Mathematical Logic
Exam
15 January 2016

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- Verranno valutati solo gli esercizi con soluzione riportata su questi fogli;
- Rispondete utilizzando una penna a inchiostro (no matite);
- Scrivete in stampatello, in modo chiaro (risposte illeggibili non saranno considerate);
- Depennate in modo chiaro il lavoro di brutta copia e le risposte che non volete siano considerate;

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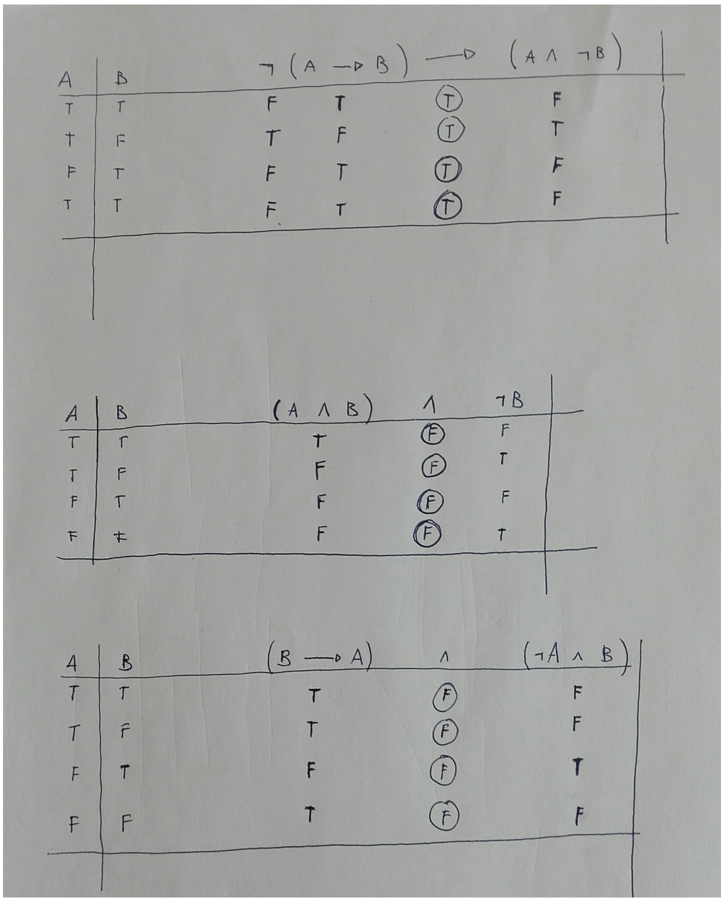
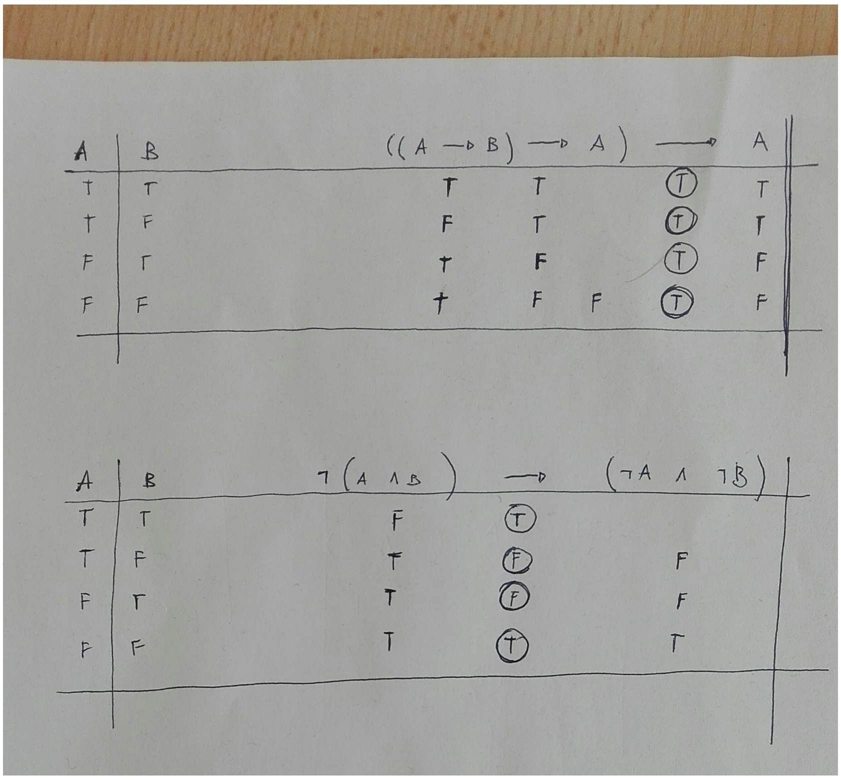
Exercise 1. [5 marks]
Say whether the propositions below are valid (VAL), satisfiable but not valid (SAT) or unsatisfiable (UNSAT) in PL:

a.	$\neg(A \rightarrow B) \rightarrow (A \wedge \neg B)$	□ VAL	□ SAT	□ UNSAT
b.	$(A \wedge B) \wedge \neg B$	□ VAL	□ SAT	□ UNSAT
c.	$(B \rightarrow A) \wedge (\neg A \wedge B)$	□ VAL	□ SAT	□ UNSAT
d.	$((A \rightarrow B) \rightarrow A) \rightarrow A$	□ VAL	□ SAT	□ UNSAT
e.	$\neg(A \wedge B) \rightarrow (\neg A \wedge \neg B)$	□ VAL	□ SAT	□ UNSAT





Exercise 2. [4 marks]
Provide the formal definition of (a) being true in a model and (b) satisfiability for a proposition P in PL

[2 marks] (a) P is true under v if v ⊨ P, where v ⊨ P iff v(P) = True [2 marks]

[2 marks] (b) P is satisfiable if there is some (at least one) truth valuation v such that v ⊨ P

Exercise 3. [2 marks]
Say whether the following propositions are "true" or "false".

a)	In PL, the interpretation of the symbol ⊥ has always the same meaning regardless the interpretation function.	□ True □ False
b)	In PL, the notation v ⊨ A has to be read "v entails A".	□ True □ False

Exercise 4. [4 marks]
Using the tableaux calculus, determine whether the formula $\neg A \wedge (C \rightarrow A) \wedge B$ is unsatisfiable. Mark each branch as open or closed. Motivate the answer.

