Mathematical Logic
- Sample Exam Papers -

Chiara Ghidini and **Luciano Serafini**

Academic Year 2013-2014

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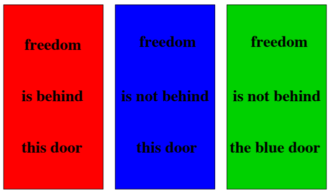
Logica Matematica

Laurea Specialistica in Informatica
DIT - Universita' degli Studi di Trento

Exam

19 June 2006

Exercise 1 (Propositional logic: modelling)**.**
The Path to Freedom
Kyle, Neal, and Grant find themselves trapped in a dark and cold dungeon (HOW they arrived there is another story). After a quick search the boys find three doors, the first one red, the second one blue, and the third one green.
Behind one of the doors is a path to freedom. Behind the other two doors, however, is an evil fire-breathing dragon. Opening a door to the dragon means almost certain death.
On each door there is an inscription:



Given the fact that at LEAST ONE of the three statements on the three doors is true and at LEAST ONE of them is false, which door would lead the boys to safety?

Provide a propositional language and a set of axioms that formalize the problem and check whether the boys can choose a door being sure it will lead them to freedom.

Exercise 2 (Propositional logic: theory)**.** *Given a set S of propositional formulas on the set $\{P_1, \ldots, P_n\}$ of primitive proposition. Show that if $|S| > 2^{(2^n)}$ then there are two formulas A and B in S such that $\models A \Leftrightarrow B$.*

Exercise 3 (First order logic: modelling)**.** *Let $C = \{c_1, .., c_k\}$ be a non empty and finite set of colors. A partially colored directed graph is a structure $\langle N, R, c\rangle$ where*

- *N is a non empty set of nodes*
- *R is a binary relation on N*

- *c associate color to nodes (not all the nodes are necessarily colored and each node has at most one color)*

Provide a first order language and a set of axioms that formalize partially colored graphs. Show that every model of this theory correspond to a partially colored graph, and vice-versa. For each of the following properties, write a formula which is true in all and only the graphs that satisfies the property:

1. *connected nodes don't have the same color*
2. *the graph contains only 2 yellow nodes*
3. *starting from a red node one can reach in at most 4 steps a green node*
4. *for each color there is at least a node with this color*
5. *the graph is composed of $|C|$ disjoint non empty subgraphs, one for each color*

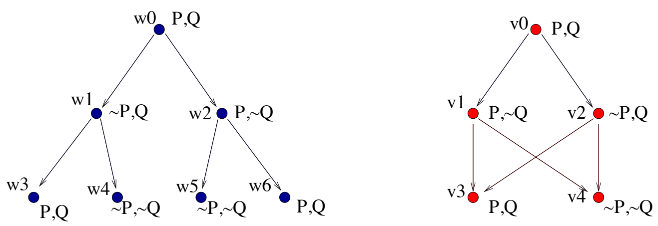
Exercise 4 (First order logic: theory)**.** *For each of the following formulas either prove its validity via natural deduction or provide a countermodel*

1. $\forall x(P(x) \supset \exists y P(y))$
2. $\exists x(P(x) \supset \forall y P(y))$
3. $\neg\neg\forall x.P(x) \supset \forall x.\neg\neg P(x)$

Exercise 5 (Modal logic)**.** *For each of the following formulas either prove that it is valid or find a counter-example. Note that if your attempts to produce a falsifying model always end in incoherent pictures, it may be because the formula is valid.*

1. $A \supset \Box A$
2. $(\neg\Diamond A \wedge \Diamond B) \supset \Diamond(\neg A \wedge B)$
3. $\Box\Diamond A \supset \Diamond\Box A$

Exercise 6 (Modal logic: Theory)**.** *Check if the following two models bisimulate. If this is the case, describe the bismulation relation.*



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Solution of the exam

27 July 2006

Exercise 1 (Propositional logic: modelling (max 5 marks)). *Consider the finite set of binary strings*

$$\left\{ \begin{array}{l} (000000), (100000), (110000), (111000), (111100), (111110), \\ (111111), (011111), (001111), (000111), (000011), (000001) \end{array} \right\} \quad (1)$$

Explain how it is possible to represent such a set in a propositional formula. and find the most compact representation, and show that it is a sound and complete representation.

Solution 1. *A standard way to represent a set of binary strings with a given finite length is by associating an atomic formula p_i for $1 \leq i \leq n$, and interpreting p_i in the proposition "the i-th digit of the string is 1". Since there are only two digits, the formula $\neg p_i$ encodes the proposition "the i-th digit of the string is 0".*

The language to describe the set of six digit binary strings with is therefore is the propositional language built on the set $P = \{p_1, \ldots, p_6\}$ of propositional letters.

Any interpretation of this language corresponds to a string. For instance the interpretation I with

x	p_1	p_2	p_3	p_4	p_5	p_6
$I(x)$	*true*	*false*	*true*	*false*	*false*	*false*

corresponds to the string "101011".

To represent the set of strings (1) we can define a theory T, such that all the models of such a theory corresponds to all the element of the set (1).

A first definition of T can be done by enumerating all the strings. i.e., T contains the following axioms

$$\begin{array}{l} (\neg p_1 \wedge \neg p_2 \wedge \neg p_3 \wedge \neg p_4 \wedge \neg p_5 \wedge \neg p_6) \vee \\ (p_1 \wedge \neg p_2 \wedge \neg p_3 \wedge \neg p_4 \wedge \neg p_5 \wedge \neg p_6) \vee \\ (p_1 \wedge p_2 \wedge \neg p_3 \wedge \neg p_4 \wedge \neg p_5 \wedge \neg p_6) \vee \\ \vdots \\ (\neg p_1 \wedge \neg p_2 \wedge \neg p_3 \wedge \neg p_4 \wedge \neg p_5 \wedge p_6) \end{array}$$

However, this definition is rather "long" and there is a more compact theory T'. that formalizes the same set of strings. Notice that the set (1) can be characterized like this:

(1) contains all the strings s, which are of one of the form 0...01...1, or 1...10...0

we can formalize this definition by the following 8 axioms $(1 < i < 6)$.

$$p_i \supset \bigwedge_{j<i} p_j \vee \bigwedge_{j>i} p_j \tag{2}$$

$$\neg p_i \supset \bigwedge_{j<i} \neg p_j \vee \bigwedge_{j>i} \neg p_j \tag{3}$$

This theory T' is better than the theory T for the following reasons:

- *T' is more* compact *than T, as the former contains 8 axioms of length 6* $= 8 * 6$. *while the latter contains one (big) axiom of length* $12 * 6$.
- *the theory T' better capture the* intensional definition *of the set (1).*

To show that this set of axioms is sound an complete, we have to prove that an interpretation $I \models T'$ if and only if I corresponds to a string of (1).

Suppose that $T \models T'$ and let s be the string corresponding to I, then s. Suppose that $s \notin (1)$, then it must be of the form "$\dots 0 \dots 1 \dots 0 \dots$" or of the form "$\dots 1 \dots 0 \dots 1 \dots$", but in the first case one of the axiom (3) is satisfied, and in the second case, one of the axioms (2) is satisfied.

Vice-versa, suppose that s is a string contained in (1) and I its corresponding interpretation, let us show that $I \models T'$. Let $s[i]$ denote the i-th digit of s. To show that $I \models (2)$ suppose that $s[i] = 0$, in this case $I \not\models p_i$ and therefore $I \models (2)$. If $s[i] = 1$, then either all the digits after $s[i]$ must be 1's or all the digits before $s[i]$ must be 1. This implies that either $I \models \bigwedge_{j>i} p_j$ or $I \models \bigwedge_{j<i} p_j$. This implies that $I \models (2)$.

Exercise 2 (Propositional logic: natural deduction (max 2 marks)**.** *Derive the following formulas via Natural Deduction,*

$$\neg(A \supset \neg B) \supset (A \wedge B)$$

Exercise 3 (Propositional logic: theory (max 3 marks)**.** *Provide the definition of* maximally consistent set of formulas *and show that if Γ is maximally consistent and $\Gamma \vdash \phi$, then $\phi \in \Gamma$.*

Exercise 4 (First order logic: modelling (max 5 marks))**.** *Minesweeper is a single-player computer game invented by Robert Donner in 1989. The object of the game is to clear a minefield without detonating a mine.*

The game screen consists of a rectangular field of squares. Each square can be cleared, or uncovered, by clicking on it. If a square that contains a mine is clicked, the game is over. If the square does not contain a mine, one of two things can happen: (1) A number between 1 and 8 appears indicating the amount of adjacent (including diagonally-adjacent) squares containing mines, or (2) no number appears; in which case there are no mines in the adjacent cells. An example of game situation is provided in the following figure Provide a first order language that allows to formalize the knowledge of a player in a game state. In such a language you should be able to formalize the following knowledge:

1. *there are exactly n mines in the minefield*

Figure 1: An example of a state in the Mines game

2. *if a cell contains the number 1, then there are exactly two mines in the adjacent cells.*

3. *show by means of deduction that there must be a mine in the position (3,3) of the game state of picture 1.*

Suggestion: define the predicate $Adj(x, y)$ to formalize the fact that two cells x and y are adjacent

Exercise 5 (First order logic: theory (max 5 marks))**.** *Show that if an interpretation I satisfies the formula*

$$\forall x_0, x_1, \ldots x_n \left(\bigvee_{0 \le i \ne j \le n} x_i = x_j \right)$$

then the domain contains at most n elements.

Exercise 6 (Modal logic (max 5 marks))**.** *For each of the following formulas either prove that it is valid or find a counter-example. Note that if your attempts to produce a falsifying model always end in incoherent pictures, it may be because the formula is valid.*

1. $\Box\Diamond A \supset A$

2. $\Box\Box A \supset \Box A$

3. $(\Diamond A \supset \Box B) \supset (\Box A \supset \Box B)$

Exercise 7 (Modal logic: Theory (max 5 marks))**.** *Show that in the frame $\mathcal{F} = (W, R)$ if R is an equivalence relation then the schema $\phi \supset \Box\Diamond\phi$ is valid*

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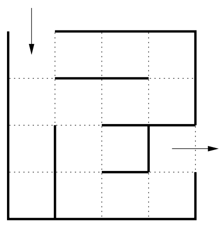
24 October 2006

Exercise 1 (Propositional logic: modelling (max 5 marks)). *Consider the 4×4 maze in the following picture*



provide a formalization of the problem such that finding a path of maximum length of 16 from the entrance to the exit of the maze, is encoded in a satisfiability problem.

Exercise 2 (Propositional logic theory (Max 5 marks)). *Use the DPLL procedure to verify weather the following formula is satisfiable:*

$$(p \vee (\neg q \wedge r)) \supset ((q \vee \neg r) \supset p)$$

Exercise 3 (First order logic: representation). *A labelled graph is a triple $\langle V, A, L\rangle$ where V is a set of vertex, A is a set of directed arcs between vertexes and L is a function that associates a label to each arc. An example of labelled graph is shown in the Figure 1 Provide a language and a theory for labelled graphs (2 marks).*
For each of the following conditions on graph write the corresponding axioms.

1. *R_a is transitive;*
2. *$R_c = R_a \circ R_b$;*
3. *All the arcs exiting from a node has different labels*

for every label x, R_x denotes the binary relation between vertexes defined as:

$$R_x(v_1, v_2) \text{ if and only there is an arc labeled with } x \text{ from } v_1 \text{ to } v_2.$$

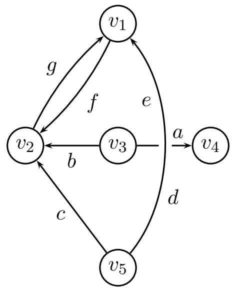


Figure 1: an example of labeled graph

Exercise 4 (First order logic: natural deduction (max 2 marks). *Derive the following formulas via Natural Deduction,*

$$\neg\exists y \forall x (P(x, y) \equiv \neg P(x, x))$$

Exercise 5 (First order logic: natural deduction (max 3 marks). *Show that the following inference rule is sound*

$$\frac{\forall x(\phi(x) \vee \psi(x)) \qquad \neg\phi(a)}{\psi(a)}$$

Exercise 6 (Modal logic representation (max 5 marks)). *Consider the modal language with two modalities K_1 and K_2 such that the formula $K_1\phi$ means that* agent 1 knows that ϕ is true *and $K_2\phi$ means that* agent 2 knows that ϕ is true. *Provide a set of axioms that allow to represent the following conditions:*

1. *what is known by any agent must be true*
2. *if something is true then it is known at least by one agent*
3. *agent 1 and agent 2 never have contradicting knowledge*
4. *agent 2 knows all what is known by agent 1*
5. *if agent one knows that ϕ is true, then agent 2 knows that agent 1 knows that ϕ is true, and vice-versa*

Exercise 7 (Modal logic theory (max 5 marks)). *Show that if a frame $\langle W, R \rangle$ satisfy the schema $\Box\phi \supset \Box\Box\phi$ then R is transitive.*

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16 January 2007

Exercise 1 (Propositional logic: modelling (Max 5 marks)).
The Labyrinth Guardians.
You are walking in a labyrinth and all of a sudden you find yourself in front of three possible roads: the road on your left is paved with gold, the one in front of you is paved with marble, while the one on your right is made of small stones. Each street is protected by a guardian. You talk to the guardians and this is what they tell you:

- *The guardian of the gold street:* "This road will bring you straight to the center. Moreover, if the stones take you to the center, then also the marble takes you to the center."
- *The guardian of the marble street:* "Neither the gold nor the stones will take you to the center."
- *The guardian of the marble street:* "Follow the gold and you'll reach the center, follow the marble and you will be lost."

Given that you know that all the guardians are liars, can you choose a road being sure that it will lead you to the center of the labyrinth? If this is the case, which road you choose?

Provide a propositional language and a set of axioms that formalize the problem and show whether you can choose a road being sure it will lead to the center.

Exercise 2 (Propositional Logic: theory (Max 5 marks)). *Show that for any pair of maximally consistent set* Γ *and* Σ, *if* $\Gamma \cup \Sigma$ *is maximally consistent then* $\Gamma = \Sigma$.

Exercise 3 (First order logic: modelling (Max 5 marks)).
The Draughts game.
The game of Draughts is played on a standard Chess board 64 black and white chequered squares. Each player has 12 pieces (men) normally in the form of fat round counters. One player has black men and the other has white men.

When starting, each player's men are placed on the 12 black squares nearest to that player (see Figure 3). The white squares are not used at all in the game - the men only move diagonally and so stay on the black squares throughout. Black always plays first.

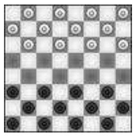

Figure 1: Starting position on a 8x8 Draughts board.

Players take turns to move a man of their own colour. There are fundamentally 4 types of move: the ordinary move of a man, the ordinary move of a king, the capturing move of a man and the capturing move of a king.

An ordinary move of a man is its transfer diagonally forward left or right from one square to an immediately neighbouring vacant square. When a man reaches the farthest row forward (the king-row or crownhead) it becomes a king: another piece of the same shade is placed on top of the piece in order to distinguish it from an ordinary man.

An ordinary move of a king is from one square diagonally forward or backward,left or right, to an immediately neighbouring vacant square.

Whenever a piece (man or king) has an opponent's piece adjacent to it and the square immediately beyond the opponent's piece is vacant, the opponent's piece can be captured. If the player has the opportunity to capture one or more of the opponent's pieces, then the player must do so. A piece is taken by simply hopping over it into the vacant square beyond and removing it from the board. Unlike an ordinary move, a capturing move can consist of several such hops - if a piece takes an opponent's piece and the new position allows it to take another piece, then it must do so straight away.

Kings are allowed to move and capture diagonally forwards and backwards and are consequently more powerful and valuable than ordinary men. However, ordinary men can capture Kings.

The game is won by the player who first manages to take all his opponent's pieces or renders them unable to move.

For each of the following conditions on Draughts game write the corresponding axioms, using an appropriate first order logic language.

1. *Each piece is either white or black.*
2. *Each piece is either a king or a man.*
3. *White squares are always empty (always: in each instant of the game).*
4. *In each instant of the game, black squares are either empty or contain a piece.*
5. *At the beginning of the game (instant zero) there are 12 white and 12 black men on the board.*

6. *Whenever a black man captures a white man, in the next instant of the game there is a white man less (and vice-versa).*

7. *If a piece in square* x *captures a piece in square* y *hopping over it into the vacant square* z*, then in the next instant of the game the square* z *contains the piece that moved while squares* x *and* y *are empty.*

Exercise 4 (First Order logic: theory (Max 3 marks)). *If the following formula is valid, show a proof in natural deduction, if not provide a countermodel.*

$$(\forall x(P(x) \supset \exists y Q(x,y))) \supset (\exists x P(x) \supset \exists y Q(x,y))$$

Exercise 5 (Modal logic representation (Max 5 marks)). *For each of the following formulas either prove that it is valid or find a counter-example. Note that if your attempts to produce a falsifying model always end in incoherent pictures, it may be because the formula is valid.*

1. $\Box\Diamond A \supset \Diamond\Diamond A$
2. $\Diamond(\Box A \wedge \Diamond B) \supset \Diamond\Diamond \mathrm{T}$
3. $\neg\Diamond\Box A \supset \Diamond\Diamond\neg A$

Exercise 6 (Modal Logic: Theory (max, 5 marks)). *Prove that the axiom schema*

$$\Diamond\Box\phi \supset \phi \tag{1}$$

is strongly complete w.r.t., the class of frames $\langle W, R\rangle$ where R is symmetric. (suggestion, you have to prove that (i) (1) is true in all the symmetric frames and that (ii) for any non symmetric frame F there is model $M = (F, V)$ and a world $w \in W$ such that $M, w \not\models (1)$.

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13 February 2007

Exercise 1 (Propositional logic: modelling (Max 5 marks)). *Three boxes are presented to you. One contains gold, the other two are empty. Each box has imprinted on it a clue as to its contents; the clues are*
(Box 1) The gold is not here
(Box 2) The gold is not here
(Box 3) The gold is in Box 2
Only one message is true; the other two are false. Which box has the gold? Formalize the puzzle in Propositional Logic and find the solution using a truth table.

Solution 1. *Let B_i with $i \in \{1,2,3\}$ stand for "gold is in the i-th box". With this language we can formalize the messages on the boxes as follows:*

$\neg B_1$	$\neg B_2$	B_2

We can also formalize the statements of the problem as follows:

1. *One box contains gold, the other two are empty.*

$$(B_1 \wedge \neg B_2 \wedge \neg B_3) \vee (\neg B_1 \wedge B_2 \wedge \neg B_3) \vee (\neg B_1 \wedge \neg B_2 \wedge B_3) \tag{1}$$

2. *Only one message is true; the other two are false.*

$$(\neg B_1 \wedge \neg\neg B_2 \wedge \neg B_2) \vee (\neg\neg B_1 \wedge \neg B_2 \wedge \neg B_2) \vee (\neg\neg B_1 \wedge \neg\neg B_2 \wedge B_2) \tag{2}$$

(2) is equivalent to:

$$(B_1 \wedge \neg B_2) \vee (B_1 \wedge B_2) \tag{3}$$

Let us compute the truth table for (1) and (3)

B_1	B_2	B_3	(1)	(3)
T	T	T	F	T
T	T	F	F	T
T	F	T	F	T
T	F	F	T	T
F	T	T	F	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

The only assignment I that verifies both (1) and (3) is the one with $I(B_1) = T$ and $I(B_2) = I(B_3) = F$, which implies that the gold is in the first box.

Exercise 2 (Propositional Logic: theory (Max 5 marks)). *Let A, B and C be propositional formulas. Show the following equivalence:*

$$(A, \neg B \models C \text{ and } A, B \models C) \iff A \models C$$

Solution 2. *There are two possible solution, one semantic based and the second syntactic based.*

Semantic based solution *We apply the definition of logical consequence, that states that $\Gamma \models \phi$ if and only if for all interpretation I, if $I \models \Gamma$, then $I \models \phi$.*

$\underline{(A, B \models C,\ A, \neg B \models C) \Longrightarrow A \models C}$ *Let I be an interpretation such that $I \models A$. There are two cases either $I \models B$ or $I \models \neg B$. If $I \models B$, then $I \models \{A, B\}$ and by the fact that $A, B \models C$, we can conclude that $I \models C$. If, instead, $I \models \neg B$, then, $I \models \{A, \neg B\}$, and from the fact that $A, \neg B \models C$, we can conclude that $I \models C$. Since ether $I \models B$ or $I \models \neg B$, then in all the cases $I \models C$.*

$\underline{A \models C \Longrightarrow (A, B \models C,\ A, \neg B \models C)}$ *For any interpretation I, if $I \models \{A, B\}$, then $I \models A$, and by the fact that $A \models C$, we can conclude that $I \models C$, and therefore $A, B \models C$ Similarly, if $I \models \{A, \neg B\}$, then $I \models A$, by the fact that $A \models C$, we have that $I \models C$, and therefore that $A, \neg B \models C$.*

Exercise 3 (First order logic: modelling (Max 5 marks)). *Formalize in first order logic the train connections in Italy. Provide a language that allows to express the fact that a town is directly connected (no intermediate train stops) with another town, by a type of train (e.g., intercity, regional, interregional). Formalize the following facts by means of axioms:*

1. *There is no direct connection from Rome to Trento*
2. *There is an intercity from Rome to Trento that stops in Firenze, Bologna and Verona.*
3. *Regional trains connect towns in the same region*
4. *Intercity trains don't stops in small towns.*

Solution 3. *We define the language as follows*

Constants *RM, FI, BO, VR, TN, . . . are identifiers of the towns of Roma, Firenze, Bologna, Verona, Trento, ... and InterCity, Regional, . . . are the identifiers of the type of trains*

Predicates *Train with arity equal to 1, where Train(x) means x is a train Town with arity equal to 1, where Train(x) means x is a town SmallTown with arity equal to 1, where Train(x) means x is a small town TrainType with arity equal to 2, where TrainType(x, y) means that the train x is of type y. IsInRegion with arity equal to 2, where IsInRegion(x, y) means that the town x is in region y. DirectConn with arity equal to 3, where DirectConn(x, y, z) means that the train x directly connects (with no intermediated stops) the towns y and z.*

Background axioms *With these set of axioms we have to formalize some background knowledge which is necessary to make the formalization more adeguate*

1. *a train has exaclty one train type;*

$$\forall x(Train(x) \supset \exists y(TrainType(x, y))) \wedge \forall xyz(TrainType(x, y) \wedge TrainType(x, z) \supset y = z) \tag{4}$$

2. *Intercity type is different from regional type:*

$$\neg(InterCity = Regional) \qquad (\text{also written as } InterCity \neq Regional) \tag{5}$$

3. *A town is associated to exactly one region*

$$\forall x(Town(x) \supset \exists y(IsInRegion(x, y))) \wedge \forall xyz(IsInRegion(x, y) \wedge IsInRegion(x, z) \supset y = z) \tag{6}$$

4. *small towns are towns:*

$$\forall x(SmallTown(x) \supset Town(x)) \tag{7}$$

5. *if a town a is connected to a town b. b is also connected to a town a.*

$$\forall xy(\exists z DirectConn(z, x, y) \supset \exists z DirectConn(z, y, x)) \tag{8}$$

Specific axioms *The axioms that formalizes the specific situation described in the exercise are the following:*

1. *There is no direct connection from Rome to Trento*

$$\exists x DirectConn(x, RM, TN)$$

2. *There is an intercity from Rome to Trento that stops in Firenze, Bologna and Verona.*

$$\exists x(DirectConn(x, RM, FI) \wedge DirectConn(x, FI, BO) \wedge DirectConn(x, BO, VR) \wedge DirectConn(x, VR, TN) \wedge TrainType(x, InterCity))$$

3. *Regional trains connect towns in the same region*

$$\forall xyz(\mathit{TrainType}(x, \mathit{Regional}) \supset (\mathit{DirectConn}(x, y, z) \supset \exists w(\mathit{IsInRegion}(y, w) \wedge \mathit{IsInRegion}(z, w))))$$

4. *Intercity trains don't stops in small towns.*

$$\forall xyz(\mathit{DirectConn}(x, y, z) \wedge \mathit{TrainType}(x, \mathit{InterCity}) \supset \neg \mathit{SmallTown}(y) \wedge \neg \mathit{SmallTown}(y))$$

Exercise 4 (First Order logic: theory (Max 5 marks)). *Either prove via Natural Deduction or show a countermodel for the following formula:*

$$(\exists x Q(x) \wedge (\forall x(P(x) \supset \neg Q(x)))) \supset \exists x \neg P(x)$$

Solution 4.

$$\dfrac{\dfrac{\dfrac{\exists x Q(x) \wedge (\forall x(P(x) \supset \neg Q(x)))}{\exists x Q(x)}\wedge E \qquad \dfrac{\dfrac{\dfrac{Q(x)^3 \qquad \dfrac{P(x)^2 \qquad \dfrac{\dfrac{\exists x Q(x) \wedge (\forall x(P(x) \supset \neg Q(x)))^1}{\forall x(P(x) \supset \neg Q(x))}\wedge E}{P(x) \supset \neg Q(x)}\forall E}{\neg Q(x)}\supset E}{\bot}\supset E}{\neg P(x)}\bot_c \textit{(disc 2)}}{\exists x \neg P(x)}\exists I}{\exists x \neg P(x)}\exists E \textit{(disc 3)}}{(\exists x Q(x) \wedge (\forall x(P(x) \supset \neg Q(x)))) \supset \exists x \neg P(x)}\supset I \textit{(disc 1)}$$

Exercise 5 (Modal logic representation (Max 5 marks)). *Show how it is possible to represent the railways connections in a country by means of a Kripke frame. First, select the schema you have to impose to capture the following fact: "if there is a direct train connection to go from a to b, then there is also a train connection in the opposite direction*

Then, provide a set of axioms to formalize the following statements.

