Proofs by induction on the structure of formula

Theorem

Any propositional formula ϕ which does not contain the symbol of negation $\neg$ and of falsehood $\bot$ is satisfiable.

Proof.

Base case Let us assume that ϕ does not contain any propositional connective, then ϕ is an atomic formula p.
The interpretation $\mathcal{I}(p) = \text{True}$ satisfies ϕ.

Inductive step Assume that the statement holds for every ψ containing a number n of connectives and prove that it holds for a formula ϕ containing $n+1$ connectives.
Three cases

- $\phi = \psi \vee \theta$.
  If ϕ contains $n+1$ connectives, then ψ and θ contain at most n connectives. They do not contain the symbol of negation $\neg$ and of falsehood $\bot$ and are therefore satisfiable. Let $\mathcal{I}_\psi$ and $\mathcal{I}_\theta$ the two interpretations that satisfy ψ and θ, respectively.
  $$\mathcal{I}(p) = \begin{cases} \mathcal{I}_\psi(p) & \text{if } p \text{ occurs in } \psi, \\ \mathcal{I}_\theta(p) & \text{if } p \text{ occurs in } \theta \text{ and does not occur in } \psi. \end{cases}$$
  satisfies ϕ

□

Proofs by induction on the structure of formula

Theorem

Any propositional formula ϕ which does not contain the symbol of negation $\neg$ and of falsehood $\bot$ is satisfiable.

Proof.

Inductive step Continued...
Three cases

- $\phi = \psi \supset \theta$. Strategy similar to $\vee$
- $\phi = \psi \wedge \theta$.
  Let $\mathcal{I}_\psi$ and $\mathcal{I}_\theta$ the two interpretations that satisfy ψ and θ, respectively.
  How do I define $\mathcal{I}$?

Another strategy of proof is needed. We need to prove a stronger theorem!

□

Proofs by induction on the structure of formula

Theorem (Stronger theorem)

Any propositional formula ϕ which does not contain the symbol of negation $\neg$ and of falsehood $\bot$ is satisfiable by an assignment that assigns True to all propositional atoms.

Proof.

Base case Let us assume that ϕ does not contain any propositional connective, then ϕ is an atomic formula p.
The interpretation $\mathcal{I}(p) = \text{True}$ satisfies ϕ and is compliant to our requirement.

Inductive step Assume that the statement holds for every ψ containing a number n of connectives and prove that it holds for a formula ϕ containing $n+1$ connectives.
Three cases

- $\phi = \psi \vee \theta$.
  ψ and θ contain at most n connectives. By induction the are satisfiable by two interpretations $\mathcal{I}_\psi$ and $\mathcal{I}_\theta$ that assign all he propositional atoms of ψ and θ to true, respectively.
  $\mathcal{I} = \mathcal{I}_\psi \cup \mathcal{I}_\theta$ is the assignment we need to prove the theorem.

□

Proofs by induction on the structure of formula

Proof.

Inductive step Continued...
Three cases

- $\phi = \psi \supset \theta$. Analogous to the above
- $\phi = \psi \wedge \theta$. Analogous to the above

□

Proofs by induction on the structure of formula

Theorem

Any propositional formula ϕ which contains a subformula at most once once is satisfiable.

Proof.

Base case Let us assume that ϕ does not contain any propositional connective, then ϕ is an atomic formula p.
The interpretation $\mathcal{I}(p) =$ True satisfies ϕ.

Inductive step Assume that the statement holds for every ψ containing a number n of connectives and prove that it holds for a formula ϕ containing $n+1$ connectives.
Three cases

- $\phi = \psi \vee \theta$.
  By inductive hypothesis let $\mathcal{I}_\psi$ and $\mathcal{I}_\theta$ the two interpretations that satisfy ψ and θ, respectively.
  Let p be a propositional atom occurring in ϕ, then it either occur in ψ or it occur in θ (but not in both).
  $\mathcal{I} = \mathcal{I}_\psi \cup \mathcal{I}_\theta$ is the assignment we need to prove the theorem.
- Similarly for $\phi = \psi \supset \theta$ and $\phi = \psi \wedge \theta$.

□