Convert the formula in CNF.

$$
\begin{array}{c}
\neg A \wedge (\neg C \vee A) \wedge B \\
| \\
\neg A \\
| \\
(\neg C \vee A) \wedge B \\
| \\
(\neg C \vee A) \\
| \\
B \\
/ \quad \backslash \\
\neg C \qquad A
\end{array}
$$

Since the first branch is open then it is satisfiable.

Exercise 5. [4 marks]
Provide the steps and the output of the DPLL algorithm (by assuming a version WITHOUT the pure literal step) for the PL formula $(C \rightarrow A) \wedge (C \rightarrow B) \wedge \neg(A \wedge B)$ and say if the formula is satisfiable or not.

Convert the formula in CNF. By implication elimination we obtain the formula: $(\neg C \vee A) \wedge (\neg C \vee B) \wedge (\neg A \vee \neg B)$

As we do not have unit clauses, we need to go for the branching literal step.
Let us choose C and first call DPLL for: $(\neg C \vee A) \wedge (\neg C \vee B) \wedge (\neg A \vee \neg B) \wedge C$
By assigning $v(C) = T$, by propagation we obtain: $A \wedge B \wedge (\neg A \vee \neg B)$

With the branching on A we can choose to call DPLL on: $A \wedge B \wedge (\neg A \vee \neg B) \wedge A$
By assigning $v(A) = T$, by propagation we obtain: $B \wedge \neg B$ that is clearly inconsistent.

Check for the other branch $(\neg C \vee A) \wedge (\neg C \vee B) \wedge (\neg A \vee \neg B) \wedge \neg C$
By assigning $v(C) = F$, by propagation we obtain $(\neg A \vee \neg B)$
We need to go for the branching literal step again:
Let us choose C and first call DPLL for: $\neg A \wedge (\neg A \vee \neg B)$
By assigning $v(A) = F$, by propagation we obtain that the formula is satisfiable.

Exercise 6. [2 marks]
Provide the formal definition of *interpretation over an assignment* in FOL

$I_a(c) = I(c)$	**for each constant c**
$I_a(x) = a(x)$	**for each variable x**
$I_a(f^n(t_1,\ldots, t_n)) = I(f^n)(I_a(t_1),\ldots, I_a(t_n))$	**for each function f of arity n**

Exercise 7. [4 marks] – FOL formalization

Given the following language:
Constants: A, B, C, D, E, F;
Predicates: On^2, $Above^2$, $Free^1$, Red^1, $Green^1$.

Translate in FOL the following natural language sentences:

[1 mark] Everything that is free has nothing on it
φ_1 : $\forall x.(Free(x) \longrightarrow \neg\exists y.On(y, x))$

[1 mark] Everything that is green is free
φ_2 : $\forall x.(Green(x) \longrightarrow Free(x))$

[1 mark] There is something that is red and is not free
φ_3 : $\exists x.(Red(x) \wedge \neg Free(x))$

[1 mark] Everything that is not green and is above B, is red
φ_4 : $\forall x.(\neg Green(x) \wedge Above(x, B) \longrightarrow Red(x))$

Exercise 8. [4 marks] – FOL interpretation/semantics

Given the language provided in the previous exercise and given the following interpretation:

- $I_1(A)$ = hat, $I_1(B)$ = Joe, $I_1(C)$ = bike, $I_1(D)$ = Jill, $I_1(E)$ = case, $I_1(F)$ = ground;
- $I_1(On)$ = {⟨hat, Joe⟩, ⟨Joe, bike⟩, ⟨bike, ground⟩, ⟨Jill, case⟩, ⟨case, ground⟩}
- $I_1(Above)$ = {⟨hat, Joe⟩, ⟨hat, bike⟩, ⟨hat, ground⟩, ⟨Joe, bike⟩, ⟨Joe, ground⟩, ⟨bike, ground⟩, ⟨Jill, case⟩, ⟨Jill, ground⟩, ⟨case, ground⟩}
- $I_1(Free)$ = {⟨hat⟩, ⟨Jill⟩}, $I_1(Green)$ = {⟨hat⟩, ⟨ground⟩}, $I_1(Red)$ = {⟨bike⟩, ⟨case⟩}

For each formula in Exercise 7, check whether it is satisfied by the interpretation I_1.

φ_1	☐ yes	☐ no
φ_2	☐ yes	☐ no
φ_3	☐ yes	☐ no
φ_4	☐ yes	☐ no

Exercise 9. [4 marks] – Modal logic
Given the Kripke model M = <W, R, I> with: W = {1, 2, 3}, R = {<1, 2>, <2, 1>, <1, 3>, <3, 3>}, I(A) = {1, 2} and I(B) = {2, 3}

a. **[2 marks]** Say whether the frame <W, R> is serial, reflexive, symmetric or transitive.
It is serial.

b. **[2 marks]** Is M, 1 ⊨ ◊(A ∧ B)? Provide a proof for your response.
Yes, because A ∧ B is true in 2 and 2 is accessible from 1.