Mathematical Logics

17 Resolution and Unification

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The rule of Propositional Resolution

$$\textbf{RES} \qquad \frac{A \vee C, \qquad \neg C \vee B}{A \vee B}$$

The formula $A \vee B$ is called a resolvent of $A \vee C$ and $B \vee \neg C$, denoted $Res(A \vee C, B \vee \neg C)$.

Exercize

Show that the Resolution rule is logically sound; i.e., that the conclusion is a logical consequence of the premise

RES inference rules assumes that the formulas are in normal form (CNF)

Soundness of Propositional Resolution

$$\textbf{RES} \quad \frac{A \vee C, \quad \neg C \vee B}{A \vee B}$$

To prove soundness of the **RES** rule we show that the following logical consequence holds:

$$(A \vee C) \wedge (\neg C \vee B) \models A \vee B$$

i.e., we have to show that, for every interpretation $\mathcal{I}$,

$$\text{if } \mathcal{I} \models (A \vee C) \wedge (\neg C \vee B), \text{ then } \mathcal{I} \models A \vee B$$

- Suppose that $\mathcal{I} \models (A \vee C) \wedge (\neg C \vee B)$, then $\mathcal{I} \models (A \vee C)$ and $\mathcal{I} \neg C \vee B)$
- This implies that $\mathcal{I} \models A \vee C$, and therefore that either $\mathcal{I} \models A$ or $\mathcal{I} \models C$
  - If $\mathcal{I} \models A$, then $\mathcal{I} \models A \vee B$
  - If $\mathcal{I} \models C$, then from the fact that $\mathcal{I} \models \neg C \vee B$ we have that $\mathcal{I} \models B$. Which implies that $\mathcal{I} \models A \vee B$.

Soundness of Propositional Resolution

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Generality of Propositional Resolution

The propositional resolution inference rule implements a very general inference pattern, that includes many inference rules of propositional logics once the formulas are transformed in CNF.

Rule Name	Original form	CNF form
Modus Ponens	$\dfrac{p \quad p \supset q}{q}$	$\dfrac{\{p\} \quad \{\neg p, q\}}{\{q\}}$
Modus tollens	$\dfrac{\neg q \quad p \supset q}{\neg p}$	$\dfrac{\{\neg q\} \quad \{\neg p, q\}}{\{\neg p\}}$
Chaining	$\dfrac{p \supset q \quad q \supset r}{p \supset r}$	$\dfrac{\{\neg p \vee q\} \quad \{\neg q, r\}}{\{\neg p, r\}}$
Reductio ad absurdum	$\dfrac{p \supset q \quad p \supset \neg q}{\neg p}$	$\dfrac{\{\neg p \vee q\} \quad \{\neg p, \neg q\}}{\{\neg p\}}$
Reasoning by case	$\dfrac{p \vee q \quad p \supset r \quad q \supset r}{r}$	$\dfrac{\dfrac{\{p, q\} \quad \{\neg p, r\}}{\{q, r\}} \quad \{\neg q, r\}}{\{r\}}$
Tertium non datur	$\dfrac{p \quad \neg p}{\bot}$	$\dfrac{\{p\} \quad \{\neg p\}}{\{\}}$

Completeness of propositinal resolution

- Using propositional resolution alone (without axiom schemata or other rules of inference), it is possible to build a theorem prover that is sound and complete for Propositional Logic.
- But we have to transform every formula in CNF.
- The search space using propositional resolution is much smaller than for Modus Ponens and the Hilbert Axiom Schemas

Clausal normal forms - (CNF)

- A clause is essentially an elementary disjunction $l_1 \vee \cdots \vee l_n$ but written as a (possibly empty) set of literals $\{l_1, \ldots, l_n\}$.
- The empty clause $\{\}$ is a clause containing no literals. and therefore it is not satisfiable
- A unit clause is a clause containing only one literal.
- A clausal form is a (possibly empty) set of clauses, written as a list: $C_1 \ldots C_k$ it represents the conjunction of these clauses.

Every formula in CNF can be re-written in a clausal form, and therefore every propositional formula is equivalent to one in a clausal form.

Example (Clausal form)

the clausal form of the CNF-formula $(p \vee \neg q \vee \neg r) \wedge \neg p \wedge (\neg q \vee r)$ is $\{p, \neg q, \neg r\}, \{\neg p\}, \{\neg q, r\}$
Note that the empty clause $\{\}$ (sometimes denoted by $\Box$) is not satisfiable (being an empty disjunction)

Clausal Propositional Resolution rule

The Propositional Resolution rule can be rewritten for clauses:

$$RES\,\frac{\{A_1, \ldots, C, \ldots, A_m\} \quad \{B_1, \ldots, \neg C, \ldots, B_n\}}{\{A1, \ldots, A_m, B_1, \ldots, B_n\}}$$

- The clause $\{A_1, \ldots, A_m, B_1, \ldots, B_n\}$ is called a resolvent of the clauses $\{A_1, \ldots, C, \ldots, A_m\}$ and $\{B_1, \ldots, \neg C, \ldots, B_n\}$.

Example (Applications of RES rule)

$$\frac{\{p, q, \neg r\} \quad \{\neg q, \neg r\}}{\{p, \neg r, \neg r\}} \qquad \frac{\{\neg p, q, \neg r\} \quad \{r\}}{\{\neg p, q\}} \qquad \frac{\{\neg p\} \quad \{p\}}{\{\}}$$

The rule of Propositional Resolution

Example

Try to apply the rule **RES** to the following two set of clauses $\{\{\neg p, q\}, \{\neg q, r\}, \{p\}, \{\neg r\}\}$

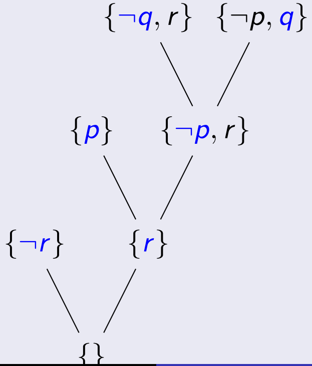
Solution

The rule of Propositional Resolution

Example

Try to apply the rule **RES** to the following two set of clauses
$\{\{\neg p, q\}, \{\neg q, r\}, \{p\}, \{\neg r\}\}$

Solution



Some remarks

$$\frac{\{p, q, \neg r\} \quad \{\neg q, \neg r\}}{\{p, \neg r, \neg r\}} \qquad \frac{\{\neg p, q, \neg r\} \quad \{r\}}{\{\neg p, q\}} \qquad \frac{\{\neg p\} \quad \{p\}}{\{\}}$$

