Mathematical Logic
2° Midterm
21 December 2015

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- Indicate con una croce gli esercizi che volete che vengano valutati;
- Verranno valutati solo gli esercizi segnati con la croce e con soluzione riportata su questi fogli;
- Rispondete utilizzando una penna a inchiostro (no matite);
- Scrivete in stampatello, in modo chiaro (risposte illeggibili non saranno considerate);
- Depennate in modo chiaro il lavoro di brutta copia e le risposte che non volete siano considerate;

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Exercise 1. [4 marks] – DPLL

Check using DPLL whether the formula P: $\neg A \wedge ((B \rightarrow C) \vee A) \wedge (\neg C \vee \neg A \vee D)$ is satisfiable. Provide the application of the steps of the algorithm and the explanation of the answer.

The formula must be converted in CNF first. For this, it is sufficient to apply $\rightarrow$ elimination.
This leads to the formula: $\neg A \wedge (\neg B \vee C \vee A) \wedge (\neg C \vee \neg A \vee D)$. We can use DPLL now.
We can assign $v(A) = 0$. By propagating the unit clause we eliminate the literal A in the second clause and we eliminate the third clause. This simplifies the formula to $(\neg B \vee C)$.
We then call DPLL on $B \wedge (\neg B \vee C)$ and on $\neg B \wedge (\neg B \vee C)$ where B is the branching literal.
The first call clearly concludes satisfiability for $v(B) = 1$, $v(C) = 1$.

Exercise 2. [8 marks] – FOL formalization
Represent the following propositions in FOL:

[2 marks] "A sad father is a male person having an unemployed child"

$\forall x\ (SadFather(x) \rightarrow (Male(x) \wedge Person(x) \wedge \exists y\ (HasChild(x, y) \wedge Unemployed(y))))$

[2 marks] "There exist at least two dogs of different color"

$\exists x\ \exists y\ (\ Dog(x) \wedge Dog(y) \wedge \neg(x = y) \wedge \neg(Color(x) = Color(y)))$

[2 marks] "There is a bicycle only if there are at least two wheels"

$\exists x\ \exists y\ ((wheel(x) \wedge wheel\ (y) \wedge \neg\ (\ x = y)) \leftrightarrow\ \exists z\ bicycle(z)$

[2 marks] "Bill takes either Analysis or Geometry (but not both)"

$(Takes(Bill, Analysis) \wedge\ \neg\ Takes(Bill, Geometery)) \vee (\neg\ Takes(Bill, Analysis) \wedge\ Takes(Bill, Geometery))$

Exercise 3. [3 marks] – FOL formalization
Express in FOL the commutative property of the sum of two natural numbers, i.e. the fact that A + B = B + A.

$\forall x\ \forall y\ ((Natural(x) \wedge Natural(y)) \rightarrow Sum(x,y) = Sum(y,x))$

Exercise 4. [4 marks] – FOL interpretation
Consider a world with objects A, B e C. Consider the logical language with constant symbols a, b and c, function symbols, f and g, and predicate symbols P, Q and R. Consider the following interpretation I:

$I(a) = A, I(b) = A, I(c) = B$
$I(f) = \{\langle A, B\rangle, \langle B, C\rangle, \langle C, C\rangle\}$
$I(P) = \{A, B\}$
$I(Q) = \{C\}$
$I(R) = \{\langle B, A\rangle, \langle C, B\rangle, \langle C, C\rangle\}$

For each of the following FOL formulas say whether it is true or false in the given interpretation I?

$Q(f(c))$	□ **True** (selected) □ **False**
$R(a, b)$	□ **True** □ **False** (selected)
$\exists x.f(x) = b$	□ **True** □ **False** (selected)
$\forall xy.R(x, y) \rightarrow (\forall z.R(z, y) \rightarrow z = y)$	□ **True** □ **False** (selected)

Exercise 5. [4 marks] – FOL semantics

For the two following formulas, (a) explain in natural language what is the meaning formally encoded in the formulas and, (b) provide an interpretation under which the second sentence is false and the first sentence is true.

