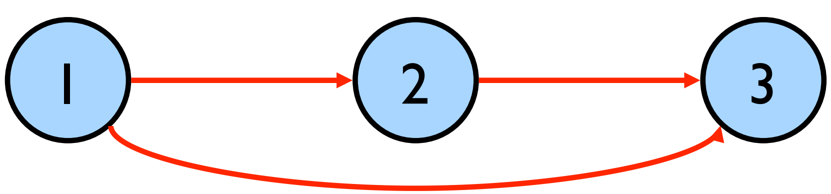


- Reflexive: for every w ∈ W, wRw

- Symmetric: for every w, w' ∈ W, if wRw' then w'Rw



- Transitive: for every w, w', w'' ∈ W, if wRw' and w'Rw'' then wRw''



Modal logic exercise

Given the Kripke model M = <W, R, I> with:

W = {1, 2, 3}   R = {<1, 2>, <2, 1>, <1, 3>, <3, 3>}   I(A) = {1, 2} and I(B) = {2, 3}

- **Say whether the frame <W, R> is serial, reflexive, symmetric or transitive.**
  - It is serial.
- **Is M, 1 ⊨ ◊(A ∧ B)? Provide a proof for your response.**
  - Yes, because A ∧ B is true in 2 and 2 is accessible from 1.
- **Is ☐A satisfiable in M? Provide a proof for your response.**
  - We should have that M, w ⊨ ☐A for all worlds w. This means that for all worlds w there is a w' such that wRw' and M, w' ⊨ A.
  - For w = 1 we have 1R3 and M, 3 ⊨ ¬A. Therefore the response is NO.