- Note that two clauses can have more than one resolvent, e.g.:

$$\frac{\{p, \neg q\} \quad \{\neg p, q\}}{\{\neg q, q\}} \qquad \frac{\{\neg p, q\} \quad \{p, \neg p\}}{\{\neg p, p\}}$$

  However, it is wrong to apply the Propositional Resolution rule for both pairs of complementary literals simultaneously as follows:

$$\frac{\{p, \neg q\} \quad \{\neg p, q\}}{\{\}}$$

Sometimes, the resolvent can (and should) be simplified, by removing duplicated literals on the fly:

$$\{A_1, \ldots, C, C, \ldots, A_m\} \Rightarrow \{A_1, \ldots, C, \ldots, A_m\}.$$

For instance:

$$\frac{\{p, \neg q, \neg r\} \quad \{q, \neg r\}}{\{p, \neg r\}} \quad \text{instead of} \quad \frac{\{p, \neg q, \neg r\} \quad \{q, \neg r\}}{\{p, \neg r, \neg r\}}$$

Propositional resolution as a refutation system

- The underlying idea of Propositional Resolution is like the one of Semantic Tableau: in order to prove the validity of a logical consequence $A_1, \ldots, A_n \models B$, show that the set of formulas $\{A_1, \ldots, A_n, \neg B\}$ is Unsatisfiable
- That is done by transforming the formulae $A_1, \ldots, A_n$ and $\neg B$ into a clausal form, and then using repeatedly the Propositional Resolution rule in attempt to derive the empty clause $\{\}$.
- Since $\{\}$ is not satisfiable, its derivation means that $\{A_1, \ldots, A_n, \neg B\}$ cannot be satisfied together. Then, the logical consequence $A_1, \ldots, A_n \vdash B$ holds.
- Alternatively, after finitely many applications of the Propositional Resolution rule, no new applications of the rule remain possible. If the empty clause is not derived by then, it cannot be derived at all, and hence the $\{A_1, \ldots, A_n, \neg B\}$ can be satisfied together, so the logical consequence

Problem solving using resolution

- For direct inference, resolution cannot be used, even when the goal is a simple clause. for instance is we want to prove derive $p \wedge q$ form p and q, (i.e., we want to prove that $p, q \models p \wedge q$ directly, **RES** inference rule is useless.
- However, resolution is complete when the goal is the empty clause, (i.e., $\bot$) If $\{\phi_1, \phi_2, \dots \phi_n\}$ is a finite set of clauses, then $\{\phi_1, \phi_2, \dots \phi_n\} \models \bot$ iff there is a sequence of resolutions which may be applied to $\{\phi_1, \phi_2, \dots \phi_n\}$ to yield the empty clause.
- Therefore we cannot prove that $p, q \models p \wedge q$ directly, but we have to transform the problem in the form accepted by Resolution, i.e., in the equivalent form

$$\{p\}, \{q\}, \{\neg p, \neg q\} \models \bot$$

Problem solving using resolution

Example

- To prove $p \supset p$ in Hilbert system is extremely difficult. In the resolution system, it is trivial.
- $p \supset p$ is equivalent to $\neg p \vee p$.
- To prove the validity of this formula, convert its negation to CNF: $\neg(\neg p \vee p)$ obtaining $\{p\}, \{\neg p\}$
- with a single application of **RES**

$$\textbf{RES}\,\frac{\{p\} \quad \{\neg p\}}{\{\}}$$

- we obtain the empty clause.

Propositional resolution - Examples

Example

- Check whether $(\neg p \supset q), \neg r \vdash p \vee (\neg q \wedge \neg r)$ holds.
- Check whether $p \supset q, q \supset r \models p \supset r$ holds.
- Show that the following set of clauses is unsatisfiable $\{\{A, B, \neg D\}, \{A, B, C, D\}, \{\neg B, C\}, \{\neg A\}, \{\neg C\}\}$

Problem solving with Propositional Resolution

Six sculptures $\{C, D, E, F, G, H\}$ are to be exhibited in rooms $\{1, 2, 3\}$ of an art gallery.

1. Sculptures C and E may not be exhibited in the same room.
2. Sculptures D and G must be exhibited in the same room.
3. If sculptures E and F are exhibited in the same room, no other sculpture may be exhibited in that room.
4. At least one sculpture must be exhibited in each room, and
5. no more than three sculptures may be exhibited in any room.

1. If sculpture D is exhibited in room 1 and sculptures E and F are exhibited in room 2, which of the following must be true?
   1. Sculpture C must be exhibited in room 1.
   2. Sculpture H must be exhibited in room 3.
   3. Sculpture G must be exhibited in room 1.
   4. Sculpture H must be exhibited in room 2.
   5. Sculptures C and H must be exhibited in the same room.

Problem solving with Propositional Resolution

Six sculptures $\{C, D, E, F, G, H\}$ are to be exhibited in rooms $\{1, 2, 3\}$ of an art gallery.

$$P = \{Exhibits(X, n) \mid X \in \{C, \ldots, H\},\ n \in \{1, 2, 3\}\}$$

$$\bigwedge_{\substack{X \in \{C,\ldots,H\} \\ n \in \{1,2,3\}}} Exhibits(X, n) \equiv \neg Exhibits(X, (n \mod 3)+1) \wedge \neg Exhibits(X, (n \mod 3)+2)$$

1. Sculptures C and E may not be exhibited in the same room.

   no formalization = no information

2. Sculptures D and G must be exhibited in the same room.

$$\bigwedge_{n \in \{1,2,3\}} Exhibits(D, n) \equiv Exhibits(G, n)$$

Problem solving with Propositional Resolution

3. If sculptures E and F are exhibited in the same room, no other sculpture may be exhibited in that room.

$$\bigwedge_{n\in\{1,2,3\}}\left(Exhibits(E,n)\wedge Exhibits(F,n)\supset\bigwedge_{X\in\{C,\dots,H\}\setminus\{E,F\}}\neg Exhibits(X,n)\right)$$

4. At least one sculpture must be exhibited in each room

$$\bigwedge_{n\in\{1,2,3\}}\bigvee_{X\in\{C,\dots,H\}}Exhibits(X,n)$$

5. no more than three sculptures may be exhibited in any room.

$$\bigwedge_{n\in\{1,2,3\}}\bigwedge_{\substack{\mathcal{S}\subset\{C,\dots,H\}\\|\mathcal{S}|=4}}\neg\left(\bigwedge_{X\in E}Exhibits(X,n)\right)$$

Problem solving with Propositional Resolution

1. If sculpture D is exhibited in room 1 and sculptures E and F are exhibited in room 2, which of the following must be true?