1. *You cannot be at the same time in Roma and Firenze*
2. *There is no direct train connection from Roma to Trento.*
3. *From Rome you can reach Trento with 2 changes.*
4. *At Riva del Garda there is no train station.*

Solution 5. *If the train direct connections is represented by the relation R of a Kripke frame, and each world is considered as a train stop, then then the condition*

"if there is a direct train connection to go from a to b, then there is also a train connection in the opposite direction"

can be imposed by requiring that R is symmetric. Symmetry of the accessibility relation can be strongly represented by means of the schema

$$\phi \supset \Box\Diamond\phi$$

As far as the other conditions, they can be represented by means axioms, on a language that contains the propositions RM, TN, FI... (meaning that we are at in Rome, Trento, Firenze, ...).

1. *You cannot be at the same time in Roma and Firenze*

$$RM \supset \neg FI$$

2. *There is no direct train connection from Roma to Trento.*

$$RM \supset \neg\Diamond TN$$

3. *From Rome you can reach Trento with 2 changes.*

$$RM \supset \Diamond\Diamond\Diamond TN$$

4. *At Riva del Garda there is no train station.*

$$RivaDelGarda \supset \Box\bot$$

Exercise 6 (Modal Logic: Theory (max, 5 marks)). *Show that the schema*

$$\Box\phi \equiv \Diamond\phi$$

is strongly complete with respect to frames (W, R) in which the accessibility relation R is a total function, i.e, for all $v \in W$ there is exactly one $w \in W$ such that vRw.

Solution 6. *First we show that every Kripke Frame $F = (W, R)$ where R is a totat function satisfies the axiom $\Box\phi \equiv \Diamond\phi$, Let I be any truth assignment on F and let $M = (F, I)$. Suppose that $M, w \models \Box\phi$ and for all v with wRv, $M, v \models \phi$, by the fact that R is a function, there is at least such a v, and therefore $M, w \models \Diamond\phi$. This means that $M, w \models \Box\phi \supset \Diamond\phi$, for all $w \in W$. Viceversa, suppose that $M, w \models \Diamond\phi$, this means that there is one v with wRv, such that $M, v \models \phi$. From the fact that R is a function there is no other v' different from v such that wRv', and therefore we can conclude that for all v with wRv $M, v \models \phi$. This implies that $M, v \models \Box\phi$ and therefore that $M, w \models \Diamond\phi \supset \Box\phi$. Since we have proved that for all M, and for all w $M, w \models \Box\phi \equiv \Diamond\phi$ we can conclude that $F \models \Box\phi \equiv \Diamond\phi$.*

In the second part we prove that in the frame $F = (W, R)$, R is not symmetric, then there is a world w and an assignment I such that the model $M = F, I$, is such that $M, w \not\models \Box p \equiv \Diamond p$.

If R is not a total function then either there is a world w with two successors v_1 and v_2, or there is a world w with no successors. See the following picture.

If wRv_1 and wRv_2, we can set the assignment so that $M, v_1 \models p$ and $M, v_2 \models \neg p$. This implies that $M, v \models \Diamond p$ and $M, w \not\models \Box p$. If there is no v such that wRv, we have that $M, w \models \Box p$ and $M, w\neg \models \Diamond p$. In both cases we have a countermodel of the axioms schema $\Box\phi \equiv \diamond\phi$.

Logica Matematica

Laurea Specialistica in Informatica
DIT - Universita' degli Studi di Trento

Trento, 28 Maggio 2007

Exercise 1 (Propositional logic: modelling (Max 5 marks)). *Three boxes are presented to you. One contains gold, the other two are empty. Each box has imprinted on it a clue as to its contents; the clues are*
(Box 1) The gold is not here
(Box 2) The gold is not here
(Box 3) The gold is in Box 2
Only one message is true; the other two are false. Which box has the gold? Formalize the puzzle in Propositional Logic and find the solution using a truth table.

Exercise 2 (Propositional Logic: theory (Max 5 marks)). *Let A, B and C be propositional formulas. Show the following equivalence:*

$$(A, \neg B \models C \text{ and } A, B \models C) \iff A \models C$$

Exercise 3 (First order logic: modelling (Max 5 marks)). *Formalize in first order logic the train connections in Italy. Provide a language that allows to express the fact that a town is directly connected (no intermediate train stops) with another town, by a type of train (e.g., intercity, regional, interregional). Formalize the following facts by means of axioms:*

There is no direct connection from Rome to Trento

There is an intercity from Rome to Trento that stops in Firenze, Bologna and Verona.

Regional trains connect towns in the same region

Intercity trains don't stops in small towns.

Exercise 4 (First Order logic: theory (Max 5 marks)). *Either prove via Natural Deduction or show a countermodel for the following formula:*

$$(\exists x Q(x) \wedge (\forall x (P(x) \supset \neg Q(x)))) \supset \exists x \neg P(x)$$

Exercise 5 (Modal logic representation (Max 5 marks)). *Show how it is possible to represent the railways connections in a country by means of a Kripke frame.*

First, select the schema you have to impose to capture the following fact: "if there is a direct train connection to go from a to b, then there is also a train connection in the opposite direction

Then, provide a set of axioms to formalize the following statements.

1. *You cannot be at the same time in Roma and Firenze*
2. *There is no direct train connection from Roma to Trento.*
3. *From Rome you can reach Trento with at least 2 changes.*
4. *At Riva del Garda there is no train station.*

Exercise 6 (Modal Logic: Theory (max, 5 marks)). *Show that the schema*

$$\Box\phi \equiv \Diamond\phi$$

is strongly complete with respect to frames (W, R) *in which the accessibility relation* R *is a function, i.e, for all* $v \in W$ *there is exactly one* $w \in W$ *such that* vRw.

Logica Matematica

Laurea Specialistica in Informatica
DIT - Universita' degli Studi di Trento

3 July 2007

Exercise 1 (Propositional logic: modelling (Max 5 marks)). *Four married couples of friends, Aldo(M), Beatrice(F), Cinzia(F), Dario(M), Enrico(M), Federico(M), Giada(F), and Helena(F), go out for dinner and book two tables of four seats each. Each husband seats in front of his wife. Furthermore you know that*

- *Aldo seats on the right of Beatrice*
- *Dario and Federico are not in the same table*
- *Helena, who is married with Enrico, seats on the right of Cinzia*
- *Dario is married with Cinzia.*

Provide a logical formalization of this problem so that it is possible to logically infer the people sitting at each table. (Suggestion: model only the fact that a person sits in a table and forget about all the other details)

Exercise 2 (Propositional Logic: theory (Max 5 marks)). *Use the Davis-Putnam procedure to compute models for the following clause sets or to prove that no model exists.*

$$\{P, Q, S, T\}, \{P, S, \neg T\}, \{Q, \neg S, T\}, \{P, \neg S, \neg T\}, \{P, \neg Q\}, \{\neg R, \neg P\}, \{R\}$$

Exercise 3 (First order logic: modelling (Max 7 marks)). *Provide a formalization of the scenario in exercise 1. The following facts should be derivable from your axiomatization:*

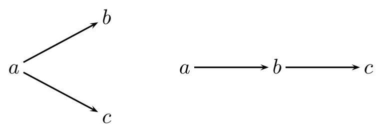
- *If a husband is on the right of another husband then the wife of the first is on the right of the wife of the second;*
- *If two people seat on two different tables they are not married;*
- *If x sits on right of y, and y sits on right of z, and z sits on right of w, then w sits on right of x.*

Exercise 4 (First Order logic: theory (Max 5 marks)). *Use natural deduction to show that the following formula is valid*

$$\exists x(A \supset B(x)) \supset (A \supset \exists x B(x))$$

where x does not occur free in A. Explain why you need the condition of x not being free in A, to prove that the above formula is valid.

Exercise 5 (Modal logic: Modelling (Max 5 marks)). *Given the two frames (a, b, and c are worlds, Provide a formula che is always true in the world a of the first model and always false in the world a of the second model.*



Exercise 6 (Modal logic: theory (Max 3 marks)). *Find a counteremodel for the formula $\Box A \supset \Box\Box A$.*

Logica Matematica

Laurea Specialistica in Informatica
DIT - Universita' degli Studi di Trento

11 September 2007

Exercise 1 (Propositional logic: modelling (Max 5 marks)). *Brown, Jones, and Smith are suspected of a crime. They testify as follows:*

Brown: *Jones is guilty and Smith is innocent.*

Jones: *If Brown is guilty then so is Smith.*

Smith: *I'm innocent, but at least one of the others is guilty.*

Let B, J , and S be the statements "Brown is innocent", "Jones is innocent", "Smith is inno cent". Express the testimony of each suspect as a propositional formula.

Write a truth table for the three testimonies.

Use the above truth table to answer the following questions:

1. *Are the three testimonies consistent?*
2. *The testimony of one of the suspects follows from that of another. Which from which?*
3. *Assuming everybody is innocent, who committed perjury?*
4. *Assuming all testimony is true, who is innocent and who is guilty?*
5. *Assuming that the innocent told the truth and the guilty told lies, who is innocent and who is guilty?*

Solution 1.

	B	J	S	$J \wedge \neg S$	$B \supset S$	$\neg S \wedge (B \vee J)$
(1)	T	T	T	F	T	F
(2)	T	T	F	T	F	T
(3)	T	F	T	F	T	F
(4)	T	F	F	F	F	T
(5)	F	T	T	F	T	F
(6)	F	T	F	T	T	T
(7)	F	F	T	F	T	F
(8)	F	F	F	F	T	F

1. *Yes the assigment (6) makes them all true*
2. *Yes the assigment (6) makes them all true*
3. $\wedge\neg S \models \neg S \wedge (B \vee J)$
4. *Everybody is innocent corresponds to assignment (8), and in this cae Brown and Smith statements are false.*
5. *From assignment (6) you have that Jones is guilty and the others are innocents*
6. *We have to search for an assignment such that if B (resp. J and S) is true, then the sentence of B (resp. J and S) is false, and if B (resp. J and S) is false then the sentence of B (resp. J and S) is true. The only assignment satisfying this restriction is assignment (3) in which Jones is innocent and Brown and Smith are guilty.*

⋆⋆⋆

Exercise 2 (Propositional Logic: theory (Max 5 marks)). *Show via the DPLL procedure that the following set of clauses is unsatisfiable.*

$$
\begin{array}{l}
((\neg P \vee \neg R) \\
(\neg P \vee S) \\
(P \vee \neg Q \vee \neg R) \\
(P \vee \neg Q \vee R) \\
(P \vee Q \vee \neg R) \\
(Q \vee R \vee S) \\
(R \vee \neg S))
\end{array}
$$

Solution 2. 1. *Initial set of clauses*

$$
\begin{array}{l}
(\neg P \vee \neg R) \\
(\neg P \vee S) \\
(P \vee \neg Q \vee \neg R) \\
(P \vee \neg Q \vee R) \\
(P \vee Q \vee \neg R) \\
(Q \vee R \vee S) \\
(R \vee \neg S)
\end{array}
$$

2. *By considering P we can generate the following new clauses*

$$
\begin{array}{l}
(\neg Q \vee \neg R) \\
(Q \vee \neg R) \\
(\neg Q \vee \neg R \vee S) \\
(\neg Q \vee R \vee S) \\
(Q \vee \neg R \vee S)
\end{array}
$$

3. *The new set of clauses obtained by adding the derived clauses and deleting the clauses containing P from the previous ones are:*

$$(Q \vee R \vee S)$$
$$(R \vee \neg S)$$
$$(\neg Q \vee \neg R)$$
$$(Q \vee \neg R)$$
$$(\neg Q \vee \neg R \vee S)$$
$$(\neg Q \vee R \vee S)$$
$$(Q \vee \neg R \vee S)$$

4. *By considering Q we can generate the following new clauses:*

$$(R \vee S)$$
$$(\neg R)$$
$$(\neg R \vee S)$$

5. *The new set of clauses obtained by adding the derived clauses and deleting the clauses containing Q from the previous ones are:*

$$(R \vee \neg S)$$
$$(R \vee S)$$
$$(\neg R)$$
$$(\neg R \vee S)$$

6. *By considering R we can generate the following new clauses:*

$$(S)$$
$$(\neg S)$$

7. *The new set of clauses obtained by adding the derived clauses and deleting the clauses containing R from the previous ones are:*

$$(S)$$
$$(\neg S)$$

8. *By considering S we can generate the empty clause*

⋆ ⋆ ⋆

Exercise 3 (First order logic: modelling (Max 7 marks)). *Assume the following predicates:*

$H(x)$:	*x is a human*
$C(x)$:	*x is a car*
$T(x)$:	*x is a truck*
$D(x, y)$:	*x drives y*

Write formulas representing the obvious assumptions: no human is a car, no car is a truck, humans exist, cars exist, only humans drive, only cars and trucks are driven, etc. Write formulas representing the following statements:

1. *Everybody (man) drives a car or a truck.*
2. *Some people drive both.*
3. *Some people don't drive either*
4. *Nobody drives both*
5. *Every car has at most one driver*
6. *Everybody drives exactly one vehicle (car or truck)*

Solution 3. *Obvious assumptions can be formalized as follows:*

no human is a car	$\forall x.(H(x) \supset \neg C(x))$
no car is a truck	$\forall x.(C(x) \supset \neg T(x))$
humans exist	$\exists x.H(x)$
cars exist	$\exists x.C(x)$
only humans drive	$\forall x.(\exists y.D(x,y) \supset H(x))$
only cars and trucks are driven	$\forall x.(\exists y.D(y,x) \supset C(x) \vee T(x))$

The formulas for the above statements are the following

1.	*Everybody drives a car or a truck*	$\forall x.(H(x) \supset \exists y.(D(x,y) \wedge (C(y) \vee T(y)))$
2.	*Some people drive both*	$\exists xyz.(D(x,y) \wedge C(y) \wedge D(x,z) \wedge T(z))$
3.	*Some people don't drive either*	$\exists x \forall y.\neg D(x,y)$
4.	*Nobody drives both*	$\forall xyz.(D(x,y) \wedge D(x,z) \supset \neg(C(y) \wedge T(z)))$
5.	*Every car has at most one driver*	$\forall xyz.(C(z) \wedge D(x,z) \wedge D(y,z) \supset x = y)$
b 6	*Everybody drives exactly one vehicle*	$\forall x.\exists y(D(x,y) \wedge \forall z.(D(x,z) \supset y = z))$

⋆ ⋆ ⋆

Exercise 4 (First Order logic: theory (Max 5 marks)). *If the following formula is valid, show a proof in natural deduction, if not provide a counter-model.*

$$\neg\neg\forall x.P(x) \supset \forall x.\neg\neg P(x)$$

Solution 4.

$$
\begin{prooftree}
\AxiomC{$\neg\neg\forall x.P(x)^3$}
\AxiomC{$\neg P(x)^2$}
\AxiomC{$\forall x.P(x)^1$}
\RightLabel{$\forall E$}
\UnaryInfC{$P(x)$}
\RightLabel{$\rightarrow E$}
\BinaryInfC{$\bot$}
\RightLabel{$\bot_c^1$}
\UnaryInfC{$\neg\forall x.P(x)$}
\RightLabel{$\rightarrow E$}
\BinaryInfC{$\bot$}
\RightLabel{$\bot_c^2$}
\UnaryInfC{$\neg\neg P(x)$}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall x.\neg\neg P(x)$}
\RightLabel{$\rightarrow I^3$}
\UnaryInfC{$\neg\neg\forall x.P(x) \rightarrow \forall x.\neg\neg P(x)$}
\end{prooftree}
$$

⋆ ⋆ ⋆

Exercise 5 (Modal logic representation (Max 8 marks)). *For each of the following sentence, which express a property on the binary relation R, find the axiom schema in modal logics that formalises the corresponding property. Explain your choice.*

1. $\forall x.R(x.x)$
2. $\forall xyz.(R(x,y) \wedge R(y,z) \supset R(x,z))$
3. $\forall x \exists y.R(x,y)$
4. $\forall xy.(R(x,y) \supset R(y.x)$

Solution 5.

1. $\forall x.R(x.x)$ *expresses reflexivity which can be formalized with the axiom schema* $\Box\phi \supset \phi$
2. $\forall xyz.(R(x,y) \wedge R(y,z) \supset R(x,z))$ *expresses transitivity which is formalized by the axiom schema* $\Box\phi \supset \Box\Box\phi$.
3. $\forall x \exists y.R(x,y)$ *expresses seriality, which can be represented by the axiom* $\Diamond\top$
4. $\forall xy.(R(x,y) \supset R(y.x)$ *expresses simmetry, which can be formalized by the axioms schema* $\phi \supset \Box\Diamond\phi$.

Mathematical logic
1^{st} assessment – Propositional Logic

Laurea Specialistica in Informatica
Universitá degli Studi di Trento

Prof. Luciano Serafini

March 21, 2013

Exercise 1. *(3 mins) List all the subformulas of the formula* $\neg p \supset (q \wedge (r \wedge \neg\neg q))$

Solution 1.

$\neg p \supset (q \wedge (r \wedge \neg\neg q))$
$\neg p$
p
$(q \wedge (r \wedge \neg\neg q))$
q
$(r \wedge \neg\neg q)$
r
$\neg q$

Exercise 2. *A (undirected) graph is defined as* $G = (V, E)$*, where* $V = \{v_1, v_2, \dots v_n\}$ *the set of vertices and* $E = \{(v_i, v_j), \dots (v_k, v_l)\}$ *the set of edges connecting pairs of vertices. (i.e. a set of connected vertices.) such that, if* $(v_i, v_j) \in E$*, then also* $(v_j, v_i) \in E$*.*

Propose a propositiona language to represent a graph with n nodes, and write a set of axioms that characterizes the graphs in which nodes has the maximal degree of 3 (i.e., a node has at most 3 neighbors.

Solution 2. *(10 mins) Let* $\mathcal{L}$ *be the propositional language conposed of the* $n(n-1)$ *propositional letters* v_{ij} *with* $1 \leq i \neq j \leq n$*. The intuitive interpretation of* v_{ij} *is that in the graph there is an arc from vertex* v_i *to vertex* v_j*.*

The set of axioms are the following

1. *for every* $i \neq j$*, we add the axioms* $v_{ij} \equiv v_{ji}$*, this represents the fact that the graph is undirected, and therefore having an arc form* i *to* j *is the same as aving the arc in the opposite direction.*
2. *the fact that the degree of the graph is less then tree, can be represented by a set of formulas of the form:*

$$v_{ij} \wedge v_{ik} \wedge v_{ih} \supset \neg v_{il}$$

for each 5-tuple (i, j, k, h, l) *of pairwise distinct numbers* $\leq n$

Exercise 3. *Among the two inference rules show that one is correct and the other not*

$$\frac{(A \vee B \vee C) \quad (\neg A \vee \neg B \vee C)}{C}\textsf{\textit{Rule1}} \qquad \frac{(A \vee C) \quad (\neg A \vee C)}{C}\textsf{\textit{Rule2}}$$

Solution 3. *The rule Rule1 is not correct since there is an interpretation where the premises are both true and the conclusion is false. Consider the interpretation* I *Nin which* A *is true,* B *is false and* C *is false. We have that* $I \models A \vee B \vee C$ *since* $I \models A$*.* $I \models \neg A \vee \neg B \vee C$*, since* $I \not\models B$*. Furthermore we have that* C *(the conclusion) is not satisfied by* I*.*

The rule Rule2, instead is correct. To show the correctness we have to prove that any interpretation that satisfies the premises, satisfies also the conclusion of the rule. Let I *be an interpretation that satisifes the premises of Rule2. I.e.,*

$$I \models A \vee C \tag{1}$$
$$I \models \neg A \vee C \tag{2}$$

From (1) we have two cases: (a) $I \models A$ *and (b)* $I \models C$*. In case (a), from the fact (2), we have that* $I \models C$*, i.e.,* I *satisfies the conclusion of Rule2. In case (b) we already have that* I *satisfies the conclusion of Rule2. Since in both cases,* I *satisfies the conclusion of Rule2 we can conclude that the rule is correct.*

Exercise 4. *Show that if* $A, B \models C$ *and* $A, \neg B \models C$*, then* $A \models C$.

Exercise 5. *Let* $\mathcal{L}$ *be the propositional language with the set* $p_1, \dots, p_n$ *of propositions. Show how many maximally consistent sets of formulas of* $\mathcal{L}$ *exist, and explain why.*

Exercise 6. *Provide an example of two sets of formulas* Γ *and* Σ *which are consistent, and such that* $\Gamma \cup \Sigma$ *is not consistent. Then show that, for every pair of consistent sets of formulas* Γ, Σ, *if* $\Gamma \cup \Sigma$ *is inconsistent, then there is a formula* ϕ *such that* $\Gamma \models \phi$ *and* $\Sigma \models \neg\phi$.