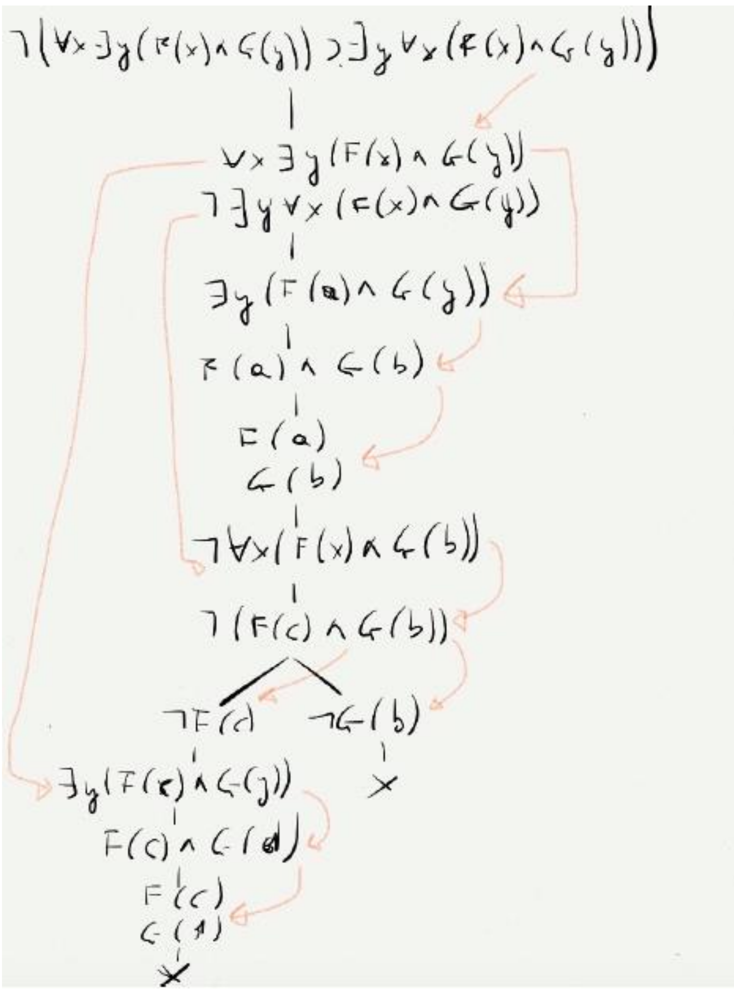
∀x∃y.f(x, y) Per ogni individuo x del dominio esiste un y per cui la funzione f ha un valore
∃y∀x.f(x, y) Esiste un individuo y che con tutti gli individui x del dominio ha un valore della funzione f

Nota: come dalle slide, f è un simbolo che indica tipicamente una funzione, in questo caso di arità = 2.
U = {A, B, C}
I(f) = {(A, B, C), (B, C, D), (C, A, D)} Nota: l'interpretazione di f ha 3 argomenti (arità + 1)
I(A)=a; I(B)=b; I(C)=c

Exercise 6. [4 marks] – FOL reasoning

Check the validity of the following formula with tableau method:

$$\forall x \exists y(F(x) \wedge G(y)) \longrightarrow \exists y \forall x(F(x) \wedge G(y))$$



Exercise 7. [6 marks] – Modal logic
Given the Kripke model M = <W, R, I> with W = {1, 2}, R = {<1, 2>, <2, 2>}, I(A) = {1,2} and I(B) = {1}

a. **[2 marks]** Say whether the frame <W, R> is serial, reflexive, symmetric or transitive
It is serial and transitive (it is actually also Euclidian).

b. **[2 marks]** Is $M, 1 \models \Diamond\neg B$? Provide the proof for your response.
Yes, because 2 is accessible from 1 and $M, 2 \models \neg B$

c. **[2 marks]** Prove that $\Box A$ is satisfiable in M
It is satisfiable because $M, 2 \models A$, and for all worlds w in {1, 2} we have that 2 is accessible from w.