$$Exhibites(D, 1) \wedge Exhibites(E, 2) \wedge Exhibites(F, 3) \supset \phi$$

1. Sculpture C must be exhibited in room 1. $\phi = Exhibits(C, 1)$
2. Sculpture H must be exhibited in room 3. $\phi = Exhibits(B, 3)$
3. Sculpture G must be exhibited in room 1. $\phi = Exhibits(G, 1)$
4. Sculpture H must be exhibited in room 2. $\phi = Exhibits(H, 2)$
5. Sculptures C and H must be exhibited in the same room. $\phi = \bigvee_{n \in \{1,2,3\}} Exhibits(C, n) \equiv Exhibits(H, n)$

Problem solving with Propositional Resolution

$$CNF\left(\bigwedge_{\substack{X\in\{C,\dots,H\}\\ n\in\{1,2,3\}}} Exhibits(X,n) \equiv \left(\begin{array}{c}\neg Exhibits(X,(n \mod 3)+1)\wedge\\ \neg Exhibits(X,(n \mod 3)+2)\end{array}\right)\right) =$$

$$\left\{\begin{array}{l}\{\neg Exhibits(X,n), \neg Exhibits(X,m)\},\\ \{Exhibits(X,1), Exhibits(X,2), Exhibits(X,3)\}\end{array}\;\middle|\;\begin{array}{l}X\in\{C,\dots,H\}\\ n\neq m\in\{1,2,3\}\end{array}\right\}$$

$$CNF\left(\bigwedge_{n\in\{1,2,3\}} Exhibits(D,n)\equiv Exhibits(G,n)\right) =$$

$$\left\{\begin{array}{l}\{\neg Exhibits(D,n), Exhibits(G,n)\}\\ \{\neg Exhibits(G,n), Exhibits(D,n)\}\end{array}\;\middle|\; n\in\{1,2,3\}\right\}$$

Problem solving with Propositional Resolution

$$CNF\left(\bigwedge_{n\in\{1,2,3\}}\left(Exhibits(E,n)\wedge Exhibits(F,n)\supset\bigwedge_{\substack{X\in\{C,\dots,H\}\\X\notin\{E,F\}}}\neg Exhibits(X,n)\right)\right)=$$

$$\left\{\left\{\begin{array}{c}\neg Exhibits(E,n),\neg Exhibits(F,n),\\ \neg Exhibits(X,n)\end{array}\right\}\ \middle|\ \begin{array}{l}n\in\{1,2,3\}\\ X\in\{C,\dots,H\}\setminus\{E,F\}\end{array}\right\}$$

Problem solving with Propositional Resolution

$$CNF\left(\bigwedge_{n\in\{1,2,3\}}\bigvee_{X\in\{C,\dots,H\}} Exhibits(X,n)\right) =$$

$$\{\{Exhibits(X,n) \mid X \in \{C,\dots,H\}\} \mid n \in \{1,2,3\}\} =$$

$$\left\{\begin{array}{c}\{Exhibits(C,1), Exhibits(C,2), Exhibits(C,3)\}\\ \{Exhibits(D,1), Exhibits(D,2), Exhibits(D,3)\}\\ \vdots \\ \{Exhibits(H,1), Exhibits(H,2), Exhibits(H,3)\}\end{array}\right\}$$

Problem solving with Propositional Resolution

$$CNF\left(\bigwedge_{n\in\{1,2,3\}}\ \bigwedge_{\substack{\mathcal{S}\subset\{C,\dots,H\}\\|\mathcal{S}|=4}} \neg\left(\bigwedge_{X\in E} Exhibits(X,n)\right)\right) =$$

$$\left\{\left\{\begin{array}{l}\neg Exhibits(X_1,n),\neg Exhibits(X_2,n),\\ \neg Exhibits(X_3,n),\neg Exhibits(X_4,n),\end{array}\right\}\ \middle|\ \begin{array}{l}\{X_1,X_2,X_3,X_4\}\subset\{C,\dots,H\}\\ X_i\neq X_j \text{ for } i\neq j,\quad n\in\{1,2,3\}\end{array}\right\} =$$

$$\left\{\begin{array}{c}\{\neg Exhibits(C,1),\neg Exhibits(D,1),\neg Exhibits(E,1),\neg Exhibits(F,1)\}\\ \{\neg Exhibits(C,1),\neg Exhibits(D,1),\neg Exhibits(E,1),\neg Exhibits(G,1)\}\\ \{\neg Exhibits(C,1),\neg Exhibits(D,1),\neg Exhibits(E,1),\neg Exhibits(H,1)\}\\ \vdots\\ \{\neg Exhibits(E,1),\neg Exhibits(F,1),\neg Exhibits(G,1),\neg Exhibits(H,1)\}\end{array}\right\}$$

$$CNF(\neg(Exhibites(D,1) \wedge Exhibites(E,2) \wedge Exhibites(F,3) \supset \phi) =$$

$$\{\{Exhibites(D,1)\}, \{Exhibites(E,2)\}, \{Exhibites(F,3)\}, \{\neg\phi\}\}$$

where ϕ is one of the following formulas

1. $Exhibits(C,1)$ NO
2. $Exhibits(B,3)$ NO
3. $Exhibits(G,1)$ YES
4. $Exhibits(H,2)$ NO
5. We consider the last case separately

Problem solving with Propositional Resolution

$Exhibits(D,1) \equiv Exhibits(G,1)$	assumption	(1)
$Exhibits(D,1) \wedge Exhibits(E,2) \wedge Exhibits(F,2) \supset$		
$Exhibits(G,1)$	goal	(2)
$\neg Exhibits(D,1), Exhibits(G,1)$	clausify (1)	(3)
$Exhibits(D,1)$	deny (10)	(4)
$\neg Exhibits(G,1)$	deny (10)	(5)
$Exhibits(G,1)$	RES (3), (4)	(6)
$\bot$	RES (6), (5)	(7)

Problem solving with Propositional Resolution

5. Sculptures C and H must be exhibited in the same room.

$$\bigvee_{n\in\{1,2,3\}} Exhibits(C,n) \equiv Exhibits(H,n)$$

$$CNF\left(\neg\left(\begin{array}{c} Exhibites(D,1) \wedge Exhibites(E,2) \wedge Exhibites(F,3) \supset \\ \bigvee_{n\in\{1,2,3\}} Exhibits(C,n) \equiv Exhibits(H,n) \end{array}\right)\right) =$$

$$\left\{\begin{array}{c} \{Exhibites(D,1)\}, \{Exhibites(E,2)\}, \{Exhibites(F,3)\} \\ \{Exhibites(C,1), Exhibites(H,1)\}, \{\neg Exhibites(C,1), \neg Exhibites(H,1)\}, \\ \{Exhibites(C,2), Exhibites(H,2)\}, \{\neg Exhibites(C,2), \neg Exhibites(H,2)\}, \\ \{Exhibites(C,3), Exhibites(H,3)\}, \{\neg Exhibites(C,3), \neg Exhibites(H,3)\} \end{array}\right\}$$