Solution 4. *If* $\Gamma \cup \Sigma$ *is inconsistent then* $\Gamma \cup \Sigma \vdash \bot$. *This means that there is a deduction of* $\bot$ *from a finite subset* $\Gamma_0 \cup \Sigma_0$ *of* $\Gamma \cup \Sigma$. *We suppose, w.l.o.g. that* $\Gamma_0 \subseteq \Gamma$ *and* $\Sigma_0 \subseteq \Sigma$. *Consider the formula* $\sigma_1 \wedge \dots \wedge \sigma_n$, *obtained by making a conjunction with all the formulas in* $\Sigma_0 = \{\sigma_1, \dots, \sigma_n\}$. *From the fact that* $\Gamma_0 \cup \Sigma_0 \vdash \bot$ *we can infer that* $\Gamma_0, \sigma_1 \wedge \dots \wedge \sigma_n \vdash \bot$ *and therefore that* $\Gamma_0, \vdash \neg(\sigma_1 \wedge \dots \wedge \sigma_n)$. *The fact that* $\Gamma_0 \subseteq \Gamma$, *implies that* $\Gamma \vdash \neg(\sigma_1 \wedge \dots \wedge \sigma_n)$. *On the other hand we have that* $\Sigma \vdash (\sigma_1 \wedge \dots \wedge \sigma_n)$. *So the formula* A *we are looking for is indeed* $(\sigma_1 \wedge \dots \wedge \sigma_n)$

Exercise 7. *Show by induction that every formula that does not contain the negation symbol "*$\neg$*" is satisfiable.*

Exercise 8. *Convert the following propositional logic sentences into Conjunctive Normal From:*

- $(p \wedge q) \vee ((\neg r \vee \neg s) \wedge (p \supset q))$
- $(a \supset \neg b) \wedge (\neg b \vee c) \wedge (a \vee \neg c)$

Exercise 9. *Determine via DPLL if the following set of clauses is satisfiable*

- $p \vee q \vee r$
- $p \vee \neg q$
- $q \vee \neg r$
- $r \vee \neg p$
- $\neg p \vee \neg q \vee \neg r$

Exercise 10. *Prove by means of natural deduction:*

1. $((A \supset B) \supset A) \supset A$
2. $((A \supset B) \vee (C \supset D)) \supset ((A \supset D) \vee (C \supset B))$
3. $((A \supset B) \supset B) \supset ((B \supset A) \supset A)$

Mathematical logic
2^{nd} assessment – First order Logic

Laurea Specialistica in Informatica
Universitá degli Studi di Trento

Prof. Luciano Serafini

January 21, 2008

Exercise 1. *Let L be a propositional language that allows to express weather forecasts. L contains the primitive propositions $Raining_i$ and $Sunny_i$, for $i \in \{0, 1, 2, \dots\}$. $Sunny_i$ (resp. $Raining_i$) means that in i-th days from now (0 is today, 1 is tomorrow, and so on ...) we will have a sunny (resp. raining) day. Define a first order language for describing the same domain. Then for each of the following sentences write*

- *a formalization in propositional language, if it exists, and if it does not exists then explain why.*
- *a formalization in first order language*

1. *Tomorrow we will have the same weather as today*
2. *eventually we will have a sunny day*
3. *within 5 days we will have two sunny days in a raw*

Solution 1. ***First order language*** *Constants 0; function succ with arity = 1; Predicate Sunny, Raining with arity = 1*

Sent.	*PD*	*FOL*
1.	$Sunny_0 \equiv Sunny_1 \wedge Raining_0 \equiv Raining_1$	$Sunny(0) \equiv Sunny(s(0)) \wedge Raining(0) \equiv Raining(s(0))$
2	*this fact cannot be expressed in propositional logic since this would result in an infinite disjunction of the form $Sunny_1 \vee Sunny_2 \vee Sunny_3 \vee \dots$, which is not a well formed formula in PL. Furthermore none of the finite disjunction would be OK. Indeed the formula $Sunny_1 \vee \dots \vee Sunny_n$, expresses that fact that the sunny day will happens within n days. However in this statements to do not commit to any particular upper-bound of raining days.*	$\exists x Sunny(x)$
3.	$\bigvee_{i=1}^{4}(Sunny_i \wedge Sunny_{i+1})$	$\exists x(s(0) \leq x \wedge x \leq s(s(s(s(0)))) \wedge Sunny(x) \wedge Sunny(s(x))$

Exercise 2. *Write a first order formula which is true in all the interpretations whose domain contains exactly 3 elements.*

Solution 2.

$$\exists x, y, z(x \neq y \wedge y \neq z \wedge x \neq z \wedge \forall w(x = w \vee y = w \vee z = w))$$

Exercise 3. *Let P be the only binary predicate (predicate on arity 2) of a first order language. Suppose that we consider only the interpretations of the previous exercise (i.e., the interpretations whose domain contains exactly 3 elements). Propose a propositional language, and show a way to transform the following FOL formulas in such a language*

- $\forall xy P(x, y)$
- $\exists xy P(x, y)$
- $\forall x \exists y(P(x, y))$
- $\exists x \forall y(P(x, y))$

Solution 3. ***Propositional Language*** P_{ij} *with* $i, j \in \{1, 2, 3\}$.

$$\forall xy P(x, y) \implies \bigwedge_{i=1}^{3} \bigwedge_{j=1}^{3} P_{ij} \tag{1}$$

$$\exists xy P(x, y) \implies \bigvee_{i=1}^{3} \bigvee_{j=1}^{3} P_{ij} \tag{2}$$

$$\forall x \exists y(P(x, y)) \implies \bigwedge_{i=1}^{3} \bigvee_{j=1}^{3} P_{ij} \tag{3}$$

$$\exists x \forall y(P(x, y)) \implies \bigvee_{i=1}^{3} \bigwedge_{j=1}^{3} P_{ij} \tag{4}$$

Exercise 4. *For each of the following formulas, say if they are valid, satisfiable, or unsatisfiable. For valid formulas provide a proof of validity; For satisfiable formulas provide an interpretation and an assignment; For unsatisfiable formulas provide a proof of unsatisfiability.*

1. $\forall xy(Q(x, y) \supset Q(y, x)) \supset \forall x \exists y Q(x, y)$
2. $\forall xy \exists z(P(x, y) \supset Q(y, z)) \supset \forall x(\exists y Q(x, y) \vee \forall y \neg P(y, x))$
3. $(\exists x P(x) \supset \forall y Q(y)) \supset \exists xy((P(x) \vee \neg Q(y)))$

Solution 4.

1. $\forall xy(Q(x, y) \supset Q(y, x)) \supset \forall x \exists y Q(x, y)$ *is not valid since the interpretation of I with $I(Q) = \emptyset$ satisfies the premise but not the conclusion of the implication.*
2. *The formula $\forall xy \exists z(P(x, y) \supset Q(y, z)) \supset \forall x(\exists y Q(x, y) \vee \forall y \neg P(y, x))$. In the following you can see a ND proof of it.*

```
                              P(ab)**   P(ab) -> Q(bc)***
                              -------------------------  ->E
Axy(Ez(P(xy) -> Q(yz))*            Q(bc)
------------------                --------- EI
 Ez(P(ab) -> Q(bz))                EyQ(by)
 ----------------------------------------------- EE disc ***
             EyQ(by)
     -----------------------------
        EyQ(by) \/ Ay ~P(by)           ~(EyQ(by) \/ Ay ~P(by))****
-------------------------------------------------------------- ->E
                         _|_
                  --------------- ->I disc **
                       ~ P(ab)
                  ---------------- AI
                      Ay ~P(yb)
                 --------------------
                EyQ(by) \/ Ay ~P(by)    ~(EyQ(by) \/ Ay ~P(by))****
               ------------------------------------------------ ->E
                                      _|_
                         ------------------------------ _|_c disc ****
                            EyQ(by) \/ Ay ~P(by)
                          -------------------------- AI
                          Ax((EyQ(xy) \/ Ay ~P(xy))
         ---------------------------------------------------- ->I disc *
         Axy Ez(P(xy) -> Q(yz)) -> Ax(Ey Q(xy) \/ Ay ~P(yx))
```

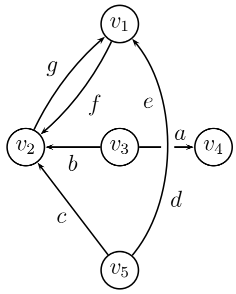


Figure 1: an example of labeled graph

3. $(\exists x P(x) \supset \forall y Q(y)) \supset \exists xy((P(x) \vee \neg Q(y)))$ *is also valid and the following is a proof:*

```
                             ~Q(x)**   Q(x)***
                            ------------------- ->1
                                 _|_
                               --------
Ax(P(x) v Q(x)) *                P(x)                 P(x) ****
----------------- AE         ----------              ------
  P(x) v Q(x)                  Ex P(x)               Ex P(x)
-------------------------------------------------------------- vE disc *** ****
                    Ex P(x)
               ------------------
                Ex P(x) v Ax Q(x)   ~(Ex P(x) v Ax Q(x)) *****
              -------------------------------------------- ->E
                                 _|_
                           ---------------- _|_ disc **
                                 Q(x)
                             ---------- AI-
                               Ax Q(x)
                        ------------------- vI
                          Ex P(x) v Ax Q(x)      ~(Ex P(x) v Ax Q(x)) ******
                         -------------------------------------------- ->E
                                        _|_
                                   ------------------ _|_ disc ******
                                  (Ex P(x) v Ax Q(x))
                      ----------------------------------------------- ->I  disc *
                            Ax(P(x) v Q(x)) -> Ex P(x) v Ax Q(x)
```

Exercise 5 (First order logic: representation). *A labelled graph is a triple* $\langle V, A, L\rangle$ *where* V *is a set of vertex,* A *is a set of directed arcs between vertexes and* L *is a function that associates a label to each arc. An example of labelled graph is shown in the Figure 1 Provide a language and a theory for labelled graphs.*
For each of the following conditions on graph write the corresponding axioms.

1. R_a is transitive;

2. $R_c = R_a \circ R_b$;

3. All the arcs exiting from a node has different labels

for every label x, R_x denotes the binary relation between vertexes defined as:

$$R_x(v_1, v_2) \text{ if and only there is an arc labeled with } x \text{ from } v_1 \text{ to } v_2. \ p$$

Solution 5. *For representing graphs see also exercises in the course handouts.*

Language *For each label a, there is a binary relation $R_a(x, y)$ which means that there is an arc labelled with a from vertex x to vertex y. To formalize conditions on graphs we can add the following axioms*

- *Any arc form x to y has only one label. For each R_a and R_b with $a \neq b$ we add the axiom*

$$\forall xy(R_a(x, y) \supset \neg R_b(x, y))$$

- *There are no reflexive arcs. For every R_a we add the following axiom*

$$\neg \exists x. R_a(x, x)$$

Formalization of the conditions *Given the language and the above theory, the axioms that formalize conditions 1–3,*

1. R_a *is transitive;*

$$\forall xyz(R_a(x, y) \wedge R_a(y, z) \supset R_a(x, z))$$

2. $R_c = R_a \circ R_b$, *which means that R_c is the composition of R_a and R_b, can be formalized by the axiom*

$$\forall xy(R_c(x, y) \equiv \exists z(R_a(x, z) \wedge R_b(z, y)))$$

3. *All the arcs exiting from a node has different labels. For any pair of different labels a, and b, we add the following axiom:*

$$\forall x(\exists y R_a(x, y) \supset \forall z \neg R_b(x, z))$$

This solution does not represent explicitly the labels as element of the domain; lables are simulated by the index of the predicates R's. This representation has some restriction in modeling universally and existentially quantified statements. For instance, to say some property that holds for all the lables, we have to add a (possibly infinite) set of axioms, one for each label a. Similarly, an existentially quantified statement over infinite set of labels is impossible. For instance, to say that between vertext x and y there is an arc, regardeless of the label, we would need a disjunction of the form

$$R_a(x, y) \vee R_b(x, y) \vee R_c(x, y) \ldots$$

These is a first order formula only if we have a finite and fixed number of labels. If there are infinite many labels we cannot write such a formula.

This solution is adequate if the labels are finite and known, so that every universally and existentially quantified statement on label can be represented as a finite conjunction and finite disjunction, respectively.

Alternative Solution 5. *If we don't know how many labels can be used in a graph, the solution provided below introduced labels as element of the domain of interpretation, and therefore allow to quantify over lables.*

Language *A unary predicate $V(x)$ for x is a vertex. A unary predicate $L(x)$ for x is a label Constants a, b, c ... for labels. A ternary predicate $A(x, y, l)$ for there is an arc from x to y labeled with l.*

We can formalize basic property of graphs by the following axioms:

- *Arcs are only between vertexes and are labelled with labels*

$$\forall xyl(A(x, y, l) \supset (V(x) \wedge V(y) \wedge L(l)))$$

- *the set of vertexes and labels are disjoint*

$$\forall xl(L(x) \supset \neg V(x))$$

- *a, b and c are distinct labels*

$$L(a) \wedge L(b) \wedge L(c) \wedge a \neq b \wedge a \neq c \wedge b \neq c$$

- *Any arc form x to y has only one label.*

$$\forall xy(A(x, y, l) \wedge A(x, y, l') \supset l = l')$$

- *There are no reflexive arcs.*

$$\neg \exists xl.A(x, x, l)$$

With this axiom we can formalize the conditions 1–3 by the following axioms

1. *R_a is transitive;*

$$\forall xyz(A(x, y, a) \wedge A(y, z, a) \supset A(x, z, a))$$

2. *$R_c = R_a \circ R_b$, which means that R_c is the composition of R_a and R_b, can be formalized by the axiom*

$$\forall xy(A(x, y, c) \equiv \exists z(A(x, z, a) \wedge A(z, y, b)))$$

3. *All the arcs exiting from a node has different labels.*

$$\forall xyy'll'(A(x, y, l) \wedge A(x, y', l') \supset l = l')$$

Mathematical logic
3^{rd} assessment – Modal Logic

Laurea Specialistica in Informatica
Universitá degli Studi di Trento

Prof. Luciano Serafini

January 22, 2008

Exercise 1 (3 marks). *Consider the model in figure 1. For each of the following formulas and say whether it is true or false in each world.*

1. $\Diamond_a p \supset \Box_b q$
2. $\Diamond_b \Diamond_b (p \wedge q) \supset \Box_a \Box_a (\neg p \wedge \neg q)$
3. $A \equiv \Box_b A$ *for any formula* A

Solution 1.

	w_1	w_2	w_3	w_4	w_5
(1)	*true*	*true*	*true*	*true*	*true*
(2)	*false*	*true*	*true*	*true*	*true*
(3)	*true*	*true*	*true*	*true*	*true*

Exercise 2 (3 marks). *A relation* R *is said to be the* identity relation *on a set* W *if,*

$$wRw' \quad \text{if and only if } w = w'$$

Propose a schematic formula ϕ *that is valid in a frame* $\mathcal{F} = (W, R)$ *if and only if* R *is the identity relation. More formally* ϕ *should be such that*

$$\mathcal{F} \models \phi \text{ if and only if } R \text{ is the identity relation on } W$$

Solution 2. $\phi = \Box A \equiv A$

1. *In the first part of the proof, we show that* $(W, R) \models \Box A \equiv A \Longrightarrow R$ *is the identity relation on* W*. Suppose that* R *is not the identity relation. This means that either* $(w, w) \notin R$ *for some* w *or* $(v, w) \in R$ *for some* v *different from* w*.*
   - *In the first case* w *is an isolated point, and we have that* $F, w \models \Box\bot$ *but* $F, w \not\models \bot$*. This implies that* $F \not\models \Box A \equiv A$*.*
   - *In the second case, consider the assignment* I *that set* p *to be true in* w *and false in* v*, then* $(F, I), w \models p$ *and* $(F, I), w \not\models \Box p$ *which means that* $F \not\models A \equiv \Box A$*.*

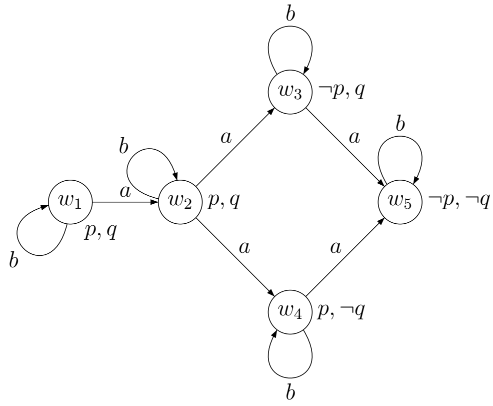


Figure 1:

We can therefore conclude that if R is not the identity function then it does not satisfy the schema $\Box A \equiv A$.

2. *in the second part, we show that R is the identity relation on $W \Longrightarrow (W, R) \models \Box A \equiv A$*

   $F \models \Box A \equiv A$ if and only if for all interpetation I and for all world w $(F, I), w \models A$ iff

   Let us prove this fact

$$\begin{aligned} (F, I), w \models \Box A \quad iff \quad & (F, I), v \models A \textit{ For all } v, \textit{ such that } (w, v) \in R \\ iff \quad & (F, I), w \models A \textit{ since } w \textit{ is the only world accessible from } w \end{aligned}$$

Exercise 3 (3 marks). *Prove that the following formulas are or are not valid in the class of all frames.*

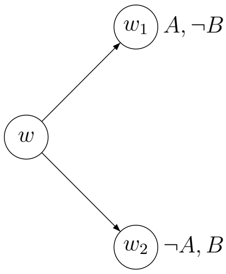
- $\Diamond A \vee \Diamond \neg A$
- $\Box A \supset (\Diamond B \supset \Diamond (A \wedge B))$
- $\Box (A \vee B) \supset (\Box A \vee \Box B)$

Solution 3.

$\Diamond A \vee \Diamond \neg A$ *is not valid in the frame* $F = (W = \{w\}, R = \emptyset\}$ *(the single isolated world)*

$\Box A \supset (\Diamond B \supset \Diamond (A \wedge B))$ *is valid. The proof is in some of the previous exercises.*

$\Box (A \vee B) \supset (\Box A \vee \Box B)$ *is not valid consider the following model*



$M, w \models \Box(A \vee B)$ *but* $M, w \not\models \Box A$ *and* $M, w \not\models \Box B$.

Exercise 4 (3 marks). *Let $F = (W, R)$ be a graph, W is the set of nodes and R is the set of undirected arcs (we admit reflexive arcs). Let $\mathcal{L}$ be a propositional language containing the proposition R, B, Y, G for Red, Blue, Yellow and Green. A model $M = (F, I)$ is a* coloring *of the graph F if and only if for every $w \in W$ there is exactly one primitive proposition $p \in \{R, B, Y, G\}$ such that $M, w \models p$ (the other three are false)*

1. *Write an axiom that is valid in all the models that are colorings of F.*
2. *Write a schematic formula that simulates the fact that the arcs in R are undirected (suggestion, an indirected arc from a to b, can be thought as two directed arcs one from a to b and the other in the opposite direction)*
3. *For each of the following sentences write a formula that is true in the worlds that satisfies it*
   (a) *I can reach a blue world in at most three steps*
   (b) *all the nodes reachable in one step from a blue node are either red or green*

Solution 4.

1. *Write an axiom that is valid in all the models that are colorings of F.*
$$(B \wedge \neg G \wedge \neg Y \wedge \neg R) \vee (\neg B \wedge G \wedge \neg Y \wedge \neg R) \vee (\neg B \wedge \neg G \wedge Y \wedge \neg R) \vee (\neg B \wedge \neg G \wedge \neg Y \wedge R)$$
2. *Write a schematic formula that simulates the fact that the arcs in R are undirected (suggestion, an indirected arc from a to b, can be thought as two directed arcs one from a to b and the other in the opposite direction)*
$$A \supset \Box\Diamond A$$
3. *For each of the following sentences write a formula that is true in the worlds that satisfies it*
   (a) *I can reach a blue world in at most three steps*
$$\Diamond(B \vee \Diamond(B \vee \Diamond(B \vee \Diamond B)))$$

(b) all the nodes reachable in one step from a blue node are either red or green

$$B \supset \Box(R \vee G)$$

Exercise 5 (3 marks). *Let $\mathcal{F} = (W, R_1, R_2)$ be a frame. Prove that $R_1 = R_2^{-1}$ if and only if*

1. $\mathcal{F} \models A \supset \Box_1 \Diamond_2 A$ *and*
2. $\mathcal{F} \models A \supset \Box_2 \Diamond_1 A$

($R^{-1} = \{(w, v) | (v, w) \in R\}$

Solution 5.