$Exhibits(E,2) \wedge Exhibits(F,2) \supset \neg Exhibits(C,2)$	assumption	(8)
$Exhibits(E,2) \wedge Exhibits(F,2) \supset \neg Exhibits(H,2)$	assumption	(9)
$Exhibits(D,I) \wedge Exhibits(E,2) \wedge Exhibits(F,2) \supset$		(10)
$(Exhibits(C,1) \equiv Exhibits(H,1)) \vee$		
$(Exhibits(C,2) \equiv Exhibits(H,2)) \vee$		
$(Exhibits(C,3) \equiv Exhibits(H,3))$	goal	
$\{\neg Exhibits(E,2), \neg Exhibits(F,2), \neg Exhibits(C,2)$	clausify (8)	(11)
$\{\neg Exhibits(E,2), \neg Exhibits(F,2), \neg Exhibits(H,2)$	clausify (9)	(12)
$Exhibits(E,2)$	deny (10)	(13)
$Exhibits(F,2)$	deny (10)	(14)
$Exhibits(C,2), Exhibits(H,2)$	deny (10)	(15)
$\neg Exhibits(F,2), \neg Exhibits(H,2)$	RES (12), (13)	(16)
$\neg Exhibits(H,2)$	RES (16), (14)	(17)
$\neg Exhibits(F,2), \neg Exhibits(C,2)$	RES (11), (13)	(18)
$\neg Exhibits(C,2)$	RES (18), (14)	(19)
$Exhibits(H,2)$	RES (15), (19)	(20)
$\bot$	RES (20), (17)	(21)

First-order resolution

- The Propositional Resolution rule in clausal form extended to first-order logic:

$$\frac{\{A_1,\ldots,Q(s_1,\ldots,s_n),\ldots,A_m\} \quad \{B_1,\ldots,\neg Q(s_1,\ldots,s_n),\ldots,B_n\}}{\{A_1,\ldots,A_m,B_1,\ldots,B_n\}}$$

  this rule, however, is not strong enough.
- **example:** consider the clause set

$$\{\{p(x)\},\{\neg p(f(y))\}\}$$

  is not satisfiable, as it corresponds to the unsatisfiable formula

$$\forall x \forall y.(p(x) \wedge \neg p(f(y)))$$

- however, the resolution rule above cannot derive an empty clause from that clause set, because it cannot unify the two clauses in order to resolve them.
- so, we need a stronger resolution rule, i.e., a rule capable to understand that x and $f(y)$ can be instantiated to the same

Unification

Finding a common instance of two terms.

Intuition in combination with Resolution

$$S = \left\{ \begin{array}{c} friend(x, y) \supset friend(y, x) \\ friend(x, y) \supset knows(x, mother(y)) \\ friend(Mary, John) \\ \neg knows(John, mother(Mary)) \end{array} \right\}$$

$$cnf(S) = \left\{ \begin{array}{c} \neg friend(x, y) \vee friend(y, x) \\ \neg friend(x, y) \vee knows(x, mother(y)) \\ friend(Mary, John) \\ \neg knows(John, mother(Mary)) \end{array} \right\}$$

Is $cnf(S)$ satisfiable or unsatisfiable?
The key point here is to apply the right substitutions

Substitutions: A Mathematical Treatment

A substitution is a finite set of replacements

$$\sigma = [t_1/x_1, \dots, t_k/x_k]$$

where $x_1, \dots, x_k$ are distinct variables and $t_i \neq x_i$.

$t\sigma$ represents the result of the substitution σ applied to t.

$c\sigma = c$	(non) substitution of constants
$x[t_1/x_1, \dots t_n/x_n] = t_i$ if $x = x_i$ for some i	substitution of variables
$x[t_1/x_1, \dots t_n/x_n] = x$ if $x \neq x_i$ for all i	(non) substitution of variables
$f(t,u)\sigma = f(t\sigma, u\sigma)$	substitution in terms
$P(t,u)\sigma = P(t\sigma, u\sigma)$	... in literals
$\{L_1, \dots, L_m\}\sigma = \{L_1\sigma, \dots, L_m\sigma\}$	... in clauses

Composing Substitutions

Composition of σ and θ written $\sigma \circ \theta$, satisfies for all terms t

$$t(\sigma \circ \theta) = (t\theta)\sigma$$

If $\sigma = [t_1/x_1, \ldots t_n/x_n]$ and $\theta = [u_1/x_1, \ldots u_n/x_n]$, then

$$\sigma \circ \theta = [t_1\theta/x_1, \ldots t_n\theta/x_n]$$

Identity substitution

$$[x/x, t_1/x_1, \ldots t_n/x_n] = [t_1/x_1, \ldots t_n/x_n]$$

$$\sigma \circ [] = \sigma$$

Associativity

$$\sigma \circ (\theta \circ \phi) = (\sigma \circ \theta) \circ \phi = \sigma \circ \theta \circ \phi =$$

Non commutativity, in general we have that

$$\sigma\theta \neq \theta\sigma$$

Composition of substitutions - example

$$f(g(x), f(y, x))[f(x, y)/x][g(a)/x, x/y] =$$
$$f(g(f(x, y)), f(y, f(x, y)))[g(a)/x, x/y] =$$
$$f(g(f(g(a), x)), f(x, f(g(a), x)))$$

$$f(g(x), f(y, x))[g(a)/x, x/y][f(x, y)/x] =$$
$$f(g(g(a)), f(x, g(a)))[f(x, y)/x] =$$
$$f(g(g(a)), f(f(x, y), g(a)))$$

Computing the composition of substitutions

The composition of two substitutions $\tau = [t_1/x_1, \dots, t_k/x_k]$ and σ

1. Extend the replaced variables of τ with the variables that are replaced in σ but not in τ with the identity substitution x/x
2. Apply the substitution simultaneously to all terms $[t_1, \dots, t_k]$ to obtaining the substitution $[t_1\sigma/x_1, \dots, t_k\sigma/x_k]$.
3. Remove from the result all cases x_i/x_i , if any.

Example

$$
\begin{aligned}
[f(x,y)/x, x/y][y/x, a/y, g(y)/z] &= \\
[f(x,y)/x, x/y, z/z][y/x, a/y, g(y)/z] &= \\
[f(y,a)/x, y/y, g(y)/z] &= \\
[f(y,a)/x, g(y)/z]
\end{aligned}
$$

Unifiers and Most General Unifiers

σ is a unifier of terms t and u if $t\sigma = u\sigma$.
For instance

- the substitution $[f(y)/x]$ unifies the terms x and $f(y)$
- the substitution $[f(c)/x, c/y, c/z]$ unifies the terms $g(x, f(f(z)))$ and $g(f(y), f(x))$
- There is no unifier for the pair of terms $f(x)$ and $g(y)$, nor for the pair of terms $f(x)$ and x.

σ is more general than θ if $\theta = \sigma \circ \phi$ for some substitution ϕ.

σ is a most general unifier for two terms t and u if it a unifier for t and u and it is more general of all the unifiers of t and u.

If σ unifies t and u then so does $\sigma \circ \theta$ for any θ.

A most general unifier of $f(a, x)$ and $f(y, g(z))$ is $\sigma = [a/y, g(z)/x]$. The common instance is

$$f(a, x)\sigma = f(a, g(z)) = f(y, g(z))\sigma$$

Unifier

Example

The substitution $[3/x, g(3)/y]$ unifies the terms $g(g(x))$ and $g(y)$. The common instance is g(g(3)). This is not however the most general unifier for these two terms. Indeed, these terms have many other unifiers, including the following:

unifying substitution	**common instance**
$[f(u)/x, g(f(u))/y]$	$g(g(f(u)))$
$[z/x, g(z)/y]$	$g(g(z))$
$[g(x)/y]$	$g(g(x))$

$[g(x)/y]$ is also the most general unifier.

Examples of most general unifier

Notation: $x, y, z \dots$ are variables, $a, b, c, \dots$ are constants $f, g, h, \dots$ are functions $p, q, r, \dots$ are predicates.

terms	MGU	result of the substitution
$p(a, b, c)$ $p(x, y, z)$	$[a/x, b/y, c/z]$	$p(a, b, c)$
$p(x, x)$ $p(a, b)$	*None*	
$p(f(g(x, a), x)$ $p(z, b)$	$[b/x, f(g(b, a))/z]$	$p(f(g(b, a), b)$
$p(f(x, y), z)$ $p(z, f(a, y))$	$[f(a, y)/z, a/x]$	$p(f(a, y), f(a, y))$

Unification Algorithm: Preparation

We shall formulate a unification algorithm for literals only, but it can easily be adapted to work with formulas and terms.

Sub expressions Let L be a literal. We refer to formulas and terms appearing within L as the *subexpressions* of L. If there is a subexpression in L starting at position i we call it $L^{(i)}$ (otherwise (i) is undefined.

Disagreement pairs. Let L_1 and L_2 be literals with $L_1 \neq L_2$. The disagreement pair of L_1 and L_2 is the pair $(L_1^{(i)}, L_2^{(i)})$ of subexpressions of L_1 and L_2 respectively, where i is the smallest number such that $L_1^{(i)} \neq L_2^{(i)}$).

Example The disagreement pair of

$$P(g(c), f(a, g(x), h(a, g(b))))$$
$$P(g(c), f(a, g(x), h(k(x, y), z)))$$

is $(a, k(x, y))$

Robinson's Unification Algorithm

Imput: a set of literals Δ
Output: $\sigma = MGU(\Delta$ or Undefined!

$\sigma := []$
while $|\Delta\sigma| > 1$ **do**
    pick a disagreement pair p in $\Delta\sigma$'
    if no variable in p **then**
        return 'not unifiable';
    else
        let $p = (x, t)$ with x being a variable;
        if x occurs in t **then**
            return 'not unifiable';
        else $\sigma := \sigma \circ [t/x]$;
return σ

Substitution

Exercize

Let $\sigma = [a/x, f(b)/y, c/z]$ and $\theta = [f(f(a))/v, x/z, g(y)/x]$

- compute $\sigma \circ \theta$ and $\theta \circ \sigma$
- For every of the following formulæ, compute (i) $\phi\sigma$; (ii) $\phi\theta$; (iii) $\phi\sigma \circ \theta$; and (iv) $\phi\theta \circ \sigma$
  1. $\phi = p(x, y, z)$
  2. $\phi = p(h(v)) \vee \neg q(z, x)$
  3. $\phi = q(x, z, v) \vee \neg q(g(y), x, f(f(a)))$
- are σ and θ and their compositions idempotent?

Definition

A function $f : X \longrightarrow X$ on a set X is idempotent if and only if $f(x) = f(f(x))$

An example of idempotent function are $round(\cdot) : \mathbb{R} \longrightarrow \mathbb{R}$, that returns the closer integer $round(x)$ to a real number x.

Unification

Exercize

For every C_1, C_2 and σ, decide whether (i) σ is a unifier of C_1 and C_2; and (ii) σ is the MGU of C_1 and C_2

C_1	C_2	σ
$P(a, f(y), z)$	$Q(x, f(f(v)), b)$	$[a/x, f(b)/y, b/z]$
$Q(x, h(a, z), f(x))$	$Q(g(g(v)), y, f(w))$	$[g(g(v))/x, h(a, z)/y, x/w]$
$Q(x, h(a, z), f(x))$	$Q(g(g(v)), y, f(w))$	$[g(g(v))/x, h(a, z)/y, g(g(v))/w]$
$R(f(x), g(y))$	$R(z, g(v))$	$[a/x, f(a)/z, v/y]$

Unification

Exercize

Consider the signature $\Sigma = \langle a, b, f(\cdot,\cdot), g(\cdot,\cdot), P(\cdot,\cdot,\cdot)\rangle$ Use the algorithm from the previous lecture to decide whether the following clauses are unifiable.

1. $\{P(f(x,a), g(y,y), z), P(f(g(a,b),z), x, a)\}$
2. $\{P(x,x,z), P(f(a,a), y, y)\}$
3. $\{P(x, f(y,z), b), P(g(a,y), f(z, g(a,x)), b)\}$
4. $\{P(a, y, U), P(x, f(x,U), g(z,b))\}$

Unification of $P(f(x,a), g(y,y), z)$, $P(f(g(a,b),z), x, a)$

- $\{P(f(x,a), g(y,y), Z), P(f(g(a,b), Z), x, a)\}$
- $\sigma = [g(a,b)/x]$
- $\{P(f(x,a), g(y,y), Z), P(f(g(a,b), Z), x, a)\}\sigma = \{P(f(g(a,b),a), g(y,y), z), P(f(g(a,b),z), g(a,b), a)\}$.
- $\{P(f(g(a,b),a), g(y,y), z), P(f(g(a,b),z), g(a,b), a)\}$.
- $\sigma = [g(a,b)/x, a/z]$
- $\{P(f(g(a,b),a), g(y,y), z), P(f(g(a,b),z), g(a,b), a)\}\sigma = \{P(f(g(a,b),a), g(y,y), a), P(f(g(a,b),a), g(a,b), a)\}$
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- $\{P(f(g(a,b),a), g(a,a), a), P(f(g(a,b),a), g(a,b), a)\}$
- a and b are two constants and they are not unificable. So the