1. *We first show 1-2 $\Longrightarrow R_1 = R_2^{-1}$. Suppose that there $R_1 = R_2^{-1}$, this means that either there is a $(v, w) \in R_1$ with $(w, v) \notin R_2$, or that there is a $(v, w) \notin R_1$ with $(w, v) \in R_2$. In the first case consider the interpetation I that set p true in v and false in all the worlds reachable with R_2 from w. Then we have that $(F, I), v \models p$ and $(F, I), w \not\models \Diamond_2 p$, which implies that $(F, I), v \not\models \Box_1 \Diamond_2 p$. The second case is symmetric.*

2. *in the second part of the proof we show $R_1 = R_2^{-1} \Longrightarrow F \models A \supset \Box_1 \Diamond_2 A$ and $F \models A \supset \Box_2 \Diamond_1 A$*

$$\begin{aligned} (F, I), v \models A &\Rightarrow (F, I), w \models \Diamond_2 A \text{ for all } (v, w) \in R_1 \\ &\Rightarrow (F, I), v \models \Box_1 \Diamond_2 A \end{aligned}$$

$$\begin{aligned} (F, I), v \models A &\Rightarrow (F, I), w \models \Diamond_1 A \text{ for all } (v, w) \in R_2 \\ &\Rightarrow (F, I), v \models \Box_2 \Diamond_1 A \end{aligned}$$

Mathematical Logic

Exam: Laurea Specialistica in Informatica
Universita' degli Studi di Trento

24 January 2008

Exercise 1 (Propositional logic (Max 6 marks)). *Let L be a propositional language with the primitive propositions $p_1, \dots p_n$ and let $\mathcal{I}$ be any subset of all the interpretations of L. Explain how to build a single formula $\phi_{\mathcal{I}}$ such that the following property holds*

$$\textit{For all intepretation } I \textit{ of } L,\ I \models \phi_{\mathcal{I}} \textit{ if and only if } I \in \mathcal{I} \tag{1}$$

Prove (1) and explain why it is not possible to find such a ϕ when the language L contains an infinite set of propositions $p_1, p_2, \dots$

Exercise 2 (Propositional logic (Max 6 marks)). *Apply the Devis-Putnam procedure to the following clauses to compute the models or to prove their unsatisfiability. If a set of clauses are satisfiable, then provide* **all** *its models.*

1. $\{P, \neg Q\}, \{\neg P, Q\}, \{Q, \neg R\}, \{S\}, \{\neg S, \neg Q, \neg R\}, \{S, R\}$
2. $\{P, Q, S, T\}, \{P, S, \neg T\}, \{Q, \neg S, T\}, \{P, \neg S, \neg T\}, \{P, \neg Q\}, \{\neg R, \neg P\}, \{R\}$

Exercise 3 (First order logic (mas 6 marks)). *A tree is a structure $T = (N, \prec)$, where N is a non empty set, $n_1 \prec n_2$ means that the node n_1 is the parent note of n_2, and the following properties hold:*

1. *there is a unique element $n_0 \in N$, called the* root of T *which does not have any parent node.*
2. *every node of T different from the root has a unique parent.*

Provide a first order language for representing tree structures and use it to formalizes the above two properties. With the same language formalize also the following properties

1. *the degree of the tree is 2, i.e. every node has at most 2 children*
2. *the maximal depth of the tree is 3, i.e. there is no branch of T with more than 3 nodes*

3. *T is binary tree. i.e., every node is either a leaf (and does not have any children) or it has exactly two children.*

Exercise 4 (First order logic (mas 6 marks)). *Either prove by ND or show a counter-model for the following formulas*

1. $\neg P(a) \vee Q(b) \supset \exists x(P(x) \supset Q(x))$
2. $\exists xy.P(x,y) \supset \exists x P(x,x)$
3. $\forall x_1, x_2, x_3(P(x_1,x_2,x_3) \supset P(x_3,x_1,x_2)) \supset (P(a,b,c) \supset P(c,b,a))$

Exercise 5 (Modal logics (mas 6 marks)). *A frame (W,R) is an S4 frame if and only if R is a reflexive and transitive relation. for each of the following formula check if it is valid in an S4 frame. If it is not valide provide a countermodel*

1. $\Box A \supset \Diamond A$
2. $\Diamond A \supset A$
3. $A \supset \Diamond A$
4. $\Diamond\Diamond A \supset \Diamond A$
5. $\Box A \wedge \Box\Box B \supset \Box\Box(A \wedge B)$
6. $\Box\Diamond A \supset \Diamond\Box A$

Mathematical Logic

Exam: Laurea Specialistica in Informatica
Universita' degli Studi di Trento

June 17, 2008

Exercise 1 (Propositional logic: modelling (Max 5 marks)). *Four married couples of friends, Aldo(M), Beatrice(F), Cinzia(F), Dario(M), Enrico(M), Federico(M), Giada(F), and Helena(F), go to play tennis. They book two fields, and every couple plays one match against one of the others.*

- *Aldo plays against Beatrice*
- *Dario and Federico are not in the same field*
- *Helena, who is married with Enrico, plays against Cinzia*
- *Dario is married with Cinzia.*

Provide a logical formalization of this problem so that it is possible to logically infer the teams and who is playing against whom.

Exercise 2 (Propositional Logic: theory (Max 5 marks)). *Use the Davis-Putnam procedure to compute models for the following clause sets or to prove that no model exists. At each step, indicate which rule you have applied.*

$$\{P, \neg Q\}, \{\neg P, Q\}, \{Q, \neg R\}, \{S\}, \{\neg S, \neg Q, \neg R\}, \{S, R\}$$

Exercise 3 (First order logic: modelling (Max 5 marks)). *Formalize the following statements, by using* only *the following first order predicates:*

$F(x)$	*x is female*
$M(x)$	*x is a male*
$MW(x, y)$	*x is married with y*
$PA(x, y)$	*x plays against y*

1. *everybody must be either a male or (exclusively) a female*

2. *Mans can be married with womens and viceversa*

3. *One can be married with at most a person*

4. *Games can be single or double. I.e., either one plays against one or two against two*

5. *married people play always in team*

6. *being married and playing against are symmetric and irreflexive relations*

7. *Married couples always plays doubles against other married couples*

Exercise 4 (First order logic: theory (Max 5 marks))**.** *Use natural deduction to show that the following formula is valid*

$$\exists x(\neg A \vee B(x)) \supset (A \supset \exists x B(x))$$

where x does not occur free in A. Explain why you need the condition of x not being free in A, to prove that the above formula is valid.

Exercise 5 (Modal logics theory (Max 5 marks))**.** *Provide a model M and a world w that falsify the following formulas*

1. $\Diamond A \wedge \Diamond B \supset \Diamond(A \wedge B)$

2. $\Box A \supset \Diamond A$

3. $\Diamond\Box A \supset \Box\Diamond A$

4. $\Box A \supset \Box\Box A$

5. $\diamond A \vee \diamond\neg A$

Exercise 6 (Modal logics Representation (Max 5 marks))**.** *Let C be the class of frames (W, R) such that every w has at most two R-successor. Provide a schema Φ such that $\mathcal{F} \models \Phi$ if and only if $\mathcal{F} \in \mathsf{C}$.*

Mathematical Logic Exam

Laurea Specialistica in Informatica
DISI - Universita' degli Studi di Trento

July 22, 2008

Exercise 1 (Propositional logic: modelling (Max 5 marks)). *There are three men in front of a jury, suspected for the theaft of a car. They testify as follows:*

Bob: *Alan stole the car. Jack cannot be guilty: he was all the time with me at Mc Doe's pub.*

Alan: *If Bob stole the car, then Jack helped him.*

Jack: *I was at home that night, you can ask my wife. I'm sure at least one of the others is guilty.*

Express the testimony of each suspect as a propositional formula, trying to use the fewest propositions as possible.

Write a truth table for the three testimonies and use it to answer the following questions:

1. *Are the three testimonies consistent?*
2. *The testimony of one of the suspects follows from that of another. Which from which?*
3. *Assuming everybody is innocent, who committed perjury?*
4. *Assuming all testimony is true, who is innocent and who is guilty?*
5. *Assuming that the innocent told the truth and the guilty told lies, who is innocent and who is guilty?*

Exercise 2 (Propositional logic: DPLL (Max 5 marks)). *Determine via DPLL whether the following formula is valid, unsatisfiable or satisfiable.*

$$(p \vee (\neg q \wedge r)) \rightarrow ((q \vee \neg r) \rightarrow p)$$

Exercise 3 (First order logic: modelling (Max 5 marks)). *Formalize the following statements, by using* only *the following first order predicates:*

$S(x)$	*x is a student*
$T(x)$	*x is a teacher*
$C(x)$	*x is a course*
$teach(x, y)$	*x teaches course y*
$attend(x, y)$	*x attends course y*
$pass(x, y)$	*x passed course y*

1. *Each teacher has at most two courses.*
2. *Each course has exactly one teacher and at least one student.*
3. *A teacher can attend a course as a student, provided that he's not teaching in that course.*
4. *Every student that attends Logica2 must have passed Logica1.*
5. *No student failed Geometry but at least one student failed Analysis.*
6. *Nobody ever passed a course taught by Prof. Attila.*

Exercise 4 (First order logic: theory (Max 5 marks)). *Use natural deduction to show that the following formula is valid*

$$(\exists x P(x) \wedge \exists y Q(y)) \equiv \exists x \exists y (P(x) \wedge Q(y))$$

Exercise 5 (Modal logics theory (Max 5 marks)). *For each of the following formulas either prove that it is valid or find a counter-example. Note that if your attempts to produce a falsifying model always end in incoherent pictures, it may be because the formula is valid.*

1. $\Box A \wedge \Diamond B \supset \Diamond(A \wedge B)$
2. $\Diamond\Box A \supset \Box\Box A$
3. $\Diamond(\Box A \wedge \Diamond B) \supset \Diamond\Diamond \mathrm{T}$

Exercise 6 (Modal logic theory (max 5 marks)). *Show that if a frame $\langle W, R\rangle$ satisfy the schema $\Box\phi \supset \Box\Box\phi$ then R is transitive.*

Mathematical logic
1^{st} assessment – Propositional Logic

Laurea Specialistica in Informatica
Universitá degli Studi di Trento

Prof. Luciano Serafini

Solutions

Exercise 1. *Explain the difference between the following two statements*

1. $\models A \vee B$
2. $\models A$ *or* $\models B$

Solution 1. $\models A \vee B$ *means that for every interpretation* m*, either* $m \models A$ *or* $m \models B$

$\models A$ *or* $\models B$ *means that either for every interpretation* m*,* $m \models A$ *or for every interpretation* m*,* $m \models B$.

To highlight the difference between 1. and 2. you can write their definition by using a more formal notation,

$$\models A \vee B \iff \forall m, (m \models A \text{ or } m \models B) \quad (1)$$
$$\models A \text{ or } \models B \iff (\forall m, m \models A) \text{ or } (\forall m, m \models B) \quad (2)$$
$$(3)$$

An example that shows the difference can be constructed by taking A *equal to the atomic formula* p *and* B *the negated atomic formula* $\neg p$*. You have that* $\models p \vee \neg p$*, but neither* $\models p$ *nor* $\models \neg p$

Exercise 2. *Provide a propositional language describing the bus transport system of a town so that you can express the following propositions:*

- *bus #5 goes from A to B and back*
- *bus #4 and bus #5 intersect at some bus station*
- *every bus intersect with at least another bus*
- *bus #1 makes a round trip (i.e., i.e., it goes from A to A without passing twice from the same station)*

Furthermore: write an axiom that states that the bus route is linear.

Solution 2. *Let* Lines *be the finite set of bus lines (e.g.,* $\{\#1, \#2 \dots\}$*; let* Stops *be the finite set of bus stops (e.g,* $\{$*Povo-Piazza-Manci,*$)$*Povo-IRST, Trento-Stazione-FS,*$\dots\}$ *we define the propositional language that contains the following set of propositional variables*

$$\{l(s_1, s_2) \mid l \in \mathsf{Lines}, \quad s_1, s_2 \in \mathsf{Stops}, \quad \textit{and} \quad s_1 \neq s_2\}$$

Intuitively $l(s_1, s_2)$ *means that the line* l *directly connects the bus stop* s_1 *with the bus stop* s_2.

- *bus #5 goes from A to B and back*

$$\#5(A,B)\vee \quad \textit{\#5 goes directly from A to B, or} \ldots \tag{4}$$
$$\bigvee_{s_1,\ldots,s_n\in Stops} \ldots \quad \textit{there are n stops such that} \ldots \tag{5}$$
$$\#5(A,s_1)\wedge\#5(s_n,B)\wedge \quad \ldots \textit{\#5 connects A with B} \ldots \tag{6}$$
$$\bigwedge_{i=1}^{n-1}(\#5(s_i,s_{i+1})\wedge\wedge \quad \ldots \textit{through } s_1,\ldots s_n. \textit{ And} \ldots \tag{7}$$
$$\#5(B,A)\vee \quad \textit{\#5 goes directly from B to A, or} \ldots \tag{8}$$
$$\bigvee_{\{s_1,\ldots,s_m\}\subset Stops} \ldots \quad \textit{there are m stops such that} \ldots \tag{9}$$
$$\#5(B,s_1)\wedge\#5(s_m,A)\wedge \quad \ldots \textit{\#5 connects B with A} \ldots \tag{10}$$
$$\bigwedge_{i=1}^{m-1}(\#5(s_i,s_{i+1})\wedge\wedge \quad \ldots \textit{through } s_1,\ldots s_m. \tag{11}$$

- *bus #4 and bus #5 intersect at some bus station. Let* $\#5(s)$ *denote the fomrula*

$$\bigvee_{s'\in Stops}\#5(s,s')\vee\#5(s',s))$$

(i.e., bus number 5 stops at s), and let define $\#4(s)$ *in an analogous way. We can formalize the intersection between* $\#5$ *and* $\#4$*, by the following formula*

$$\bigvee_{s\in Stops}(\#5(s)\wedge\#4(s)) \tag{12}$$

- *every bus intersect with at least another bus*

$$\bigwedge_{l\in Lines}\left(\bigvee_{l'\in Lines\setminus\{l\}}\left(\bigvee_{s\in Stops}(l(s)\wedge l'(s)\right)\right)$$

- *bus #1 makes a round trip (i.e., it goes from A to A without passing twice from the same stateion)*

$$\bigvee_{\{s_1,\ldots x_n\}\subseteq Stops} \ldots \quad \textit{There are n intermediate stopst such that} \tag{13}$$
$$\#1(A,s_1)\wedge \quad \textit{the first one is reachable from A and} \ldots \tag{14}$$
$$\#(s_n,A)\wedge \quad \textit{A is reachable form the last one, and} \ldots \tag{15}$$
$$\left(\bigwedge_{i=1}^{n-1}\left(\#1(s_i,s_{i+1})\wedge\bigwedge_{s\in Stops}^{s\neq s_{i+1}}\neg\#1(s_i,s)\right)\right)\wedge \quad \textit{from each intermediate stop you can reach only the successive one, and} \ldots \tag{16}$$
$$\bigwedge_{s,s'\notin S}\neg\#1(s,s') \quad \textit{\#1 does not connect any stop outside S} \tag{17}$$

Exercise 3. *Prove by induction that if a formula* ϕ *does not contain two or more occurrences of the same propositional letter, then it is satisfiable.*

Solution 3. *We prove by induction the following property:*

> *for every formula ϕ that contains only single occurrences of propositional variables, there is an interpretation $\mathcal{I}^+$ that satisfies it and an interpretation $\mathcal{I}^-$ that falsifies it.*

Notice that, the property we want to prove is stronger than the one in exercise 3. Sometimes, in proving theorems by induction, this turns to be inevitable, in order to prove some specific inductive step. In this case, proving also the fact that the formula has a counter-model (i.e., an interpretation that does not satisfy it) turns out to be necessary in order to prove the inductive step in which ϕ is of the form $\neg\phi_i$. indeed, to prove that ϕ is satisfiable, i.e., that there is a model $\mathcal{I}$ that $\mathcal{I} \models \phi$, we have to find a counter-model for ϕ_1, i.e., a model $\mathcal{I}_1$ that $\mathcal{I}_1 \not\models \phi_1$. So the inductive hypothesis should guarantee the existence of such a counter model. If we don't prove such an existence by induction, then we cannot perform the step case. To understand this please check the case of $\neg\phi_1$.

Base Case *if ϕ is an atomic formula, say p, then it is satisfiable by the interpretation $\mathcal{I}^+$, with $I^+(p) = \text{True}$, and $\mathcal{I}^-$, with $I^+(p) = \text{False}$.*

Step Case If ϕ **is** $\phi_1 \wedge \phi_2$, *then by induction there there is an $\mathcal{I}_i^+$ and $\mathcal{I}_i^-$ such that $\mathcal{I}_i^+ \models \phi_i$ and $\mathcal{I}_i^- \not\models \phi_i$ with $i = 1, 2$. Then the interpretation $\mathcal{I}^+$ defined as:*

$$\mathcal{I}^+(p) = \begin{cases} \mathcal{I}_1^+(p) & \text{if } p \text{ occurs in } \phi_1 \\ \mathcal{I}_2^+(p) & \text{if } p \text{ occurs in } \phi_2 \end{cases} \tag{18}$$

Since there is no propositional variable p that occurs both in ϕ_1 and ϕ_2, the definition of $\mathcal{I}^+$ is coherent. Furthermore, since $\mathcal{I}^+$ coincides with $\mathcal{I}_i^+$ on the variables of ϕ_i we have that $\mathcal{I} \models \phi_i$ for $i = 1, 2$ and therefore that $\mathcal{I}^+ \models \phi_1 \wedge \phi_2$.

As far as $\mathcal{I}^-$, let's take it to be one among $\mathcal{I}_1^-$ and $\mathcal{I}_2^-$, no matter which you chose, you have that $\mathcal{I}^- \not\models \phi_1 \wedge \phi_2$ as $\mathcal{I} \not\models \phi_i$ for some $i = 1, 2$.

If ϕ **is** $\phi_1 \supset \phi_2$, *then by induction ϕ_2 is satisfiable by the interpretation $\mathcal{I}^+$, which implies that $\mathcal{I}^+$ satisfy also $\phi_1 \supset \phi_2$. As far as $\mathcal{I}^-$ we proceed as in the case of $\wedge$. Let $\mathcal{I}_1^+$ be an interpretation that satisfies ϕ_1 and $\mathcal{I}_2^-$ be an intterpretation that does not satisfy ϕ_2; they exists by inductive hypothesis. We define $\mathcal{I}^-$ as in (18), obtaining that $\mathcal{I}^- \models \phi_1$ and $\mathcal{I}^- \not\models \phi_2$. This implies that $\mathcal{I}^- \not\models \phi_1 \supset \phi_2$, and therefore that $\mathcal{I}^- \not\models \phi$.*

If ϕ **is** $\phi_1 \vee \phi_2$, *We proceed as in the case of $\supset$ by taking $\mathcal{I}^+$ to be either $\mathcal{I}_1^+$ or $\mathcal{I}_2^+$, and $\mathcal{I}^-$ to be the composition via (18) of $\mathcal{I}_1^-$ or $\mathcal{I}_2^-$.*

If ϕ **is** $\phi_1 \equiv \phi_2$, *We proceed as in the case of $\supset$ by taking $\mathcal{I}^+$ to be either the composition via (18) of either $\mathcal{I}_1^+$ and $\mathcal{I}_2^+$, or $\mathcal{I}_1^-$ and $\mathcal{I}_2^-$, and for and $\mathcal{I}^-$ to be the composition via (18) of either $\mathcal{I}_1^-$ and $\mathcal{I}_2^+$ or $\mathcal{I}_1^+$ and $\mathcal{I}_2^-$.*

If ϕ **is** $\neg\phi_1$, *then let $\mathcal{I}_1^+$ be a model that satisfies ϕ_1 and $\mathcal{I}_1^-$ be a model that does not satisfy ϕ_1; they exists by inductive hypothesis. By defining $\mathcal{I}^+ = \mathcal{I}_1^-$ and $\mathcal{I}^- = \mathcal{I}_1^+$, we have that $\mathcal{I}^+ \models \neg\phi_1$ and $\mathcal{I}^- \not\models \neg\phi_1$.*

Exercise 4. *Show that if $A, B \models C$ and $A, \neg B \models C$, then $A \models C$.*

Solution 4. *We apply the definition of logical consequence, i.e. $\Gamma \models \phi$ if for every interpretation $\mathcal{I}$, $\mathcal{I} \models \Gamma$ implies that $\mathcal{I} \models \phi$.*

To prove that $A \models C$, let $\mathcal{I}$ be any interpretation with $\mathcal{I} \models A$ Since, for every formula B, either $\mathcal{I} \models B$ or $\mathcal{I} \models \neg B$, we consider the two cases:

If $\mathcal{I} \models B$ *then $\mathcal{I} \models \{A, B\}$ and by the hypothesis that $A, B \models C$, we have that $\mathcal{I} \models C$;*

If $\mathcal{I} \models \neg B$ *then $\mathcal{I} \models \{A, \neg B\}$ and by the hypothesis that $A, \neg B \models C$, we have that $\mathcal{I} \models C$.*

Since in both cases $\mathcal{I} \models C$, we can conclude that $A \models C$.

Exercise 5. *Let Γ be a maximally consistent set, show that for all ϕ either $\phi \in \Gamma$ or $\neg\phi \in \Gamma$.*

Solution 5. *Suppose by absurdum that Γ is maximally consistent and that it does not contain neither ϕ nor $\neg\phi$. By definition of maximally consistent, this implies that $\Gamma, \phi \models \bot$ and $\Gamma, \neg\phi \models \bot$. By exercise 4, we have that $\Gamma \models \bot$. which contradicts the fact that Γ is consistent. This implies that either ϕ or $\neg\phi$ belongs to Γ.*

Exercise 6. *Provide an example of two sets of formulas Γ and Σ which are consistent, and such that $\Gamma \cup \Sigma$ is not consistent. Then show that, for every pair of consistent sets of formulas Γ, Σ, if $\Gamma \cup \Sigma$ is inconsistent, then there is a formula ϕ such that $\Gamma \models \phi$ and $\Sigma \models \neg\phi$.*

Solution 6. *If $\Gamma = \{p\}$ and $\Sigma = \{\neg p\}$, then Γ and Σ are separately consistent, but $\Gamma \cup \Sigma = \{p, \neg p\}$ is not consistent.*

If $\Gamma \cup \Sigma$ is inconsistent then $\Gamma \cup \Sigma \vdash \bot$. This means that there is a deduction of $\bot$ from a finite subset $\Gamma_0 \cup \Sigma_0$ of $\Gamma \cup \Sigma$. We suppose, w.l.o.g. that $\Gamma_0 \subseteq \Gamma$ and $\Sigma_0 \subseteq \Sigma$. Consider the formula $\sigma_1 \wedge \cdots \wedge \sigma_n$, obtained by making a conjunction with all the formulas in $\Sigma_0 = \{\sigma_1, \ldots, \sigma_n\}$. From the fact that $\Gamma_0 \cup \Sigma_0 \vdash \bot$ we can infer that $\Gamma_0, \sigma_1 \wedge \cdots \wedge \sigma_n \vdash \bot$ and therefore that $\Gamma_0, \vdash \neg(\sigma_1 \wedge \cdots \wedge \sigma_n)$. The fact that $\Gamma_0 \subseteq \Gamma$, implies that $\Gamma \vdash \neg(\sigma_1 \wedge \cdots \wedge \sigma_n)$. On the other hand we have that $\Sigma \vdash (\sigma_1 \wedge \cdots \wedge \sigma_n)$. So the formula A we are looking for is indeed $(\sigma_1 \wedge \cdots \wedge \sigma_n)$

Exercise 7. *Prove hy Hilbert calculus that*

$$(\neg A \supset A) \supset A$$

Suggestion: suppose that you have already proven that $\neg A \supset \neg A$

Solution 7. *The proof of $\neg A \supset \neg A$ is as follows*

1.	$\neg A \supset ((\neg A \supset \neg A) \supset \neg A)$	*Axiom A1*
2.	$(\neg A \supset ((\neg A \supset \neg A) \supset \neg A)) \supset ((\neg A \supset (\neg A \supset \neg A)) \supset (\neg A \supset \neg A))$	*Axiom A2*
3.	$(\neg A \supset (\neg A \supset \neg A)) \supset (\neg A \supset \neg A)$	*From 1. and 2. by MP*
4.	$(\neg A \supset (\neg A \supset \neg A))$	*Axiom A1*
5.	$\neg A \supset \neg A$	*From 3. and 4. by MP*

Then we can continue with the proof of

6.	$\neg A \supset \neg A$	*Already proved*
7.	$(\neg A \supset \neg A) \supset ((\neg A \supset A) \supset A)$	*Axiom (A3)*
8.	$(\neg A \supset A) \supset A$	*From 6. and 7. by MP)*

Exercise 8. *Convert the following propositional logic sentences into Conjunctive Normal From:*

$$(a \vee \neg b) \wedge (\neg b \vee \neg c) \vee (a \vee \neg c)$$

Solution 8.

$(a \vee \neg b) \wedge (\neg b \vee \neg c) \vee (a \vee \neg c)$
$((a \vee \neg b) \vee (a \vee \neg c)) \wedge ((\neg b \vee \neg c) \vee (a \vee \neg c))$
$(a \vee \neg b \vee \neg c) \wedge (a \vee \neg b \vee \neg c)$
$(a \vee \neg b \vee \neg c)$

Exercise 9. *Determine via DPLL if the following set of clauses is satisfiable*

$$(A, B, C)(A, \neg C)(\neg A, D)(\neg A, E)(B, \neg D, \neg E)$$

If yes provide the assignment.