Unification of $P(f(x,a), g(y,y), z)$, $P(f(g(a,b),z),x,a)$

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- $\{P(f(g(a,b),a), g(y,y), a), P(f(g(a,b),a), g(a,b), a)\}\sigma =$
  $\{P(f(g(a,b),a), g(a,a), a), P(f(g(a,b),a), g(a,b), a)\}$
- $\{P(f(g(a,b),a), g(a,a), a), P(f(g(a,b),a), g(a,b), a)\}$
- a and b are two constants and they are not unificable. So the

Unification of $\{P(x, x, z), P(f(a, a), y, y)\}$

- $\{P(x, x, z), P(f(a, a), y, y)\}$
- $\sigma = [f(a, a)/x]$
- $\{P(x, x, z), P(f(a, a), y, y)\}\sigma = \{P(f(a, a), f(a, a), z), P(f(a, a), y, y)\}$
- $\{P(f(a, a), f(a, a), z), P(f(a, a), y, y)\}$
- $\sigma = [f(a, a)/x, f(a, a)/y]$
- $\{P(f(a, a), f(a, a), z), P(f(a, a), y, y)\}\sigma = \{P(f(a, a), f(a, a), z), P(f(a, a), f(a, a), f(a, a))\}$
- $\{P(f(a, a), f(a, a), z), P(f(a, a), f(a, a), f(a, a))\}$
- $\sigma = [f(a, a)/x, f(a, a)/y, f(a, a)/z]$
- $\{P(f(a, a), f(a, a), z), P(f(a, a), f(a, a), f(a, a))\}\sigma = \{P(f(a, a), f(a, a), f(a, a)), P(f(a, a), f(a, a), f(a, a))\}$
- the two terms are equal, so the initial terms are unifiable with the mgu equal to $\sigma = [f(a, a)/x, f(a, a)/y, f(a, a)/z]$

Unification

Exercize

Find, when possible, the MGU of the following pairs of clauses.

1. $\{q(a), q(b)\}$
2. $\{q(a, x), q(a, a)\}$
3. $\{q(a, x, f(x)), q(a, y, y,)\}$
4. $\{q(x, y, z), q(u, h(v, v), u)\}$
5. $\left\{ \begin{array}{l} p(x_1, g(x_1), x_2, h(x_1, x_2), x_3, k(x_1, x_2, x_3)), \\ p(y_1, y_2, e(y_2), y_3, f(y_2, y_3), y_4) \end{array} \right\}$

Theorem-Proving Example

$$(\exists y \forall x R(x, y)) \supset (\forall x \exists y R(x, y))$$

Negate $\neg((\exists y \forall x R(x, y)) \supset (\forall x \exists y R(x, y)))$

NNF $\exists y \forall x R(x, y), \quad \exists x \forall y \neg R(x, y)$

Skolemize $R(x, b), \quad \neg R(a, y)$

Unify $MGU(R(x, b), R(a, y)) = [a/x, b/y]$

Contrad.: We have the contradiction $R(b, a), \neg R(b, a)$, so the formula is valid

Theorem-Proving Example

$$(\forall x \exists y R(x, y)) \supset (\exists y \forall x R(x, y))$$

Negate $\neg((\forall x \exists y R(x, y)) \supset (\exists y \forall x R(x, y)))$

NNF $\forall x \exists y R(x, y), \quad \forall y \exists x \neg R(x, y)$

Skolemize $R(x, f(x)), \quad \neg R(g(y), y)$

Unify $MGU(R(x, f(x)), \quad R(g(y), y)) = Undefined$

Contrad.: We do not have the contradiction, so the formula is not valid.

Resolution for first order logic

The resolution rule for Propositional logic is

$$\frac{\{l_1, \ldots, l_n, p\} \qquad \{\neg p, l_{n+1}, \ldots, l_m\}}{\{l_1, \ldots l_m\}}$$

The binary resolution rule

In first order logic each l_i and p are formulas of the form $P(t_1, \dots, t_n)$ or $\neg P(t_1, \dots, t_n)$.

When two opposite literals of the form $P(t_1, \dots, t_n)$ and $P(u_1, \dots, u_n)$ occur in the clauses C_1 and C_2 respectively, we have to find a way to partially instantiate them, by a substitution σ, in such a way the resolution rule can be applied, to to $C_1\sigma$ and $C_2\sigma$, i.e., such that $P(t_1, \dots, t_n)\sigma = P(u_1, \dots, u_n)\sigma$.

$$\frac{\{l_1, \dots, l_n, P(t_1, \dots, t_n)\}\{\neg P(u_1, \dots, u_n), l_{n+1}, \dots, l_m\}}{\{l_1, \dots l_m\}\sigma}$$

where σ is the $MGU(P(t_1, \dots, t_n), P(u_1, \dots, u_n))$.

The factoring rule

$$\frac{\{l_1, \ldots, l_n, l_{n+1}, \ldots, l_m\}}{\{l_1, l_{n+1}, \ldots l_m\}\sigma} \quad \text{If } l_1\sigma = \cdots = l_n\sigma$$

Example

Prove $\forall x \exists y \neg (P(y, x) \equiv \neg P(y, y))$

Clausal form $\{\neg P(y, a),\ \neg P(y, y)\},\ \{P(y, y),\ P(y, a)\}$

Factoring yields $\{\neg P(a, a)\},\ \{P(a, a)\}$

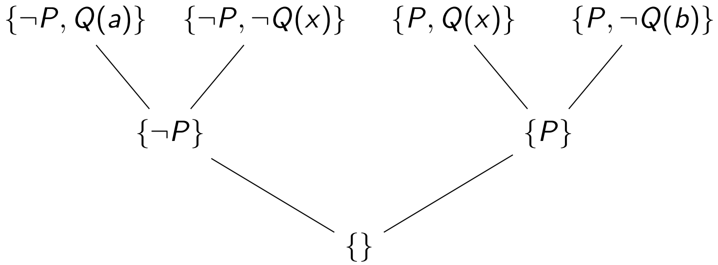
By resolution rule we obtain the empty clauses $\square$

A Non-Trivial Proof

$$\exists x[P \supset Q(x)] \wedge \exists x[Q(x) \supset P] \supset \exists x[P \equiv Q(x)]$$

Clauses are $\{P, \neg Q(b)\}$, $\{P, Q(x)\}$, $\{\neg P, \neg Q(x)\}$, $\{\neg P, Q(a)\}$

Apply resolution



Example

Assumptions:

- $\forall x(P(x) \supset P(f(x)))$
- $\forall x, y(Q(a, y) \wedge R(y, x) \supset P(x))$
- $\forall z R(b, g(a, z))$
- $Q(a, b)$

Goal $= P(f(g(a, c)))$

Inference

Example

Assumptions:

- $\forall x(P(x) \supset P(f(x)))$
- $\forall x, y(Q(a, y) \wedge R(y, x) \supset P(x))$
- $\forall z R(b, g(a, z))$
- $Q(a, b)$

Goal $= P(f(g(a, c)))$

1. clausify the assumptions

Inference

1. $\neg P(x), P(f(x))$
2. $\neg Q(a, y), \neg R(y, x), P(x)$
3. $R(b, g(a, z))$
4. $Q(a, b)$

Example

Assumptions:

- $\forall x(P(x) \supset P(f(x)))$
- $\forall x, y(Q(a, y) \wedge R(y, x) \supset P(x))$
- $\forall z R(b, g(a, z))$
- $Q(a, b)$

Goal $= P(f(g(a, c)))$

1. clausify the assumptions
2. negate and clausify the goal

Inference

1. $\neg P(x), P(f(x))$
2. $\neg Q(a, y), \neg R(y, x), P(x)$
3. $R(b, g(a, z))$
4. $Q(a, b)$
5. $\neg P(f(g(a, c)))$

Example

Assumptions:

- $\forall x(P(x) \supset P(f(x)))$
- $\forall x, y(Q(a, y) \wedge R(y, x) \supset P(x))$
- $\forall z R(b, g(a, z))$
- $Q(a, b)$

Goal $= P(f(g(a, c)))$

1. clausify the assumptions
2. negate and clausify the goal

Inference

1. $\neg P(x), P(f(x))$
2. $\neg Q(a, y), \neg R(y, x), P(x)$
3. $R(b, g(a, z))$
4. $Q(a, b)$
5. $\neg P(f(g(a, c)))$

Example

Assumptions:

- $\forall x(P(x) \supset P(f(x)))$
- $\forall x, y(Q(a, y) \wedge R(y, x) \supset P(x))$
- $\forall z R(b, g(a, z))$
- $Q(a, b)$

Goal $= P(f(g(a, c)))$

1. clausify the assumptions
2. negate and clausify the goal
3. $mgu(Q(a, y), Q(a, b)) = [y/b]$

Inference

1. $\neg P(x), P(f(x))$
2. $\neg Q(a, y), \neg R(y, x), P(x)$
3. $R(b, g(a, z))$
4. $Q(a, b)$
5. $\neg P(f(g(a, c)))$

Example

Assumptions:

- $\forall x(P(x) \supset P(f(x)))$
- $\forall x, y(Q(a, y) \wedge R(y, x) \supset P(x))$
- $\forall z R(b, g(a, z))$
- $Q(a, b)$

Goal $= P(f(g(a, c)))$

1. clausify the assumptions
2. negate and clausify the goal
3. $mgu(Q(a, y), Q(a, b)) = [y/b]$

Inference

1. $\neg P(x), P(f(x))$
2. $\neg Q(a, y), \neg R(y, x), P(x)$
3. $R(b, g(a, z))$
4. $Q(a, b)$
5. $\neg P(f(g(a, c)))$
6. $\neg R(b, x), P(x)$

Example

Assumptions:

- $\forall x(P(x) \supset P(f(x)))$
- $\forall x, y(Q(a, y) \wedge R(y, x) \supset P(x))$
- $\forall z R(b, g(a, z))$
- $Q(a, b)$

Goal $= P(f(g(a, c)))$

1. clausify the assumptions
2. negate and clausify the goal
3. $mgu(Q(a, y), Q(a, b)) = [y/b]$
4. $mgu(R(b, g(a, z)), R(b, x)) = [x/g(a, z)]$

Inference

1. $\neg P(x), P(f(x))$
2. $\neg Q(a, y), \neg R(y, x), P(x)$
3. $R(b, g(a, z))$
4. $Q(a, b)$
5. $\neg P(f(g(a, c)))$
6. $\neg R(b, x), P(x)$

Example

Assumptions:

- $\forall x(P(x) \supset P(f(x)))$
- $\forall x, y(Q(a, y) \wedge R(y, x) \supset P(x))$
- $\forall z R(b, g(a, z))$
- $Q(a, b)$

Goal $= P(f(g(a, c)))$

1. clausify the assumptions
2. negate and clausify the goal
3. $mgu(Q(a, y), Q(a, b)) = [y/b]$
4. $mgu(R(b, g(a, z)), R(b, x)) = [x/g(a, z)]$

Inference

1. $\neg P(x), P(f(x))$
2. $\neg Q(a, y), \neg R(y, x), P(x)$
3. $R(b, g(a, z))$
4. $Q(a, b)$
5. $\neg P(f(g(a, c)))$
6. $\neg R(b, x), P(x)$
7. $P(g(a, z))$

Example

Assumptions:

- $\forall x(P(x) \supset P(f(x)))$
- $\forall x, y(Q(a, y) \wedge R(y, x) \supset P(x))$
- $\forall z R(b, g(a, z))$
- $Q(a, b)$

Goal $= P(f(g(a, c)))$

1. clausify the assumptions
2. negate and clausify the goal
3. $mgu(Q(a, y), Q(a, b)) = [y/b]$
4. $mgu(R(b, g(a, z)), R(b, x)) = [x/g(a, z)]$
5. $mgu(P(x), P(g(a, z)) = [x/g(a, z)]$

Inference

1. $\neg P(x), P(f(x))$
2. $\neg Q(a, y), \neg R(y, x), P(x)$
3. $R(b, g(a, z))$
4. $Q(a, b)$
5. $\neg P(f(g(a, c)))$
6. $\neg R(b, x), P(x)$
7. $P(g(a, z))$

Example

Assumptions:

- $\forall x(P(x) \supset P(f(x)))$
- $\forall x, y(Q(a, y) \wedge R(y, x) \supset P(x))$
- $\forall z R(b, g(a, z))$
- $Q(a, b)$

Goal $= P(f(g(a, c)))$

1. clausify the assumptions
2. negate and clausify the goal
3. $mgu(Q(a, y), Q(a, b)) = [y/b]$
4. $mgu(R(b, g(a, z)), R(b, x)) = [x/g(a, z)]$
5. $mgu(P(x), P(g(a, z)) = [x/g(a, z)]$