Solution 9.

$$(A, B, C), (A, \neg C), (\neg A, D), (\neg A, E), (B, \neg D, \neg E)$$

Considering A you obtain the clauses

$$(D, B, C), (B, C, E), (\neg C, D), (\neg C, E)$$

which are added to all the clauses that don't contain A, obtaining

$$(D, B, C), (B, C, E), (\neg C, D), (\neg C, E), (B, \neg D, \neg E)$$

considering B you obtain no clauses, so you can remove all the clauses with B obtaining

$$(\neg C, D), (\neg C, E)$$

by considering C you are not able to deriva any clauses so you reach the empty set. without being able to infer the empty clause. Which implies that the set of clauses are satisfiable. An a possible assignment is $A, B, \neg C, D, E$

Exercise 10. *Prove by means of natural deduction at least one of the following formulas*

1. $(A \supset B) \supset (\neg B \supset \neg A)$
2. $((A \supset B) \supset C) \lor ((B \supset A) \supset C)$
3. $(\not A \lor \neg B) \supset \neg(A \land B)$

Solution 10.

1. $(A \supset B) \supset (\neg B \supset \neg A)$*:See esercize 1.52 on the "propositional logic exercise" collection.*
2. $((A \supset B) \supset C) \lor ((B \supset A) \supset C)$*: This formula is not valid so it cannot be proved.*
3. $(\neg A \lor \neg B) \supset \neg(A \land B)$

$$\dfrac{\dfrac{\dfrac{\neg A \lor \neg B^4 \qquad \dfrac{\neg A^1 \qquad \dfrac{A \land B^3}{A}\land E}{\bot}\supset E \qquad \dfrac{\neg B^2 \qquad \dfrac{A \land B^3}{B}\land E}{\bot}\supset E}{\bot}\lor E_{(1,2)}}{\neg(A \land B)}\supset I_{(3)}}{(\neg A \lor \neg B) \supset \neg(A \land B)}\supset I_{(4)}$$

Mathematical logic
2^{nd} assessment – First order Logic

Laurea Specialistica in Informatica
Universitá degli Studi di Trento

Prof. Luciano Serafini

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Exercise 1 (3 marks). *Show that the following formulae are not valid:*

$$\forall y \exists x P(x,y) \supset \exists x \forall y P(x.y) \quad (1)$$

$$\exists x P(x) \wedge \exists x Q(x) \supset \exists x (P(x) \wedge Q(x)) \quad (2)$$

$$\forall x (P(x) \vee Q(x)) \supset \forall x P(x) \vee \forall x Q(x) \quad (3)$$

Exercise 2 (6 marks). *For each of the following formula say if it is valid(V), unsatisfiable (U) or satisfiable (S). If the fomula is satisfiable, provide an interpretation that makes it true, If the formula is valid, provide a proof in ND. If the formula is unsatisfiable, show it by resolution*

1. $\forall x (P(x) \vee Q(x)) \supset (\forall x P(x) \vee \forall x Q(x))$
2. $\exists x \forall y P(x,y) \wedge \exists z \forall w \neg P(w,z))$
3. $\forall xy (P(x,y) \supset \neg P(y,x)) \supset \forall x \exists y \neg P(x,y)$

Exercise 3 (3 marks). *Show that if $\neg forallxphi(x)$ is satisfiable, then $\neg\phi(c)$ is satisfiable for some constant c not appearing in ϕ. Show also that $\neg\forall x \phi(x) \supset \neg\phi(c)$ is not valid.*

Exercise 4 (2 marks). *Is the following inference rule sound?*

$$\frac{\forall x (A(x) \supset \exists y B(x,y)) \qquad \neg B(a,b)}{\neg A(a)}$$

Explain why.

Exercise 5 (4 marks). *Express the following knowledge in a set K of first-order logic formulas and add enough common sense statements (e.g. everyone has at most one spouse, nobody can be married to himself or herself, Tom, Sue and Mary are different people) to make K entail a formula expressing the fact that "Mary is not married". Show this either either by means of a proof or by semantic reasoning.*

Knowledge: *There are exactly three people in the club, Tom, Sue and Mary. Tom and Sue are married. If a member of the club is married, their spouse is also in the club.*

Exercise 6 (4 marks). *Formulate the requirements below as sentences of first order logic and show that the two of them cannot be true together in any interpretation. (This is the barber's paradox by Bertrand Russell)*

1. *Anyone who does not shave himself must be shaved by Figaro (The Barber of Seville)*

2. *Whomever the barber shaves, must not shave himself.*

Then show by means of resolution that the two sentences are unsatisfiable

Exercise 7 (2 marks). *What can you say on the cardinality (i.e., the number of elements) of the domain of the models of* $\forall xyz(x = y \vee y = z \vee z = w)$*? Is* $\exists x \exists y \exists z(x \neq y \wedge y \neq z \wedge z \neq w)$ *always true in such models? Explain why.*

Exercise 8 (2 marks). *Find if the two pairs of terms are unifyable and if yes provide the MGU*

1. $f(x, g(a, y))$, $f(a, g(x, z))$

2. $f(g(x), g(y))$, $f(y, g(x))$

Exercise 9 (4 marks). *Consider the following formulae asserting that a binary relation is symmetric, transitive, and total:*

$$
\begin{array}{lll}
S1 & : & \forall x \forall y (P(x, y) \supset P(y, x)) \\
S2 & : & \forall x \forall y \forall z ((P(x, y) \wedge P(y, z)) \supset P(x, z)) \\
S3 & : & \forall x \exists y P(x, y)
\end{array}
$$

Prove by resolution that

$$S1 \wedge S2 \wedge S3 \supset \forall x P(x, x).$$

Mathematical logic
3^{rd} assessment – Modal Logic

Laurea Specialistica in Informatica
Universitá degli Studi di Trento

Prof. Luciano Serafini

Solutions

Exercise 1 (3 marks)**.** *Consider the model in figure 1.*

1. *Check if* $M, w_1 \models \Diamond_a(p \wedge \Diamond_b(q \wedge \neg\Box_a(\neg p \wedge \Diamond_a\Box_b\neg q)))$
2. *If there is one, find a world* w *and a formula* ϕ *such that* $M, w \not\models \Box_b\phi \supset \phi$
3. *Write a formula that is satisfied* ***only*** *in world* w_1, w_2 *and* w_3

Solution 1. 1. *Check if* $M, w_1 \models \Diamond_a(p \wedge \Diamond_b(q \wedge \neg\Box_a(\neg p \wedge \Diamond_a\Box_b\neg q)))$

$$
\begin{array}{lcl}
M, w_1 & \models & \Diamond_a(p \wedge \Diamond_b(q \wedge \neg\Box_a(\neg p \wedge \Diamond_a\Box_b\neg q))) \Longleftarrow \\
M, w_2 & \models & p \wedge \Diamond_b(q \wedge \neg\Box_a(\neg p \wedge \Diamond_a\Box_b\neg q)) \Longleftarrow \\
M, w_2 & \models & p \textit{ and} \\
M, w_2 & \models & \Diamond_b(q \wedge \neg\Box_a(\neg p \wedge \Diamond_a\Box_b\neg q)) \Longleftarrow \\
M, w_1 & \models & q \wedge \neg\Box_a(\neg p \wedge \Diamond_a\Box_b\neg q) \Longleftarrow \\
M, w_1 & \models & q \textit{ and} \\
M, w_1 & \models & \neg\Box_a(\neg p \wedge \Diamond_a\Box_b\neg q) \Longleftarrow \\
M, w_2 & \not\models & \neg p \wedge \Diamond_a\Box_b\neg q \Longleftarrow \\
M, w_2 & \not\models & \neg p \textit{ or} \\
M, w_2 & \not\models & \Diamond_a\Box_b\neg q \Longleftarrow \\
M, w_2 & \models & p
\end{array}
$$

since $M, w_2 \models p$, *and* $M, w_1 \models q$, *by following back the arrows, we can conclude that* $M, w_1 \models \Diamond_a(p \wedge \Diamond_b(q \wedge \neg\Box_a(\neg p \wedge \Diamond_a\Box_b\neg q)))$

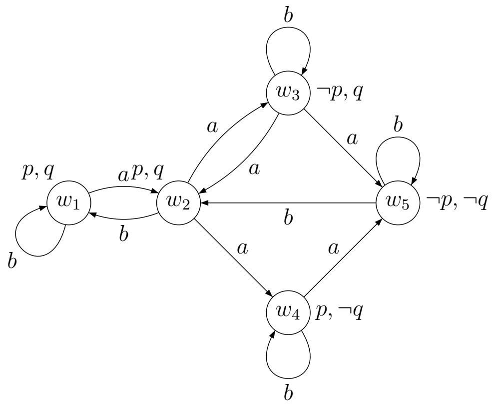


Figure 1:

2. *If there is one, find a world w and a formula ϕ such that $M, w \not\models \Box_b\phi \supset \phi$*

Since the formula schema $\Box_b\phi \supset \phi$ holds in every world w in which the relation R_b is reflexive, i.e., when $R_b(w, w)$, we have to seek for a world in the model of figure 1 such that $R_b(w, w)$ is not true. The only one is w_2. Now we have to find the formula ϕ, such that $M, w_2 \models \Box_b\phi$ but $M, w_2 \not\models \phi$.

Notice that the only world which are accessible via R_b to w_2 is w_1, and therefore we have the following equivalence:

$$M, w_2 \models \Box_b\phi \textit{ if and only if } M, w_1 \models \phi$$

So we have to search for a ϕ which is true in w_1 and falese in w_2.

Notice that, such a ϕ cannot be a propositional formula, w_1 and w_2 have the same assignment to propositional letters, and therefore they satisfies the same propositional formulas. This means that if ϕ is propositional and $w_2 \models_b \Box\phi$ then $w_1 \models \phi$ and $w_2 \models \phi$. So we have to search for a formula ϕ which contains at least a modal operator.

Consider for instance the formula $\Box_a(p \wedge q)$, we have that

$$\begin{aligned} M, w_1 &\models \Diamond_a(p \wedge q) \\ M, w_2 &\not\models \Diamond_a(p \wedge q) \\ M, w_1 &\models \Box_b(\Diamond_a(p \wedge q)) \end{aligned}$$

and therefore $w_2 \not\models \Box_b\Diamond_a(p \wedge q) \supset \Diamond_a(p \wedge q)$

3. *Write a formula that is satisfied* ***only*** *in world* w_1, w_2 *and* w_3

$$q$$

notice that the only worlds that satisfy q *are* w_1, w_2 *and* w_3.

Exercise 2 (5 marks)**.** *Suppose that* R *has the following property:*

$$\text{for all } w \in W \text{ there are at most two distinct worlds } w_1 \text{ and } w_2 \text{ such that } wRw_1 \text{ and } wRw_2 \tag{1}$$

Propose a schematic formula ϕ *that is valid in a frame* $\mathcal{F} = (W, R)$ *if and only if* R *satisfies (1) Explain why.*

Solution 2. *Intuitively we have to find a formula that imposes the following condition, written in first order logic:*

$$\forall w, w_1, w_2, w_3(R(w, w_1) \wedge R(w, w_2) \wedge R(w, w_3) \supset (w_1 = w_2 \vee w_1 = w_3 \vee w_2 = w_3)) \tag{2}$$

Suppose that A, B, *and* C, *are three formulas which are true in* w_1, w_2, *and* w_3 *respectively. The antecedent of the formula (2) could be represented with*

$$\Diamond A \wedge \Diamond B \wedge \Diamond C$$

The consequence of (2) states that two of the three worlds w_1, w_2 *and* w_3 *must coincide. Which implies that there should be a world in which* $A \wedge B$ *is true or* $A \wedge C$ *is true or* $B \wedge C$*. So the schema is*

$$\Diamond A \wedge \Diamond B \wedge \Diamond C \supset \Diamond(A \wedge B) \vee \Diamond(A \wedge C) \vee \Diamond(B \wedge C) \tag{3}$$

The discussion given above, cannot be considered as a formal proof, so we need to prove that (3) is sound and complete with respect to condition (1)

$F \models$**(3)** $\Longrightarrow$ R **satisfies (1)** *We actually prove that* R *does not satisfy (1) then* $F \not\models$*(3)*

Suppose that $F = (W, R)$ *is such that* $R(w, w_i)$ *for* $i = 1, 2, 3$, *and suppose that* w_1, w_2, w_3 *are distinct world. Let* p, q *and* r *three propositional letters, and let* M *be the model* (F, V) *with* $V(p) = \{w_1\}$, $V(q) = \{w_2\}$, *and* $V(r) = \{w_3\}$ *We have that* $M, w \models \Diamond p \wedge \Diamond q \wedge \Diamond r$ *but* $w \not\models \Diamond(p \wedge q)$, $w \not\models \Diamond(p \wedge r)$, *and* $w \not\models \Diamond(q \wedge r)$,

R **satisfies (1)** $\Longrightarrow$ $F \models$ **(3)** *Suppose that* $F = (W, R)$ *satisfies the property (1), to prove that* $F \models$ *(3), we have to show that for every model* $M = (F, V)$, *and for every world in* $w \in W$

$$M, w \models (3)$$

therefore, we proceed as follows:

$$
\begin{aligned}
M, w \models \Diamond A \wedge \Diamond B \wedge \Diamond C &\Longrightarrow M, w \models \Diamond A,\ M, w \models \Diamond B \text{ and } M, w \models \Diamond C \\
&\Longrightarrow \text{There are } w_1, w_2, w_3 \text{ with } wRw_1,\ wRw_2 \text{ and } wRw_3 \\
&\ M, w_1 \models A,\ M, w_2 \models B \text{ and } M, w_3 \models C \\
&\ \text{Since } F \models (1), \text{ either } w_1 = w_2 \text{ or } w_1 = w_3 \text{ or } w_2 = w_3 \\
&\Longrightarrow M, (w_1 = w_2) \models A \wedge B \text{ or } M, (w_1 = w_3) \models A \wedge C \text{ or} \\
&\ M, (w_2 = w_3) \models B \wedge C \\
&\Longrightarrow M, w \models \Diamond(A \wedge B) \text{ or } M, w \models \Diamond(A \wedge C) \text{ or } M, w \models \Diamond(B \wedge C) \\
&\Longrightarrow M, w \models \Diamond(A \wedge B) \vee \Diamond(A \wedge C) \vee \Diamond(B \wedge C)
\end{aligned}
$$

Exercise 3 (3 marks)**.** *Prove via tableaux that the following formulas are or are not valid in the class of all frames.*

Solution 3.

- $\Diamond A \vee \Box\neg A$ *To show that this formula is* valid *we search for a counter-model via the tableaux method. If we managed to find it the formula will not be valid, if we don't find it then the formula is valid.*

$$
\begin{array}{c}
w \not\models \Diamond A \vee \Box\neg A \\
| \\
w \not\models \Diamond A \\
| \\
w \not\models \Box\neg A \\
| \\
wRw' \\
| \\
w' \not\models \neg A \\
| \\
w' \models A \\
| \\
w' \not\models A
\end{array}
$$

CLOSED

Since the tableaux is closed, i.e., all the branches (actually there is only one branch) of the tableaux are closed, then there is no counter-model for the initial formula. This implies that the formula is valid.

- $\Box A \supset (\Diamond B \supset \Diamond(A \wedge B))$ *Same as before. We prove that the formula is valid by building* *the tableaux, and observing that it is closed*

$$w \not\models \Box A \supset (\Diamond B \supset \Diamond(A \wedge B))$$
$$w \models \Box A$$
$$w \not\models \Diamond B \supset \Diamond(A \wedge B)$$
$$w \models \Diamond B$$
$$w \not\models \Diamond(A \wedge B)$$
$$wRw' \models \Diamond B$$
$$w' \models B$$
$$w' \models A$$
$$w' \not\models A \wedge B$$
$$w' \not\models A \qquad\qquad w' \not\models B$$

CLOSED **CLOSED**

- $\Diamond(A \vee B) \equiv (\Diamond A \vee \Diamond B)$ *Same as before. We prove that the formula is valid we have to show that the two directions of the implications are valid i.e. that the following two formulas are valid*

$$\Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)$$
$$(\Diamond A \vee \Diamond B) \supset \Diamond(A \vee B)$$

$$w \not\models \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)$$
$$w \models \Diamond(A \vee B)$$
$$w \not\models \Diamond A \vee \Diamond B$$
$$w \not\models \Diamond A$$
$$w \not\models \Diamond B$$
$$wRw'$$
$$w' \models A \vee B$$
$$w' \models A \qquad\qquad w' \models B$$
$$w' \not\models A \qquad\qquad w' \not\models B$$

CLOSED **CLOSED**

$$w \not\models (\Diamond A \vee \Diamond B) \supset \Diamond(A \vee B)$$
$$w \models \Diamond A \vee \Diamond B$$
$$w \not\models \Diamond(A \vee B)$$
$$w \models \Diamond A \qquad\qquad w \models \Diamond B$$
$$wRw' \qquad\qquad wRw''$$
$$w' \models A \qquad\qquad w'' \models B$$
$$w' \models A \qquad\qquad w'' \models B$$
$$w' \not\models A \vee B \qquad\qquad w'' \not\models A \vee B$$
$$w' \not\models A \qquad\qquad w'' \not\models B$$

CLOSED **CLOSED**

Exercise 4 (2 marks)**.** *Compute the standard translation in first order logic of the formula*

$$\Box\Box P \wedge \Box\Diamond Q \supset \neg\Diamond(P \wedge \Box Q)$$

Solution 4.

$\Box\Box P\ \wedge$	$\forall xy(R(x,y) \supset \forall z(R(y,z) \supset P(z)))\ \wedge$
$\Box\Diamond Q\ \supset$	$\forall y(R(x,y) \supset \exists z(R(y,z) \wedge Q(z))))\ \supset$
$\neg\Diamond(P\ \wedge \Box Q)$	$\neg\exists y(R(x,y) \wedge (P(y) \wedge \forall z(R(y,z) \supset Q(z))))$

Exercise 5 (4 marks)**.** *Consider the following axioms schmata*

$$\begin{array}{ll} \mathbf{4}. & \Box A \supset \Box\Box A \\ \mathbf{T}. & \Box A \supset A \\ \mathbf{5}. & \Diamond A \supset \Box\Diamond A \end{array}$$

Show that if $F \models \mathbf{5}.$ *and* $F \models \mathbf{T}.$ *then* $F \models \mathbf{4}.$

Solution 5. *First notice that this is different from showing that the formula*

$$(\Box A \supset A) \wedge (\Diamond A \supset \Box\Diamond A) \supset (\Box A \supset \Box\Box A) \tag{4}$$

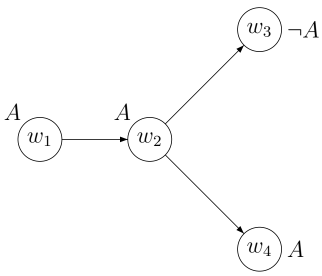
is valid. Indeed (4) is valid if and only if that

$$\text{For all } M = (F, V), \text{ and for all } w \in W,\ M, w \models (\Box A \supset A) \wedge (\Diamond A \supset \Box\Diamond A) \supset (\Box A \supset \Box\Box A) \tag{5}$$

while the exercise asks to show that

$$\begin{array}{l} \textbf{If} \text{ for all } M = (F, V), \text{ and for all } w \in W,\ M, w \models (\Box A \supset A) \\ \quad \text{and for all } M = (F, V), \text{ and for all } w \in W,\ M, w \models \Diamond A \supset \Box\Diamond A) \\ \textbf{then} \text{ for all } M = (F, V), \text{ and for all } w \in W,\ M, w \models (\Box A \supset \Box\Box A) \end{array} \tag{6}$$

Notice the difference between the quantification on models in statement (5) (of the form $\forall x(P(x) \wedge Q(x) \supset R(x)))$ *and statement (6) (of the form* $\forall x(P(x)) \wedge \forall x Q(x) \supset \forall x R(x)))$ *By the way notice that the formula (5) is not valid consider the following frame:*



*There are two ways to prove property (6) either by providing a Hilbert style deduction of **4.** from **T.** and **5.**, or semantically, by considering the property of the accessibility relation which is axiomatized by the three axiom schemata.*

Hilbert deduction *We have to prove the fact via Hilbert calculus. I.e. we have to prove that **4.** can be inferred using the Hilbert calculus for modal logic **K.** with the additional axioms **T.** and **5.**. Namely we have to prove that*

$$\vdash_{\mathbf{S5}} \Box\phi \supset \Box\Box\phi$$

This is quite complex. You can see a solution at the following web site

`http://www.logic.at/lvas/185249/EX-28.pdf`

Semantically *We know that, for every frame* $F = (W, R)$

$$F \models \Box A \supset A \iff R \text{ is reflexive}$$
$$F \models \Diamond A \supset \Box\Diamond A \iff R \text{ is Euclidean}$$

So proving (6) can be reduced to prove that

$$\text{if } R \text{ is reflexive and Euclidean then } R \text{ is transitive} \tag{7}$$

The following is a proof of (7)

Suppose that vRw and that wRu. By reflexivity we have that vRv, by Eulerianity we have that vRw and vRv, implies that wRv. Again by Eulerianity, we have that wRv and wRu, implies that vRu.

Exercise 6 (3 marks)**.** *Show that, in the frame $F = (W, R)$ if R is an equivalence relation, then $\Diamond\Diamond\phi \equiv \Diamond\phi$ is valid in F.*

Solution 6. *To show that a formula ϕ is* valid *in a frame F, i.e., that $F \models \phi$, we have to show that for every model $M = (F, V)$ based on F and for every world $w \in W$, $M, w \models \phi$.*

So we have to show that if $F = (W, R)$ and R is an equivalent relation (i.e., it is reflexive, symmetric and transitive), then for every model $M = (F, V)$ and for every world $w \in W$

$$M, w \models \Diamond\Diamond\phi \equiv \Diamond\phi$$

i.e., that $M, w \models \Diamond\Diamond\phi \supset \Diamond\phi$ and $M, w \models \Diamond\phi \supset \Diamond\Diamond\phi$

$$\begin{aligned} M, w \models \Diamond\Diamond\phi &\implies \text{there are } v \text{ and } u \text{ with } wRv \text{ and } vRu\text{, and } M, u \models \phi \\ &\implies \text{By transitivity } wRu \text{ and } M, u \models \phi \\ &\implies M, w \models \Diamond\phi \end{aligned}$$

$$\begin{aligned} M, w \models \Diamond\phi &\implies \text{By reflexivity } wRw \text{ and } M, w \models \phi \\ &\implies M, w \models \Diamond\Diamond\phi \end{aligned}$$

Mathematical logic
Final exam

Laurea Specialistica in Informatica
Universitá degli Studi di Trento

Prof. Luciano Serafini

February 6, 2009

Exercise 1 (Propositional Logic: 5 marks). *Formalize the following problems in propositional logic and solve the riddles using some form of inference, either ND, or DP, or Resolution*

Lets hear Alceo, Safo and Catulo

- *Alceo says: "The only ones who speak the truth here are Catulo and I"*
- *Safo states: "Catulo is a lier"*
- *Catulo replies: "Safo speaks the truth, or it is Alceo who lies"*

Assuming that the person who lies always lies and that the person who speaks the truth is always truthful, who is sincere? Who lies?

Exercise 2 (Propositional Logic: 5 marks). *Let* Γ *and* Σ *be two maximally consistent sets. Show that either* $\Gamma \cup \Sigma = \Gamma = \Sigma$ *or* $\Gamma \cup \Sigma \vdash \bot$

Exercise 3 (First Order Logic: 4 marks). *Is the following inference rule sound?*

$$\frac{\forall xy(A(x,y) \supset \neg\exists z B(x,z)) \qquad A(a,a) \wedge B(b,b)}{a \neq b}$$

Explain why.

Exercise 4 (First Order Logic: 6 marks)**.** *A tree is a structure $T = (N, \prec)$, where N is a non empty set, $n_1 \prec n_2$ means that the node n_1 is the parent note of n_2, and the following properties hold:*

1. *there is a unique element $n_0 \in N$, called the* root of T *which does not have any parent node.*
2. *every node of T different from the root has a unique parent.*

Provide a first order language for representing tree structures and use it to formalizes the above two properties. With the same language formalize also the following properties

1. *the degree of the tree is 2, i.e. every node has at most 2 children*
2. *the maximal depth of the tree is 3, i.e. there is no branch of T with more than 3 nodes*
3. *T is binary tree. i.e., every node is either a leaf (and does not have any children) or it has exactly two children.*

Exercise 5 (Modal logics: 4 marks)**.** *A frame (W, R) is an S4 frame if and only if R is a reflexive and transitive relation. for each of the following formula check if it is valid in an S4 frame. If it is not valide provide a countermodel*

1. $\Box A \supset \Diamond A$
2. $A \supset \Diamond A$
3. $\Box A \wedge \Box\Box B \supset \Box\Box(A \wedge B)$

Exercise 6 (Modal logics: 6 marks)**.** *Show that in the frame $F = (W, R)$ if R is function (i.e., if forall w exists* only one *w' such that wRw') $\Diamond\phi \equiv \Box\phi$ is valid.*

Mathematical logic
Final exam

Laurea Specialistica in Informatica
Universitá degli Studi di Trento

Prof. Luciano Serafini

July 23, 2009

Exercise 1 (6 marks). *Consider the following conditional code, which returns a boolean value.*

```
f(bool a,b,c)
if (a || b && c)
   if (a && c)
   return b
   else return b
else if (a || b)
   return a;
else
   return true
```

`&&` *stands for* $\wedge$*;* `||` *stands for* $\vee$*;* `!` *stands for* $\neg$*;* `true` *stands for* $\top$*; and* `false` *stands for* $\bot$*;*

Simplify the code in a formula ϕ *with propositional variables* `a`*,* `b` *and* `c` *such that* ϕ *is equivalent to the above program. I.e.* ϕ *is true for all the truth assignments to* `a`*,* `b` *and* `c`*, for which the program returns* `true`*, and* ϕ *is false for all the truth assignments to* `a, b` *and* `c` *for which the program returns* `false`*.*

Exercise 2 (4 marks). *Prove by natural deduction at least one of the following formulas*

1. $(A \supset B) \supset (\neg B \supset \neg A)$
2. $((\neg A \vee B) \supset C) \vee ((\neg B \supset A) \supset C)$
3. $(A \supset B) \supset \neg(A \wedge \neg B)$

Exercise 3 (4 marks). *Formulate the requirements below as sentences of first order logic and show that the two of them cannot be true together in any interpretation. (This is the barber's paradox by Bertrand Russell)*

1. *Anyone who does not shave himself must be shaved by Figaro (The Barber of Seville)*

2. *Whomever the barber shaves, must not shave himself.*

Then show by means of resolution that the two sentences are unsatisfiable

Exercise 4 (2 marks). *Write a first order formula which is true in all the interpretations whose domain contains exactly 3 elements.*

Exercise 5 (4 marks). *Let P be the only binary predicate (predicate on arity 2) of a first order language. Suppose that we consider only the interpretations of the previous exercise (i.e., the interpretations whose domain contains exactly 3 elements). Propose a propositional language, and show a way to transform the following FOL formulas in such a language*

- $\forall xy P(x,y)$
- $\exists xy P(x,y)$
- $\forall x \exists y (P(x,y))$
- $\exists x \forall y (P(x,y))$

Exercise 6 (5 marks). *Let $\mathcal{F} = (W, R_1, R_2)$ be a frame. Prove that $R_1 = R_2^{-1}$ if and only if*

1. $\mathcal{F} \models A \supset \Box_1 \Diamond_2 A$ *and*

2. $\mathcal{F} \models A \supset \Box_2 \Diamond_1 A$

$(R^{-1} = \{(w,v) | (v,w) \in R\}$

Exercise 7 (5 marks). *For each of the following formulas either show that it is valid (proving via tableaux) or provide a countermodel*

1. $\Box A \supset A$

2. $(\Box A \supset \Box B) \supset (\Diamond B \supset \Diamond A)$

3. $\Box(A \wedge \Diamond B) \supset (\Box \bot \vee \Diamond(\neg A) \vee \Diamond\Diamond B))$

Mathematical logic
Final exam

Laurea Specialistica in Informatica
Universitá degli Studi di Trento

Prof. Luciano Serafini

September 10, 2009

Exercise 1 (Propositional Logic: 6 marks)**.** *Formalize the following problems in propositional logic and solve the riddles using some form of inference, either ND, or DP, or Resolution*

Lets hear Alceo, Safo and Catulo

- *Alceo says: "The only ones who speak the truth here are Safo and I"*
- *Catulo replies: "Safo is a lier"*
- *Safo states: "Catulo speaks the truth, or it is Alceo who lies"*

Assuming that the person who lies always lies and that the person who speaks the truth is always truthful, who is sincere? Who lies?

Exercise 2 (Propositional Logic: 5 marks)**.** *Let* Γ *and* Σ *be two maximally consistent sets. Show that* $\Gamma \models \Sigma$ *implies that* $\Gamma = \Sigma$.

Exercise 3 (First Order Logic: 5 marks)**.** *Is the following inference rule sound?*

$$\frac{\forall xy(x = y) \quad P(a)}{\forall x P(x)}$$

Explain why.

Exercise 4 (First Order Logic: 4 marks)**.** *A partially ordered set (poset) is a set P with a binary relation $\leq$ over a set P which is reflexive, antisymmetric, and transitive. Writhe the first order formulas corresponding to the three properties, and show that the following formulas are logical consequences of them.*

Exercise 5 (Modal logics: 5 marks)**.** *A frame (W, R) is an S4 frame if and only if R is a reflexive and transitive relation. for each of the following formula check if it is valid in an S4 frame. If it is not valide provide a countermodel*

1. $\Box A \supset \Diamond A$
2. $A \supset \Diamond A$
3. $\Box A \wedge \Box\Box B \supset \Box\Box(A \wedge B)$

Exercise 6 (Modal logics: 5 marks)**.** *Show that in the frame $F = (W, R)$ if R is function (i.e., if forall w exists* only one w' *such that wRw') $\Diamond\phi \equiv \Box\phi$ is valid.*

Mathematical logic

– 1^{st} assessment – Propositional Logic –

26 March 2013

Exercise 1. Explain, in natural language and with the usage of the appropriate definitions and examples if you need, the difference between the following statements

1. $\models A \vee B$
2. $\models A$ or $\models B$

Solution. $\models A \vee B$ means that for every interpretation m, either $m \models A$ or $m \models B$

$\models A$ or $\models B$ means that either for every interpretation m, $m \models A$ or for every interpretation m, $m \models B$.

To highlight the difference between 1. and 2. you can write their definition by using a more formal notation,

$$\models A \vee B \quad \Longleftrightarrow \quad \forall m, (m \models A \text{ or } m \models B) \tag{1}$$

$$\models A \text{ or } \models B \quad \Longleftrightarrow \quad (\forall m, m \models A) \text{ or } (\forall m, m \models B) \tag{2}$$

$$\tag{3}$$

An example that shows the difference can be constructed by taking A equal to the atomic formula p and B the negated atomic formula $\neg p$. You have that $\models p \vee \neg p$, but neither $\models p$ nor $\models \neg p$

Exercise 2. Brown, Jones, and Smith are suspected of a crime. They testify as follows:

- **Brown**: *"Jones is guilty and Smith is innocent"*.
- **Jones**: *"If Brown is guilty then so is Smith"*.
- **Smith**: *"I'm innocent, but at least one of the others is guilty"*.

Let B, J, and S be the statements "*Brown is guilty*", "*Jones is guilty*", and "*Smith is guilty*", respectively. Do the following:

1. Express the testimony of each suspect as a propositional formula.
2. Write a truth table for the three testimonies.

3. Use the above truth table to answer the following questions:
   (a) Are the three testimonies satisfiable?
   (b) The testimony of one of the suspects follows from that of another. Which from which?
   (c) Assuming that everybody is innocent, who committed perjury?
   (d) Assuming that all testimonies are true, who is innocent and who is guilty?
   (e) Assuming that the innocent told the truth and the guilty told lies, who is innocent and who is guilty?

Solution.

1. The three statements can be expressed as $J \wedge \neg S$, $B \supset S$, and $\neg S \wedge (B \vee J)$.

2.

	B	J	S	$J \wedge \neg S$	$B \supset S$	$\neg S \wedge (B \vee J)$
(1)	T	T	T	F	T	F
(2)	T	T	F	T	F	T
(3)	T	F	T	F	T	F
(4)	T	F	F	F	F	T
(5)	F	T	T	F	T	F
(6)	F	T	F	T	T	T
(7)	F	F	T	F	T	F
(8)	F	F	F	F	T	F

3. (a) Yes, assigment (6) makes them all true
   (b) $J \wedge \neg S \models \neg S \wedge (B \vee J)$
   (c) Everybody is innocent corresponds to assignment (8), and in this case the statements of Brown and Smith are false.
   (d) Assuming that all testimonies are true corresponds to assignment (6). In this case Jones is guilty and the others are innocents.
   (e) We have to search for an assignment such that if B (resp. J and S) is false then the sentence of B (resp. J and S) is true and that if B (resp. J and S) is true, then the sentence of B (resp. J

and S) is false. The only assignment satisfying this restriction is assignment (3) in which Jones is innocent and Brown and Smith are guilty.

Exercise 3. Prove the soundness of the $\wedge I$ rule of Natural Deduction.

$$\frac{\phi \quad \psi}{\phi \wedge \psi} \wedge I$$

That is, prove that $\Gamma \vdash_{ND} \phi \wedge \psi$ implies $\Gamma \models \phi \wedge \psi$ in the case that the last rule used in the deduction is a $\wedge I$ rule and assuming that $\Gamma \vdash_{ND} \alpha$ implies $\Gamma \models \alpha$ is true for all the sub-deductions (sub-trees) of $\Gamma \vdash_{ND} \phi \wedge \psi$ (inductive hypothesis).

Hint: Use a strategy of proof similar to that of the step case of the soundness proof for the Hilbert axiomatization.

Solution. Assume that $\Gamma \vdash_{ND} \phi \wedge \psi$ and the last rule used is $\wedge I$, then from the shape of the rule we know that there are two deductions of ϕ and ψ from two sets Γ_1 and Γ_2 with $\Gamma_1 \subseteq \Gamma$ and $\Gamma_2 \subseteq \Gamma$. In symbols this corresponds to

$$\Gamma_1 \vdash_{ND} \phi \tag{4}$$
$$\Gamma_2 \vdash_{ND} \psi \tag{5}$$

From the inductive hypothesis, (4) and (5) imply that

$$\Gamma_1 \models \phi \tag{6}$$
$$\Gamma_2 \models \psi \tag{7}$$

and because of the monotonicity of logical consequence in propositional logic we have that

$$\Gamma \models \phi \tag{8}$$
$$\Gamma \models \psi \tag{9}$$

Now we can prove that $\Gamma \models \phi \wedge \psi$. In fact, let $\mathcal{I}$ be an interpretation that satisfies Γ ($\mathcal{I} \models \Gamma$). From (8) and (9) we know that $\mathcal{I}$ satisfies both ϕ and ψ ($\mathcal{I} \models \phi$ and $\mathcal{I} \models \psi$). Therefore, from the definition of satisfiability of $\wedge$ we have that $\mathcal{I}$ satisfies $\phi \wedge \psi$ ($\mathcal{I} \models \phi \wedge \psi$).

Exercise 4. Show that if $A, B \models C$ and $A, \neg B \models C$, then $A \models C$.

Solution. We apply the definition of logical consequence, i.e. $\Gamma \models \phi$ if for every interpretation $\mathcal{I}$, $\mathcal{I} \models \Gamma$ implies that $\mathcal{I} \models \phi$.

To prove that $A \models C$, let $\mathcal{I}$ be any interpretation with $\mathcal{I} \models A$ Since, for every formula B, either $\mathcal{I} \models B$ or $\mathcal{I} \models \neg B$, we consider the two cases:

If $\mathcal{I} \models B$ then $\mathcal{I} \models \{A, B\}$ and by the hypothesis that $A, B \models C$, we have that $\mathcal{I} \models C$;

If $\mathcal{I} \models \neg B$ then $\mathcal{I} \models \{A, \neg B\}$ and by the hypothesis that $A, \neg B \models C$, we have that $\mathcal{I} \models C$.

Since in both cases $\mathcal{I} \models C$, we can conclude that $A \models C$.

Exercise 5. Translate the following natural language sentences into propositional logic formulas and say whether the obtained formulas are satisfiable, valid or unsatisfiable.

1. Alice comes to the party given that Bob doesn't come, but, if Bob comes, then Carl doesn't come;
2. If it is not the case that when Alice comes to the party also Bob comes, then Alice comes and Bob does not;
3. If Bob comes to the party also Alice comes, but actually Alice does not come to the party and Bob does;

Solution.

1. $(A \supset \neg B) \wedge (B \supset \neg C)$ is Satisfiable
2. $\neg(A \supset B) \supset (A \wedge \neg B)$ is Valid
3. $(B \supset A) \wedge (\neg A \wedge B)$ is Unsatisfiable

Exercise 6. Provide an example of two sets of formulas Γ and Σ which are consistent, and such that $\Gamma \cup \Sigma$ is not consistent. Then show that, for every pair of consistent sets of formulas Γ, Σ, if $\Gamma \cup \Sigma$ is inconsistent, then there is a formula ϕ such that $\Gamma \models \phi$ and $\Sigma \models \neg\phi$.

Solution. If $\Gamma = \{p\}$ and $\Sigma = \{\neg p\}$, then Γ and Σ are separately consistent, but $\Gamma \cup \Sigma = \{p, \neg p\}$ is not consistent.

If $\Gamma \cup \Sigma$ is inconsistent then $\Gamma \cup \Sigma \vdash \bot$. This means that there is a deduction of $\bot$ from a finite subset $\Gamma_0 \cup \Sigma_0$ of $\Gamma \cup \Sigma$. We suppose, w.l.o.g.

that $\Gamma_0 \subseteq \Gamma$ and $\Sigma_0 \subseteq \Sigma$. Consider the formula $\sigma_1 \wedge \cdots \wedge \sigma_n$, obtained by making a conjunction with all the formulas in $\Sigma_0 = \{\sigma_1, \ldots, \sigma_n\}$. From the fact that $\Gamma_0 \cup \Sigma_0 \vdash \bot$ we can infer that $\Gamma_0, \sigma_1 \wedge \cdots \wedge \sigma_n \vdash \bot$ and therefore that $\Gamma_0, \vdash \neg(\sigma_1 \wedge \cdots \wedge \sigma_n)$. The fact that $\Gamma_0 \subseteq \Gamma$, implies that $\Gamma \vdash \neg(\sigma_1 \wedge \cdots \wedge \sigma_n)$. On the other hand we have that $\Sigma \vdash (\sigma_1 \wedge \cdots \wedge \sigma_n)$. So the formula A we are looking for is indeed $(\sigma_1 \wedge \cdots \wedge \sigma_n)$

Exercise 7. Prove by means of natural deduction at least one of the following formulas

1. $(A \wedge B) \wedge C \models A \wedge (B \wedge C)$
2. $\models \neg(A \wedge \neg A)$
3. $\models (\neg A \vee \neg B) \supset \neg(A \wedge B)$

Solution.

1. $(A \wedge B) \wedge C \models A \wedge (B \wedge C)$

$$\dfrac{\dfrac{\dfrac{(A \wedge B) \wedge C}{A \wedge B}\,\wedge E}{A}\,\wedge E \qquad \dfrac{\dfrac{\dfrac{(A \wedge B) \wedge C}{A \wedge B}\,\wedge E}{B}\,\wedge E \quad \dfrac{(A \wedge B) \wedge C}{C}\,\wedge E}{B \wedge C}\,\wedge I}{A \wedge (B \wedge C)}\,\wedge I$$

2. $\neg(A \wedge \neg A)$

$$\dfrac{\dfrac{\dfrac{A \wedge \neg A^1}{A}\,\wedge E \quad \dfrac{A \wedge \neg A^1}{\neg A}\,\wedge E}{\bot}\,\supset E}{\neg(A \wedge \neg A)}\,\bot c_{(1)}$$

3. $(\neg A \vee \neg B) \supset \neg(A \wedge B)$

$$\dfrac{\dfrac{\dfrac{\neg A \vee \neg B^4 \qquad \dfrac{\neg A^1 \quad \dfrac{A \wedge B^3}{A}\,\wedge E}{\bot}\,\supset E \qquad \dfrac{\neg B^2 \quad \dfrac{A \wedge B^3}{B}\,\wedge E}{\bot}\,\supset E}{\bot}\,\vee E_{(1,2)}}{\neg(A \wedge B)}\,\supset I_{(3)}}{(\neg A \vee \neg B) \supset \neg(A \wedge B)}\,\supset I_{(4)}$$

Exercise 8. Check if the following formula

$$\phi = (((r \to r) \to q) \to ((r \to r) \land \neg p \land q)) \lor (p \land q)$$

is valid using DPLL.

Solution. To check if ϕ is valid we can check if $\neg\phi$ is (un)satisfiable using DPLL. As a first step we have to translate $\neg\phi$ in CNF, obtaining the formula:

$$\psi = CNF(\neg\phi) = (r \lor q) \land (\neg r \lor q) \land (r \lor p \lor \neg q) \land (\neg r \lor p \lor \neg q) \land (\neg p \lor \neg q)$$

On this formula we can apply the DPLL algorithm:

1. let $\mathcal{I} = \emptyset$
2. ψ does not contain unit clauses, so Unit Propagation is not applied
3. select the literal $p \in \psi$
4. $\mathcal{I} := \mathcal{I} \cup \{p\} = \{p\}$
5. $\psi := \psi|_p = (r \lor q) \land (\neg r \lor q) \land (\neg q)$
6. $DPLL(\psi, \mathcal{I})$
   (a) ψ contains the unit clause $(\neg q)$ and therefore we apply unit propagation
   (b) $\phi := \phi|_{\neg q} = (r) \land (\neg r)$
   (c) ψ contains the unit clause (r) and therefore we apply unit propagation
   (d) $\phi := \phi|_{\neg q} = ()$
   (e) ϕ contains the empty clause and therefore stops
7. $\mathcal{I} := \mathcal{I} \cup \{\neg p\} = \{\neg p\}$
8. $\psi := \psi|_{\neg p} = (r \lor q) \land (\neg r \lor q) \land (r \lor \neg q) \land (\neg r \lor \neg q)$
9. $DPLL(\psi, \mathcal{I})$
   (a) ψ does not contain unit clauses, so Unit Propagation is not applied
   (b) select the literal $q \in \psi$

(c) $\mathcal{I} := \mathcal{I} \cup \{q\} = \{\neg p, q\}$

(d) $\psi := \psi|_q = (r) \wedge (\neg r)$

(e) $DPLL(\psi, \mathcal{I})$

i. ϕ contains the unit clause (r) and therefore we apply unit propagation

ii. $\phi = \phi|_r = ()$

iii. ϕ contains the empty clause and therefore stops

(f) $\mathcal{I} := \mathcal{I} \cup \{\neg q\} = \{\neg p, \neg q\}$

(g) $\psi := \psi|_{\neg q} = (r) \wedge (\neg r)$

(h) $DPLL(\psi, \mathcal{I})$

i. ϕ contains the unit clause (r) and therefore we apply unit propagation

ii. $\phi = \phi|_r = ()$

iii. ϕ contains the empty clause and therefore stops

10. DPLL exits without returning an assignment, which implies that ψ is not satisfiable, and therefore that ϕ **is valid**

Mathematical logic

– 2^{st} assessment – First Order Logic – 7 May 2013 –

Exercise 1. [4 points] Show that if an interpretation I satisfies the formula

$$\forall x_0, \forall x_1, \ldots, \forall x_n \left(\bigvee_{0 \leq i \neq j \leq n} x_i = x_j \right)$$

then the domain of $\mathcal{I}$ contains at most n elements.

Solution. Let $\mathcal{I} = \langle \Delta, \mathcal{I} \rangle$ be an interpretation that satisfies the formula

$$\forall x_0, \forall x_1, \ldots, \forall x_n \left(\bigvee_{0 \leq i \neq j \leq n} x_i = x_j \right)$$

and assume that Δ contains $n+1$ distinct elements $d_0, d_1, \ldots, d_n$. For the sake of simplicity we use here $=$ and $\neq$ to denote both the equality (inequality) predicate in the language and the equality relation $\mathcal{I}(=)$ in the interpretation $\mathcal{I}$.

Let a be an arbitrary assignment to the variables $x_0, x_1, \ldots, x_n$. Since the formula is closed, its satisfiability does not depend upon the assignment. Thus

$$\mathcal{I} \models \forall x_0, x_1, \ldots x_n \left(\bigvee_{0 \leq i \neq j \leq n} x_i = x_j \right) [a]$$

and from the definition of satisfiability of universally quantified formulae we know that for all $n+1$ elements (not necessarily distinct) $d_i, d_j, \ldots d_k \in \Delta$

$$\mathcal{I} \models \left(\bigvee_{0 \leq i \neq j \leq n} x_i = x_j \right) [a[x_0/d_i, x_1/d_j, \ldots x_n/d_k]]$$

Since this holds for all the tuples of $n+1$ elements this must hold also for the tuple of $n+1$ distinct elements $d_0, d_2, \ldots, d_n$. Therefore we must have

$$\mathcal{I} \models \left(\bigvee_{0 \leq i \neq j \leq n} x_i = x_j \right) [a[x_0/d_0, x_1/d_1, \ldots x_n/d_n]]$$

which means that there are two elements d_i, d_j among the elements $d_0, d_1, \ldots d_n$ such that

$$d_i = d_j$$

But this is impossible as we have assumed that $d_0, d_1, \ldots, d_n$ are distinct. Therefore the assumption that Δ contains $n+1$ distinct elements $d_0, d_1, \ldots, d_n$ cannot be, and we have proven that Δ contains at most n elements.