Inference

1. $\neg P(x), P(f(x))$
2. $\neg Q(a, y), \neg R(y, x), P(x)$
3. $R(b, g(a, z))$
4. $Q(a, b)$
5. $\neg P(f(g(a, c)))$
6. $\neg R(b, x), P(x)$
7. $P(g(a, z))$
8. $P(f(g(a, z)))$

Example

Assumptions:

- $\forall x(P(x) \supset P(f(x)))$
- $\forall x, y(Q(a, y) \wedge R(y, x) \supset P(x))$
- $\forall z R(b, g(a, z))$
- $Q(a, b)$

Goal $= P(f(g(a, c)))$

1. clausify the assumptions
2. negate and clausify the goal
3. $mgu(Q(a, y), Q(a, b)) = [y/b]$
4. $mgu(R(b, g(a, z)), R(b, x)) = [x/g(a, z)]$
5. $mgu(P(x), P(g(a, z)) = [x/g(a, z)]$
6. $mgu(P(f(g(a, z))), P(f(g(a, c)))) = [z/c]$

Inference

1. $\neg P(x), P(f(x))$
2. $\neg Q(a, y), \neg R(y, x), P(x)$
3. $R(b, g(a, z))$
4. $Q(a, b)$
5. $\neg P(f(g(a, c)))$
6. $\neg R(b, x), P(x)$
7. $P(g(a, z))$
8. $P(f(g(a, z)))$

Example

Assumptions:

- $\forall x(P(x) \supset P(f(x)))$
- $\forall x, y(Q(a, y) \wedge R(y, x) \supset P(x))$
- $\forall z R(b, g(a, z))$
- $Q(a, b)$

Goal $= P(f(g(a, c)))$

1. clausify the assumptions
2. negate and clausify the goal
3. $mgu(Q(a, y), Q(a, b)) = [y/b]$
4. $mgu(R(b, g(a, z)), R(b, x)) = [x/g(a, z)]$
5. $mgu(P(x), P(g(a, z)) = [x/g(a, z)]$
6. $mgu(P(f(g(a, z))), P(f(g(a, c)))) = [z/c]$

Inference

1. $\neg P(x), P(f(x))$
2. $\neg Q(a, y), \neg R(y, x), P(x)$
3. $R(b, g(a, z))$
4. $Q(a, b)$
5. $\neg P(f(g(a, c)))$
6. $\neg R(b, x), P(x)$
7. $P(g(a, z))$
8. $P(f(g(a, z)))$
9. $\bot$

Equality

In theory, it's enough to add the equality axioms:

- The reflexive, symmetric and transitive laws $\{x = x\}$, $\{x \neq y, y = x\}$, $\{x \neq y, y \neq z, x = z\}$.
- Substitution laws like $\{x_1 \neq y_1, \ldots, x_n \neq y_n, f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)\}$ for each f with arity equal to n
- Substitution laws like $\{x_1 \neq y_1, \ldots, x_n \neq y_n, \neg P(x_1, \ldots, x_n),\ P(y_1, \ldots, y_n)\}$ for each P with arity equal to n

In practice, we need something special: the paramodulation rule

$$\frac{\{P(t), l_1, \ldots l_n\} \quad \{u = v, l_{n+1}, \ldots, l_m\}}{P(v), l_1, \ldots, l_m\}\sigma} \quad \text{provides that } t\sigma = u\sigma$$

Resolution

Exercize

Find the possible resolvents of the following pairs of clauses.

C	D
$\neg p(x) \vee q(x, b)$	$p(a) \vee q(a, b)$
$\neg p(x) \vee q(x, x)$	$\neg q(a, f(a))$
$\neg p(x, y, u) \vee \neg p(y, z, v) \vee \neg p(x, v, w) \vee p(u, z, w)$	$p(g(x, y), x, y)$
$\neg p(v, z, v) \vee p(w, z, w)$	$p(w, h(x, x), w)$

Solution

C	D	σ
$\neg p(x) \vee q(x, b)$	$p(a) \vee q(a, b)$	$[a/x]$
$\neg p(x) \vee q(x, x)$	$\neg q(a, f(a))$	NO
$\neg p(x, y, u) \vee \neg p(y, z, v) \vee \neg p(x, v, w) \vee p(u, z, w)$	$p(g(x', y'), x', y')$	
$\neg p(x, y, u) \vee \neg p(y, z, v) \vee \neg p(x, v, w) \vee p(u, z, w)$	$p(g(x', y'), x', y')$	
$\neg p(x, y, u) \vee \neg p(y, z, v) \vee \neg p(x, v, w) \vee p(u, z, w)$	$p(g(x', y'), x', y')$	
$\neg p(v, z, v) \vee p(w, z, w)$	$p(w', h(x', x'), w')$	

Resolution

Exercize

Find the possible resolvents of the following pairs of clauses.

C	D
$\neg p(x) \vee q(x,b)$	$p(a) \vee q(a,b)$
$\neg p(x) \vee q(x,x)$	$\neg q(a, f(a))$
$\neg p(x,y,u) \vee \neg p(y,z,v) \vee \neg p(x,v,w) \vee p(u,z,w)$	$p(g(x,y),x,y)$
$\neg p(v,z,v) \vee p(w,z,w)$	$p(w,h(x,x),w)$

Solution

C	D	σ
$\neg p(x) \vee q(x,b)$	$p(a) \vee q(a,b)$	$[a/x]$
$\neg p(x) \vee q(x,x)$	$\neg q(a, f(a))$	NO
$\neg p(x,y,u) \vee \neg p(y,z,v) \vee \neg p(x,v,w) \vee p(u,z,w)$	$p(g(x',y'),x',y')$	
$\neg p(x,y,u) \vee \neg p(y,z,v) \vee \neg p(x,v,w) \vee p(u,z,w)$	$p(g(x',y'),x',y')$	
$\neg p(x,y,u) \vee \neg p(y,z,v) \vee \neg p(x,v,w) \vee p(u,z,w)$	$p(g(x',y'),x',y')$	
$\neg p(v,z,v) \vee p(w,z,w)$	$p(w',h(x',x'),w')$	

resolution

Exercize

Apply resolution (with refutation) to prove that the following formula

$$\phi_5 \quad m(5, f(7, f(5, f(1, 0))))$$

is a consequence of the set

$$\begin{array}{ll} \phi_1 & \neg m(x, 0) \\ \phi_2 & \neg i(x, y, z) \vee m(x, z) \\ \phi_3 & \neg m(x, z) \vee \neg i(v, z, y) \vee m(x, y) \\ \phi_4 & i(x, y, f(x, y)) \end{array}$$

resolution

Solution