Exercise 2. [3 points] For each of the formulae below provide an interpretation $\mathcal{I}$ and an assignment a that satisfy it:

1. $\forall x.(sum(x, c) = x)$
2. $Person(x) \supset \neg Dog(x)$
3. $\forall x.(Employee(x) \supset \exists y.Manager(x, y))$

Solution. Possible solutions are as follows:

1. I(sum) corresponds to the function SUM: x, y $\rightarrow$ x + y. c is a constant in the domain of natural numbers such that I(c) = 0. As we have only the variable x bound, there is no need to define any assignment
2. I(Person) = {Paul, Mary}, I(Dog) = {Bobby}. We can take for instance a(x) = Paul.
3. I(Employee) = {Paul, Mary}, I(Manager) = {(Paul, Mary), (Mary, Mary)}. As both x and y are bound, there is no need to define any assignment.

Exercise 3. [4 points] Let L be a first order language used to describe a domain containing humans and vehicles by means of the following predicates:

$$H(x) : x \text{ is a human}$$
$$C(x) : x \text{ is a car}$$
$$T(x) : x \text{ is a truck}$$
$$D(x, y) : x \text{ drives } y$$

Use L to write first order formulae that represent the usual (obvious, common sense) assumptions on humans and vehicles:

1. no human is a car,
2. no car is a truck,
3. there exist at least a human person,

4. there exist at least a car,

5. only humans drive,

6. only cars and trucks are driven.

In addition, write formulas representing the following statements:

7. Everybody (human) drives a car or a truck.

8. Some people drive both.

9. Some people don't drive either

10. Nobody drives both

11. Every car has at most one driver

12. Everybody drives exactly one vehicle (car or truck)

Solution.

1. $\forall x.(H(x) \supset \neg C(x))$
2. $\forall x.(C(x) \supset \neg T(x))$
3. $\exists x.H(x)$
4. $\exists x.C(x)$
5. $\forall x.(\exists y.D(x,y) \supset H(x))$
6. $\forall x.(\exists y.D(y,x) \supset C(x) \vee T(x))$
7. $\forall x.(H(x) \supset \exists y.(D(x,y) \wedge (C(y) \vee T(y)))$
8. $\exists xyz.(D(x,y) \wedge C(y) \wedge D(x,z) \wedge T(z))$
9. $\exists x \forall y.\neg D(x,y)$
10. $\forall xyz.(D(x,y) \wedge D(x,z) \supset \neg(C(y) \wedge T(z)))$
11. $\forall xyz.(C(z) \wedge D(x,z) \wedge D(y,z) \supset x = y)$
12. $\forall x.\exists y(D(x,y) \wedge \forall z.(D(x,z) \supset y = z))$

Exercise 4. [**5 points**] Prove the soundness of the $\exists I$ rule of Natural Deduction:

$$\frac{\phi(t)}{\exists x.\phi(x)}\ \exists I$$

Solution. Assume that $\Gamma \vdash_{ND} \exists x.\phi(x)$ and the last rule used is $\exists I$, then from the shape of the rule we know that there is a deduction of $\phi(t)$ from Γ. In symbols this corresponds to

$$\Gamma \vdash_{ND} \phi(t) \tag{1}$$

Since the deduction of $\phi(t)$ from Γ is shorter than the one of $\exists x.\phi(x)$ from Γ we can use the inductive hypothesis and conclude that (1) implies that

$$\Gamma \models \phi(t) \tag{2}$$

Now we can prove that $\Gamma \models \exists x.\phi(x)$. In fact, let $\mathcal{I}$, be an interpretation and a be an assignment such that $\mathcal{I} \models \Gamma[a]$. From (2) we know that $\mathcal{I} \models \phi(t)[a]$. Therefore, taken d as the element in the domain that correspond to the interpretation of the term t under the assignment a, that is $d = \mathcal{I}(t)[a]$ from the definition of satisfiability of $\exists$ we have that $\mathcal{I} \models \phi(x)[a[x/d]]$. Therefore there is a $d \in Delta$ such that $\mathcal{I} \models \phi(x)[a[x/d]]$, but this is exactly the definition of $\mathcal{I} \models \exists x.\phi(x)[a]$. Thus, we have proved that for any $\mathcal{I}$ and a such that $\mathcal{I} \models \Gamma[a]$, then $\mathcal{I} \models \exists x.\phi(x)[a]$ and this corresponds to prove that $\Gamma \models \exists x.\phi(x)$.

Exercise 5. [**6 points**] For each of the following formulas either prove its validity via natural deduction or provide a counter-model if it is satisfiable but not valid.

1. $\forall x \exists y.Q(x,y) \supset \exists x \forall y.Q(x,y)$
2. $\neg\neg\forall x.P(x) \supset \forall x.\neg\neg P(x)$

Solution. Formula 1. is satisfiable but not valid. Formula 2. is valid.

A counter-model for formula 1. is the following.

Let us define an interpretation $\mathcal{I}$ over the domain $\Delta = \{1,2\}$ such that $\mathcal{I}(Q) = \{\langle 1,1\rangle, \langle 2,1\rangle\}$. Thus we can easily see that for each value assigned to x by and assignment a (among 1, and 2) there is a value assigned to y (the value 2) which makes $Q(x,y)$ true, but there is no value of x such that $\forall y.Q(x,y)$ can become true.

The ND proof of formula 2. is the following

$$\cfrac{\cfrac{\cfrac{\cfrac{\neg\neg\forall x.P(x)^3 \qquad \cfrac{\cfrac{\neg P(x)^2 \qquad \cfrac{\forall x.P(x)^1}{P(x)}\,\forall E}{\bot}\,\to E}{\neg\forall x.P(x)}\,\bot_c^1}{\bot}\,\to E}{\neg\neg P(x)}\,\bot_c^2}{\forall x.\neg\neg P(x)}\,\forall I}{\neg\neg\forall x.P(x) \to \forall x.\neg\neg P(x)}\,\to I^3$$

Exercise 6. [4 points]Consider a database containing the following tables:

EMPLOYEE			
NAME	**GENDER**	**CITY**	**SALARY**
Mary	Female	Rome	2200
Paul	Male	Florence	1800
George	Male	Naples	1700
Leon	Male	London	2500
Luc	Male	Rome	1800
Lucy	Female	Rome	1700

DEPARTMENT	
EMPLOYEE	**NAME**
Mary	Administration
Paul	Marketing
George	Customer Care
Leon	Production
Luc	Production
Lucy	Production

1. provide a First Order formula which retrieves the name and the city of all the employees earning more than 1750 and working at the Production department,
2. provide the possible assignments making the formula true.

Solution. $\exists y \exists w(Employee(x, y, z, w) \wedge Department(x, Production) \wedge (w > 1750))$ with assignments (Leon, London) and (Luc, Rome)

Exercise 7. [6 points] A *tree* is a structure $T = (N, \prec)$, where N is a non empty set, and $\prec$ is a binary relation such that $n_1 \prec n_2$ means that the node n_1 is the parent note of n_2, and the following properties hold:

1. there is a unique element $n_0 \in N$, called the *root* of T which does not have any parent node.

2. every node of T different from the root has a unique parent.

Provide a first order language for representing tree structures and use it to formalise the above two properties, as well as the two properties below:

1. the degree of the tree is 2 (every node has at most 2 children),
2. the maximal depth of the tree is 3 (there is no branch with more than 3 nodes)

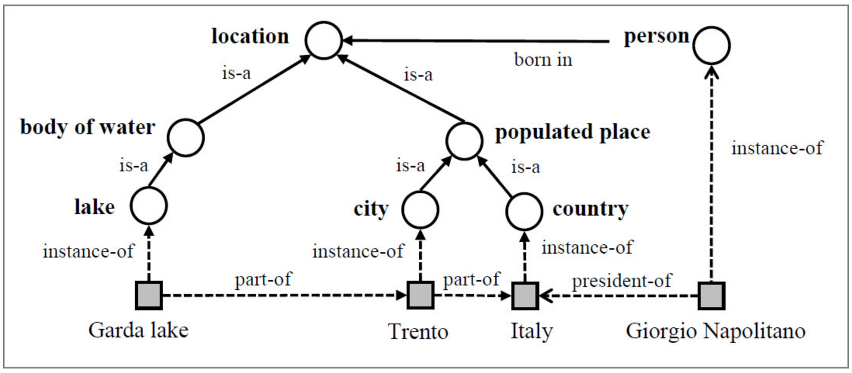
Solution. Let parent^2 be a binary predicate, such that $\mathsf{parent}(x, y)$ means that x is the parent of y, i.e., that $x \prec y$. Then we have a constant root which intuitively denotes the root of the tree.

1. $\forall x \neg \mathsf{parent}(x, \mathsf{root})$
2. $\forall x(x \neq \mathsf{root} \supset \exists y(\mathsf{parent}(y, x) \wedge \forall z \mathsf{parent}(z, x) \supset z = y))$
3. $\forall xyzw(\mathsf{parent}(x, y) \wedge \mathsf{parent}(x, z) \wedge \mathsf{parent}(x, w) \supset y = z \vee z = w \vee y = w)$
4. $\neg \exists xyzw(\mathsf{parent}(x, y) \wedge \mathsf{parent}(y, z) \wedge \mathsf{parent}(z, w))$

Mathematical logic

– 3^{rd} assessment – Description Logic – June 4, 2013 –

Exercise 1 (3 points)**.** Formalize the following semantic network into DL (TBox and ABox):



Solution.

- TBOX = {BodyOfWater $\sqsubseteq$ Location, PopulatedPlace $\sqsubseteq$ Location, Lake $\sqsubseteq$ BodyOfWater, City $\sqsubseteq$ PopulatedPlace, Country $\sqsubseteq$ PopulatedPlace, Person $\sqsubseteq$ $\exists$BornIn.Location}
- ABOX = {Person(GiorgioNapolitano), Lake(GardaLake), City(Trento), Country(Italy),
Part(GardaLake,Trento), Part(Trento, Italy), PresidentOf(GiorgioNapolitano, Italy)}

Exercise 2 (4 points)**.** Translate the following natural language sentences in DL:

1. A parent is a person having at least one child
2. Tables have exactly 4 legs
3. Germans do not have Italian friends and friends having Italian friends

4. The colour of a banana can be only yellow or red

Solution. Possible DL translations are as follows:

1. $\mathsf{PARENT} \sqsubseteq \mathsf{PERSON} \sqcap \exists \mathsf{HAS-CHILD}.\top$ or also $\mathsf{PARENT} \sqsubseteq \mathsf{PERSON} \sqcap \geq 1 \mathsf{HAS-CHILD}$
2. $\mathsf{TABLE} \sqsubseteq \geq 4 \mathsf{HAS-LEG} \sqcap \leq 4 \mathsf{HAS-LEG}$
3. $\mathsf{GERMAN} \sqsubseteq \forall \mathsf{FRIEND-OF}.(\neg \mathsf{ITALIAN} \sqcup \neg \exists \mathsf{FRIEND-OF}.\mathsf{ITALIAN})$
4. $\mathsf{BANANA} \sqsubseteq \forall \mathsf{HAS-COLOR}.\{\mathsf{yellow}, \mathsf{red}\}$

Exercise 3 (4 points)**.** Model the following problem in DL and prove its satisfiability by providing a corresponding model for it:

Lazy people are humans that work with nobody and workaholics are those humans who work with employees or bosses. An animal trainer works only with animals.

Solution. A possible TBox is:

$$\begin{aligned}
\mathsf{LazyPerson} &\equiv \mathsf{Human} \sqcap \forall \mathsf{workWith}.\bot \\
\mathsf{Workaholic} &\equiv \mathsf{Human} \sqcap \exists \mathsf{workWith}.(\mathsf{Employee} \sqcup \mathsf{Boss}) \\
\mathsf{AnimalTrainer} &\equiv \forall \mathsf{workWith}.\mathsf{Animal}
\end{aligned}$$

The model can be given in terms of Venn Diagram or as a class valuation, e.g.:

$$\begin{aligned}
I(\mathsf{Human}) &= \{a, b, c, d, e, f\} \\
I(\mathsf{LazyPerson}) &= \{a\} \\
I(\mathsf{Workaholic}) &= \{b\} \\
I(\mathsf{Employee}) &= \{c\} \\
I(\mathsf{Boss}) &= \{d\} \\
I(\mathsf{AnimalTrainer}) &= \{e\} \\
I(\mathsf{Animal}) &= \{f\} \\
I(\mathsf{workWith}) &= \{(b, c), (e, f)\}
\end{aligned}$$

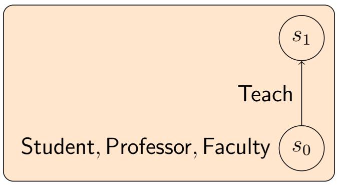
Exercise 4 (4 points)**.** Given the following TBox T

$$\mathsf{Student} \sqsubseteq \mathsf{Faculty},$$
$$\mathsf{Professor} \sqsubseteq \mathsf{Faculty} \sqcap \exists \mathsf{Teach}.\top$$

Are Student and Professor disjoint? Motivate your answer by providing a formal proof or a counterexample.

Solution. It corresponds to the problem: T $\models \mathsf{Student} \sqcap \mathsf{Professor} \sqsubseteq \bot$.

The answer is NO. There are many ways to prove it. For instance, it is enough to provide a counterexample. By using Venn diagrams we can show that there is at least a model where Student and Professor do intersect.



Exercise 5 (4 points)**.** Using the DL semantics, prove that the following inclusion axiom is valid:

$$\forall r.\forall s.A \sqcap \exists r.\forall s.B \sqcap \forall r.\exists s.C \sqsubseteq \exists r.\exists s.(A \sqcap B \sqcap C)$$

Solution. The interpretation of the first formula is given by the union of the following sets:
D = {x | ∀ y : (x,y) ∈ I(r), ∀ z : (y,z) ∈ I(s) and z ∈ I(A)}
E = {x | ∃ y : (x,y) ∈ I(r), ∀ z : (y,z) ∈ I(s) and z ∈ I(B)}
F = {x | ∀ y : (x,y) ∈ I(r), ∃ z : (y,z) ∈ I(s) and z ∈ I(C)}
The interpretation of the second formula is given by the union of the following sets:
L = {x | ∃ y : (x,y) ∈ I(r), ∃ z : (y,z) ∈ I(s) and z ∈ I(A)}
M = {x | ∃ y : (x,y) ∈ I(r), ∃ z : (y,z) ∈ I(s) and z ∈ I(B)}
N = {x | ∃ y : (x,y) ∈ I(r), ∃ z : (y,z) ∈ I(s) and z ∈ I(C)}
It can be clearly observed that D ∩ E ∩ F ⊆ L ∩ M ∩ N. In fact: D ⊆ L, E ⊆ M and F ⊆ N.

Exercise 6 (4 points)**.** Explain the steps which are needed to reformulate subsumption w.r.t. a TBox in propositional DL into a PL reasoning problem.

Solution. Given a TBox T, the problem $T \models C \sqsubseteq D$ can be reconducted to reason about satisfiability of a PL formula by:

1. Normalizing T to T' (TBox normalization)
2. Expanding C and D w.r.t. T', thus obtaining C' and D' (TBox elimination)
3. Rewriting C' and D' in PL
4. Call DPLL(CNF(C' $\rightarrow$ D')) and verify that it returns true

Exercise 7 (4 points)**.** Given the following TBox T and the ABox A:

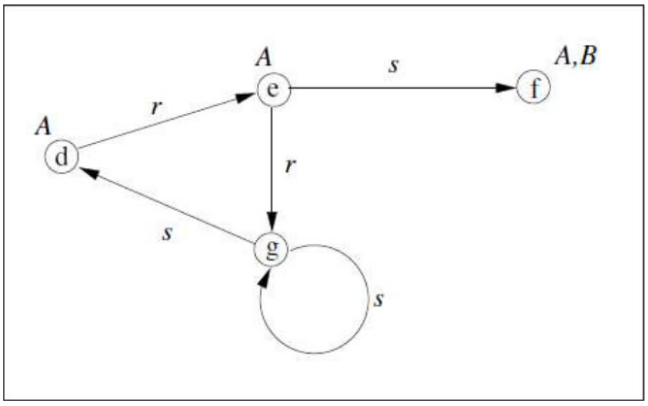
$$T = \{\mathsf{A} \sqsubseteq \neg\mathsf{B}, \mathsf{C} \equiv \mathsf{D} \sqcap \mathsf{A}, \mathsf{E} \sqsubseteq \mathsf{D}\}$$
$$A = \{\mathsf{C(a)}, \mathsf{E(c)}, \mathsf{B(b)}\}$$

1. Provide the expansion A' of A w.r.t. T without normalizing T
2. Provide the instance retrieval of D
3. Say weather by adding B(a) to A', A' is consistent or not w.r.t. T. Motivate your answer.

Solution. For the above:

1. A' = A $\cup$ {D(a), A(a), $\neg$B(a), D(c)}
2. {a, c}
3. No, it becomes inconsistent as it already contains $\neg$B(a).

Exercise 8 (6 points)**.** Consider the graph representation of the interpretation I with $\triangle^I = \{d,e,f,g\}$:



For each of the following DL concepts C, list all the elements x of $\triangle^I$ such that $x \in C^I$

1. $A \sqcup B$
2. $\exists s.\neg A$
3. $\forall s.A$
4. $\exists s.\exists s.\exists s.\exists s.A$
5. $\forall t.A \sqcap \forall t.\neg A$
6. $\neg \exists r.(\neg A \sqcap \neg B)$

Solution.

1. {d, e, f}
2. {g}
3. {e}
4. {g}
5. {d, e, f, g}
6. {d, e, f}

Mathematical logic

– Exam of 14 June 2013 –

Exercise 1 (Propositional logic: natural deduction)**.** [**3 marks**] Derive the following formulas via Natural Deduction,

$$\neg(A \supset \neg B) \supset (A \wedge B)$$

Solution. See slides of propositional reasoning part.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A^1 \quad \neg A^2}{\bot}\supset E}{\neg B}\bot c}{A \supset \neg B}\supset I_{(1)} \quad \neg(A \supset \neg B)}{\bot}\supset E}{A}\bot c_{(2)} \qquad \dfrac{\dfrac{\dfrac{\neg B^3}{A \supset \neg B}\supset I \quad \neg(A \supset \neg B)}{\bot}}{B}\bot c_{(3)}}{A \wedge B}\wedge I$$

Exercise 2 (Propositional logic:theory [**5 marks**])**.** Provide the definition of *maximally consistent set of formulas* and show that if Γ is maximally consistent and $\Gamma \vdash \phi$, then $\phi \in \Gamma$.

Solution. A set of formulae Γ is *maximally consistent* if it is consistent and any other consistent set $\Sigma \supseteq \Gamma$ is equal to Γ.

A proof that if Γ is maximally consistent and $\Gamma \vdash \phi$, then $\phi \in \Gamma$ is as follows: Since $\Gamma \vdash \phi$, then $\Gamma \cup \{\phi\}$ is consistent. In fact, assume that $\Gamma \vdash \phi$ and that $\Gamma \cup \{\phi\}$ is not consistent, then $\Gamma \cup \{\phi\} \vdash \bot$, and therefore $\Gamma \vdash \neg\phi$. But from the fact that $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$ we can build a deduction $\Gamma \vdash \bot$, which would mean that Γ is not consistent. Since Γ is maximally consistent from the hypothesis, then we have reached an absurdum and the assumption that $\Gamma \cup \{\phi\}$ is not consistent cannot be true. Thus $\Gamma \cup \{\phi\}$ is consistent. From the definition of maximally consistent we know that $\Gamma \cup \{\phi\} \subseteq \Gamma$, and therefore we can conclude that $\phi \in \Gamma$.

Exercise 3 (Propositional logic: [**3 marks**])**.** List all the subformulas of the formula $\neg p \supset (q \wedge (r \wedge \neg\neg q))$:

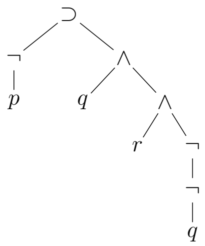
Oppure un esercizio facile du DPLL

Solution. The subformulas of $\neg p \supset (q \wedge (r \wedge \neg\neg q))$ are:

- $\neg p \supset (q \wedge (r \wedge \neg\neg q))$
- $\neg p$
- p
- $q \wedge (r \wedge \neg\neg q)$

- q
- $r \wedge \neg\neg q$
- r
- $\neg\neg q$
- $\neg q$

The tree representing all the subformulas is depicted below.



Exercise 4 (First order logic: modeling [**7 marks**]). Formalize the following statements, by using *only* the following first order predicates:

$F(x)$	x is female
$M(x)$	x is a male
$MW(x, y)$	x is married with y
$PA(x, y, z)$	x plays against y in match z

1. everybody must be either a male or (exclusively) a female
2. Men are married (only) with women and vice-versa
3. One can be married with at most one person
4. Being married is a symmetric and irreflexive relations
5. Matches can be between two players (singles) or between two teams of two players (doubles) (as in the sport of tennis). That is, in a match one can play against one or two against two.
6. married people play always together in the same team (that is, they don't play against each other and they cannot play against someone without their partner)
7. Married couples always plays doubles against other married couples

Solution. Possible formalizations are:

1. everybody must be either a male or (exclusively) a female

$$\forall x(M(x) \iff \neg F(x))$$

2. Men can be married to women and vice-versa

$$\forall xy(MW(x,y) \supset ((M(x) \wedge F(y)) \vee (F(x) \wedge M(y))))$$

3. One can be married with at most a person

$$\forall xyz(M(x,y) \wedge M(x,z) \supset y = z)$$

4. Being married is symmetric and irreflexive

$$\forall xy(MW(x,y) \supset MW(y,x))$$
$$\forall x(\neg MW(x,x))$$

5. Games can be between two players (singles) or between two teams of two players (doubles) (as in the sport of tennis). That is, in a game competition one can play against one or two against two.

   This statement can be formalized by imposing that:

   - One cannot play against more than 2 people in a match (i.e., there are only singles and doubles)

   $$\forall xywtz(PA(x,y,z) \wedge PA(x,w,z) \wedge PA(x,t,z) \supset y = w \vee y = t \vee w = t$$

   - If one plays against two different people then he/she has a team partner (ie. we are in a double)

   $$\forall xywz(PA(x,y,z) \wedge PA(x,w,z) \wedge y \neq w \supset \exists t(PA(t,y,z) \wedge PA(t,w,z) \wedge t \neq x))$$

6. married people play always in team

$$\forall xyz(MW(x,y) \supset \neg PA(x,y,z) \wedge \forall t(PA(x,t,z) \iff PA(y,t,z)))$$

7. Married couples always plays doubles against other married couples

$$\forall xyzt(MW(x,y) \wedge PA(x,t,z) \supset \exists w MW(t,w) \wedge PA(x,t,z))$$

Exercise 5 (First order logic: [**4 marks**]). Prove soundness of the $\forall I$ rule of Natural Deduction:

$$\frac{\phi(x)}{\forall x.\phi(x)}\ \forall I$$

Solution. Assume that the last rule used is $\forall I$. Then the derivation tree is of the form

$$\begin{array}{c}\Gamma\\ \Pi\\ \dfrac{A(x)}{\forall x.A(x)}\ \forall I\end{array}$$

with x not free in Γ. Let $\mathcal{I}, a$ be such that $\mathcal{I} \models \Gamma[a]$.

From the inductive hypothesis we know that $\mathcal{I} \models \phi(x)[a]$.

Since x does not appear free in Γ, then $\mathcal{I} \models \Gamma[a[x/d]]$ holds for all $d \in \Delta$.

Therefore from the inductive hypothesis $\mathcal{I} \models \phi(x)[a[x/d]]$ holds for all $d \in \Delta$.

Then for the definition of $\models$, we have that $\mathcal{I} \models \forall x.\phi(x)[a]$.

Exercise 6 (Description logic: [**3 marks**]). Using the DL semantics, prove that the following inclusion axiom is valid:

$$\forall \mathsf{r}.\forall \mathsf{s}.\mathsf{A} \sqcap (\exists \mathsf{r}.\forall \mathsf{s}.\neg \mathsf{A} \sqcup \forall \mathsf{r}.\exists \mathsf{s}.\mathsf{B}) \sqsubseteq \forall \mathsf{r}.\exists \mathsf{s}.(\mathsf{A} \sqcap \mathsf{B}) \sqcup \exists \mathsf{r}.\forall \mathsf{s}.\neg \mathsf{B}$$

Solution. Please enzo add.

Exercise 7 (Description logic: [**4 marks**]). Consider the following ABox

$$A = \left\{ \begin{array}{ccc} likes(Alice, Bob), & is\text{-}neighbour\text{-}of(Bob, Claudia) & clever(Claudia) \\ likes(Alice, Claudia), & is\text{-}neighbour\text{-}of(Claudia, Darren), & \neg clever(Darren) \end{array} \right\}$$

1. Is A satisfiable? Provide a rationale for the answer.
2. Is *Alice* an instance of $\exists likes.(clever \sqcap \exists is\text{-}neighbour\text{-}of.\neg clever)$ with respect to A? Provide a rationale for the answer.

Solution. For the above:
1. The answer is YES. An ABox is satisfiable if consistent (w.r.t. the TBox). As there is no TBox, it is enough to observe that in A there are no contradictory assertions.

2. It corresponds to the following instance checking problem:

$$\text{A} \models \exists\text{likes.(clever} \sqcap \exists\text{is-neighbour-of.}\neg\text{clever)(Alice)}$$

This corresponds to verifying the following:

$$\text{I(Alice)} \in \{\text{x} \mid \exists\text{y} : \text{(x, y)} \in \text{I(likes), y} \in \text{I(clever)} \cap \text{B}\}$$

where B = {y | ∃ y : (y, z) ∈ I(is-neighbour-of), z ∈ I(clever)}.
In other words A should contain the following assertions:
Likes(Alice, y), clever(y), is-neighbour-of(y, z), ¬clever(z).
This is true for y = Claudia and z = Darren.

Exercise 8 (Cross logics: [**4 marks**]). Translate the following natural language sentences in Propositional Logic, Description Logic and first order logic at the best of their expressiveness:

1. If Anna goes to the party then Bob does not go
2. A good apple is neither dirty nor rotten
3. A parent is a person having at least one child
4. Companies that do not have female employees are discriminatory

Solution. Possible formalizations are as follows:
Propositional logic

1. $Anna \supset \neg Bob$
2. $GoodApple \supset \neg Dirty \wedge \neg Rotten$
3. $Parent \supset Person \wedge HasChild$
4. $DiscriminatoryCompany \supset Company \wedge \neg FemaleEmployee$

First Order Logic

1. $Goes(Anna, Party) \supset \neg Goes(Bob, Party)$
2. $\forall x.GoodApple(x) \supset \neg Dirty(x) \wedge \neg Rotten(x)$
3. $\forall x.Parent(x) \supset Person(x) \wedge \exists y.HasChild(x, y)$
4. $\forall x.DiscriminatoryCompany(x) \supset Company(x) \wedge \neg \exists y.(Employee(x, y) \wedge Female(y))$

Description Logic

1. $ANNA\text{-}GOES \sqsubseteq \neg BOB\text{-}GOES$
2. $GOODAPPLE \sqsubseteq \neg DIRTY \sqcap \neg ROTTEN$
3. $PARENT \sqsubseteq PERSON \sqcap \exists HasChild.\top$
4. $DISCRIMINATORYCOMPANY \sqsubseteq COMPANY \sqcap \neg \exists HasEmployee.FEMALE$

Propositional Logic

Exercise 1. [4 marks] Let ϕ be a formula that contains only two propositional variables p and q.

1. Prove that if ϕ is satisfiable, then $\phi \wedge r$ and $\phi \wedge \neg r$ are both satisfiable for a new propositional variable r.

2. Prove that at least one of the following 4 formulas is valid:

   (a) $(p \wedge q) \rightarrow \phi$

   (b) $(p \wedge \neg q) \rightarrow \phi$

   (c) $(\neg p \wedge q) \rightarrow \phi$

   (d) $(\neg p \wedge \neg q) \rightarrow \phi$

Solution. 1. The fact that ϕ is satisfiable means that there is a truth assignment ν to p, q that satisfies ϕ, i.e., such that $\nu(\phi) = True$. Let ν_- and ν_+ be the assignments obtained by extending ν with $\nu_-(r) = False$ and $\nu_+(r) = True$ respectively. By proposition 1.2.6. of C.C. Chang and H.J. Keisler, Model Theory, Third Edition Studies in Logic and the Foundations of Mathematics North Holland, we have that, the assignment to the variables not appearing in ϕ does not affect the truth value of ϕ. This implies that $\nu_-(\phi) = True$ and $\nu_+(\phi) = True$, since ν_- and ν_+ differ form ν only from the assignment to r and r does not occur in ϕ. From this, it follows that $\nu_-(\phi \wedge \neg r) = True$ and $\nu_+(\phi \wedge r) = True$. I.e., $\phi \wedge \neg r$ and $\phi \wedge r$ are both satisfiable, by the assignment ν_- and ν_+, respectively.

2. To prove that one of the formulas in (a)–(d) is valid, we have to show that for all assignments to p and q such a formula is true. Notice that with two propositional variables we have four assignments, summarized in the following table:

	p	q
$\nu_{(a)}$	$True$	$True$
$\nu_{(b)}$	$True$	$False$
$\nu_{(c)}$	$False$	$True$
$\nu_{(d)}$	$False$	$False$

If ϕ is satisfiable and then there is an assignment $\nu_{(x)}$ with $x \in \{a, b, c, d\}$ such that $\nu_{(x)}(\phi) = True$. This implies that

$$\nu_{(x)} \models \phi \text{ and therefore } \nu_{(x)} \models (x)$$

Notice that for $y \in \{a, b, c, d\}$ and $y \neq x$, the assignment $\nu_{(y)}$ does not satisfy the antecedent of the formula (x), Which implies that for all $y \neq x$ we have that,

$$\nu_{(y)} \models (x)$$

since the antecedent of the implication (x) is not satisfied by $\nu_{(y)}$.

Exercise 2. [**3 marks**] Using the DPLL algorithm, and by providing the description of the steps followed, prove the satisfiability or unsatisfiability of the formula:

$$(\neg A \to B) \wedge (B \to A) \wedge (A \to (C \wedge D))$$

Assume a version of the DPLL without the pure literal step. Explain also what it should be done to prove its validity.

Solution. The formula needs to be first translated into CNF:

$$(\neg A \vee B) \wedge (\neg B \vee A) \wedge (\neg A \vee C) \wedge (\neg A \vee D)$$

There is no unit clause. Therefore, we need to select a literal for the branching. Let us select A. Let us call DPLL firstly on $(\neg A \vee B) \wedge (\neg B \vee A) \wedge (\neg A \vee C) \wedge (\neg A \vee D) \wedge A$. We can then associate v(A) = T and propagate thus obtaining: $B \wedge C \wedge D$. As it is a consistent conjunction of literals, after another iteration of the DPLL the algorithm clearly returns true. Therefore the formula is clearly satisfiable.
To check the validity we need to verify that the DPLL returns false when called on the negation of the original formula.

Exercise 3. [**4 marks**] Formalize the following argument into an entailment between two formulas (i.e., of the form Γ ⊨ P), and explain how to establish whether the entailment holds or not: *If Dominic goes to the racetrack, then Helen will be mad. If Ralph plays cards all night, then Carmela will be mad. If Helen or Carmela gets mad, then Veronica will be notified. Nobody notified Veronica. Consequently, Dominic did not go to the racetrack and Ralf did not play cards all night.*

Solution. If we use the following propositions:
p = Dominic goes to the racetrack
q = Helen will be mad
r = Ralph plays cards all night
s = Carmela will be mad
t = Veronica is notified
then the argument can be rewritten as $\{p \to q, r \to s, (q \vee s) \to t, \neg t\} \models \neg p \wedge \neg q$.
By the definition of entailment, we need to show that all the models satisfying all the

formulas on the left also satisfy the formula on the right. For instance, this can be reformulated in PL by reasoning about the validity of the formula:

$$((p \to q) \wedge (r \to s) \wedge ((q \vee s) \to t) \wedge \neg t) \to (\neg p \wedge \neg q)$$

The above can be asserted by using DPLL or truth tables.

First order logics

Exercise 4. [4 marks] Prove by natural deduction that the following formula is valid:

$$\forall xy(P(x) \to P(y)) \to (\exists x P(x) \to \forall x P(x))$$

Solution.

$$\dfrac{\dfrac{\dfrac{\exists x P(x)^{(1)} \qquad \dfrac{\dfrac{P(a)^{(2)} \qquad \dfrac{\forall xy(P(x) \to P(y))^{(3)}}{P(a) \to P(b)}\ \forall E}{P(b)}\ \to E}{\forall x P(x)}\ \forall I}{\forall x P(x)}\ \exists E \text{ disch. } (2)}{\exists x P(x) \to \forall x P(x)}\ \to I \text{ disch } (1)}{\forall xy(P(x) \to P(y)) \to (\exists x P(x) \to \forall x P(x))}\ \to I \text{ disch. } (3)$$

- notice that the rule "$\forall I''$ is applicable to $P(b)$ because b does not appear in any assumption $P(b)$ depends on, namely $P(a)$ and $\forall xy(P(x) \to P(y))$.
- Notice that the rule "$\exists E''$ is applicable to $\forall x P(x)$ since, this it does not contain a (the parameters of the discharged assumption) and a does not occur in any assumption $\forall x P(x)$ depends on with the exception of $P(a)$.

Exercise 5. [3 marks] Represent in FOL the following natural language sentences :

1. The Barber of Seville shaves all men who do not shave themselves.
2. There is exactly one coin in the box
3. All students get good grades if they study

Solution. The three sentences can be represented as follows:

1. $\forall x.\neg Shaves(x, x) \to Shaves(BarberOfSeville, x)$
2. $\exists x.Coin(x) \wedge InBox(x) \wedge \forall y.(Coin(y) \wedge InBox(y) \to x = y)$
3. $\forall x.Student(x) \wedge Study(x) \to GetGoodGrade(x)$

Exercise 6. [**4 marks**] Consider the FOL formula $\phi = \exists x.P(x) \rightarrow \forall x.P(x)$

1. Prove, by using the semantics of the FOL language, that ϕ is true in all the interpretations whose domain contains only one element.

2. Let $\triangle = \{a, b\}$. Find an interpretation I that does not satisfy ϕ.

Solution. 1. Let $\mathcal{I}$ be an interpretation $\mathcal{I} = \left\langle \Delta_{\mathcal{I}}, \cdot^{\mathcal{I}} \right\rangle$, such that $|\Delta_{\mathcal{I}}| = 1$ i.e.,$\Delta_{\mathcal{I}}$ contains only one element. The premise of ϕ is true in $\mathcal{I}$ iff there exists an element $d \in \Delta_{\mathcal{I}}$, such that $d \in P^{\mathcal{I}}$. The consequence of ϕ is true in $\mathcal{I}$ iff for all elements $d \in \Delta_{\mathcal{I}}$, $d \in P^{\mathcal{I}}$ Since $\Delta_{\mathcal{I}}$ contains only one element, namely d, then $\forall x P(x)$ is true in $\mathcal{I}$.

2. If we take $\mathcal{I}$ such that $P^{\mathcal{I}} = \{a\}$, the premise of ϕ is true in $\mathcal{I}$ while the consequence is false as $\mathcal{I} \not\models P(x)[x/b]$ and therefore the implication ϕ is false in $\mathcal{I}$.

Description Logics

Exercise 7. [**4 marks**] Define a TBox and ABox for the following problem: *Users have read access to files. Authorized users are special users having also write access to files. Barbara is an authorized user.*
Can Barbara read files? Give a formal proof for the answer.

Solution. (a) We can define the following TBox and ABox:
T = $\{User \sqsubseteq \exists Read.File, Authorized \sqsubseteq \exists Write.File \sqcap User\}$
A = {Authorized(Barbara)}
(b) Yes. In fact, we can expand the ABox A w.r.t. the TBox T and easily verify that:
$Authorized(Barbara) \Rightarrow User(Barbara) \Rightarrow \exists Read.File(Barbara)$

Exercise 8. [**3 marks**] Consider the following ABox A:

$$A = \left\{ \begin{array}{ccc} likes(Ralf, Claudia), & likes(Ralf, Peter), & is\text{-}neighbour\text{-}of(Claudia, Peter) \\ blond(Claudia), & \neg blond(Andrea), & is\text{-}neighbour\text{-}of(Peter, Andrea) \end{array} \right\}$$

Provide an answer and a rationale for the following questions pertaining A:

1. Does A have a model?

2. Is $Ralf$ an instance of $\exists likes.(blond \sqcap \exists is\text{-}neighbour\text{-}of.\neg blond)$?

3. Is $Ralf$ an instance of $\exists likes.(\exists is\text{-}neighbour\text{-}of.(\forall is\text{-}neighbour\text{-}of.\neg blond))$?

Solution. (1) The answer is YES. An ABox is satisfiable (has a model) if consistent (w.r.t. the TBox). As there is no TBox, it is enough to observe that in A there are no contradictory assertions.
(2) It corresponds to the following instance checking problem:

$$A \models \exists likes.(blond \sqcap \exists is\text{-}neighbour\text{-}of.\neg blond)(Ralf)$$

This corresponds to verifying that there are two individuals x and y such that A contains: $likes(Ralf, x), blond(x), is\text{-}neighbour\text{-}of(x, y), \neg blond(y)$. As it is not possible to find them, the answer is NO.
(3) It corresponds to the following instance checking problem:

$$A \models \exists likes.(\exists is\text{-}neighbour\text{-}of.(\forall is\text{-}neighbour\text{-}of.\neg blond))(Ralf)$$

This corresponds to verifying that there are two individuals x and y such that A contains: $likes(Ralf, x), is\text{-}neighbour\text{-}of(x, y)$ and that for all z such that $is\text{-}neighbour\text{-}of(y, z)$ then $\neg blond(z)$. As this is true for x = Claudia and y = Peter, the answer is YES.

Exercise 9. [4 marks] Considerthe following interpretation I = ($\triangle$,I) with:

$\triangle$ = { t1, t2, f1, f2, c1, c2, j, k, l, m, n}
I(Person) = {j, k, l ,m, n}
I(Car) = {t1, t2, f1, f2, c1, c2}
I(Ferrari) = {f1, f2}
I(Toyota) = {t1, t2}
I(likes) = {(j, f1), (k, f1), (k, t2), (l, c1), (l, c2), (m, c1), (m, t2), (n, f2), (n, c2)}

Compute the instance retrieval of the following concepts:

1. $\exists likes.Ferrari \sqcap \exists likes.Toyota$
2. $\exists likes.Car \sqcap \forall likes.\neg(Toyota \sqcup Ferrari)$

Solution. The instance retrieval of the first is {} (empty), while the instance retrieval of the second is {l}.

Mathematical logic

– 1^{st} assessment – Propositional Logic –
23 October 2013

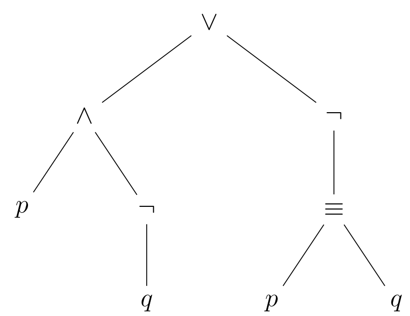
Exercise 1. [3 points] Consider the following formula

$$(p \wedge \neg q) \vee \neg(p \equiv q)$$

1. Write the formula as a tree, and
2. list all its sub-formulae

Solution.

1.



2.
   - $(p \wedge \neg q) \vee \neg(p \equiv q)$
   - $p \wedge \neg q$
   - p
   - $\neg q$
   - q
   - $\neg(p \equiv q)$
   - $p \equiv q$

Exercise 2. [6 points]:

1. Translate the following natural language sentences into propositional logic formulas:
   (a) Claudia gets a pay rise if she acquires a new customer or if she acquires a new project
   (b) Claudia does not acquire a new customer, however she gets a pay rise
   (c) Claudia acquires a new project
2. say whether (c) is a logical consequence of (a) and (b) using the truth tables and motivate your answer.

Solution.

1. Let

   $R =$ Claudia gets a pay rise
   $C =$ Claudia acquires a new customer
   $P =$ Claudia acquires a new project

   A possible formalization is the following:

   (a) $(C \vee P) \supset R$
   (b) $\neg C \wedge R$
   (c) P

2. The truth table for the formulae above is:

		(c)		(b)		(a)
R	C	P	$\neg C$	$\neg C \wedge R$	$C \vee P$	$(C \vee P) \supset R$
T	T	T	F	F	T	T
T	T	F	F	F	T	T
T	F	T	T	T	T	T
T	F	F	T	T	F	T
F	T	T	F	F	T	F
F	T	F	F	F	T	F
F	F	T	T	F	T	F
F	F	F	T	F	F	T

   As we can see from this truth table there are only two assignments that satisfy both premises: the ones in row 3 and 4. One of them (row 4) satisfies both (a) and (b) but does not satisfy (c). Therefore (c) is not a logical consequence of (a) and (b).

Exercise 3. [6 points] Let

(**MyRule1**) $$\frac{\phi \vee \psi \quad \neg\phi}{\psi}$$

and

(**MyRule2**) $$\frac{\phi \supset \psi \quad \neg\phi}{\neg\psi}$$

be two reasoning rules used to build proofs. Say (and prove) whether (**MyRule1**) and (**MyRule2**) are rules that preserve validity (i.e, that transform valid formulae in valid formulae).

Solution.

Proof.

- Let us consider (**MyRule1**).

  We have to prove that if $\phi \vee \psi$ and $\neg\phi$ are valid formulae, then ψ is a valid formula.

  Let us assume that $\phi \vee \psi$ and $\neg\phi$ are valid formulae. Then for each propositional interpretation $\mathcal{I}$ we have that $\mathcal{I} \models \phi \vee \psi$ and $\mathcal{I} \models \neg\phi$. From the definition of satisfiability of $\vee$, we have that $\mathcal{I} \models \phi$ or $\mathcal{I} \models \psi$. Since $\mathcal{I} \models \neg\phi$, and therefore $\mathcal{I} \not\models \phi$, then we can conclude that $\mathcal{I} \models \psi$. Thus ψ is a valid formula and (**MyRule1**) is a sound rule which preserves validity.

- Let us consider (**MyRule2**). Let p, q two propositional atoms, and let

  - $\phi = p \wedge \neg p$
  - $\psi = q$

  It is easy to prove that both $(p \wedge \neg p) \supset q$, and $\neg(p \wedge \neg p)$ are valid formulae. (can be done with the truth tables for instance)

  If we use them with the rule (**MyRule2**) we obtain $\neg q$ which is not a valid formula.

  Indeed let $\mathcal{I}$ be the interpretation $\mathcal{I} = q$. This interpretation does not satisfy $\neg q$.

  Thus rule (**MyRule2**) does not preserve validity.

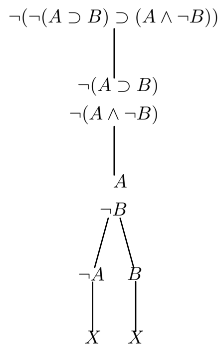
□

Exercise 4. [6 points] For each of the following formulae determine whether they are **valid**, **unsatisfiable**, or **satisfiable** (and not valid) using analytic tableaux. Report the tableau, and use it to justify your answer.

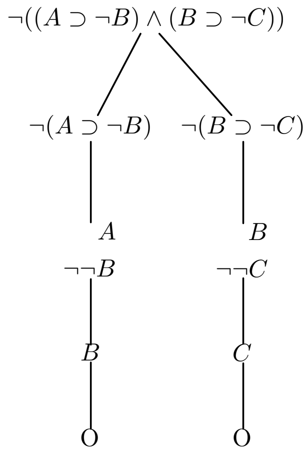
1. $\neg(A \supset B) \supset (A \wedge \neg B)$
2. $(A \supset \neg B) \wedge (B \supset \neg C)$

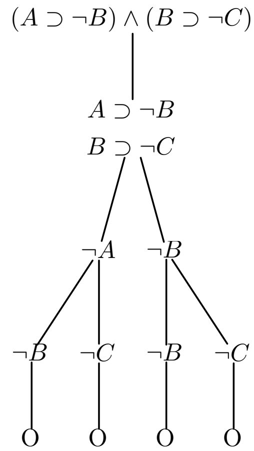
Solution.

1. The formula $\neg(A \supset B) \supset (A \wedge \neg B)$ is Valid. In fact, the tableau for its negated version is the closed tableau reported below



2. $(A \supset \neg B) \wedge (B \supset \neg C)$ is Satisfiable (and not valid). In fact, it is not valid, as shown by the tableau on the left hand side below, which remains open, and it is satisfiable, as shown by the tableau on the right hand side, which has four open branches and therefore shows at least four interpretations that make the formula true.





Exercise 5. [3 points] Apply DPLL procedure to check if the following set of clauses is satisfiable, and if it is so, return a partial assignment that makes the fomula true.

$$\phi = \{\{A, \neg B, \neg D\}, \{\neg A, \neg B, \neg C\}, \{\neg A, C, \neg D\}, \{\neg A, B, C\}\}$$

In the solution you have to specify all the application of unit propagation rule, and all the choices you take when Unit propagation is not applicable.

Solution.

1. ϕ does not contain unit clause, which implies that unit propagation is not applicable.
2. therefore, we select a literal (say A) and set $\mathcal{I}(A) = true$
3. Compute $\phi|_A$:
$$\phi|_A = \{\{\neg B, \neg C\}, \{C, \neg D\}, \{B, C\}\}$$
4. $\phi|_A$ does not contain unit clauses, therefore unit propagation is not applicable.
5. select a second literal, say $\neg B$, and set $\mathcal{I}(B) = false$
6. Compute $(\phi|_A)|_{\neg B}$ (also denoted by $\phi|_{A,\neg B}$).
$$\phi|_{A,\neg B} = \{\{C, \neg D\}, \{C\}\}$$
7. $\phi|_{A,\neg B}$ contain the unit clause $\{C\}$, we therefore extend the partial interpretation with $\mathcal{I}(C) = True$. We then apply unit propagation with $\{C\}$ as unit clause, obtaining $\phi|_{A,\neg B,C} = \{\}$, the empty set of clauses. Which means that the initial formula is satisfiable. The partial assignment is $\mathcal{I}(A) = True$, $\mathcal{I}(B) = false$ and $\mathcal{I}(C) = true$

Exercise 6. [3 points] Say when a formula ϕ is equi-satisfiable of a formula ψ. and show that the two formulas:

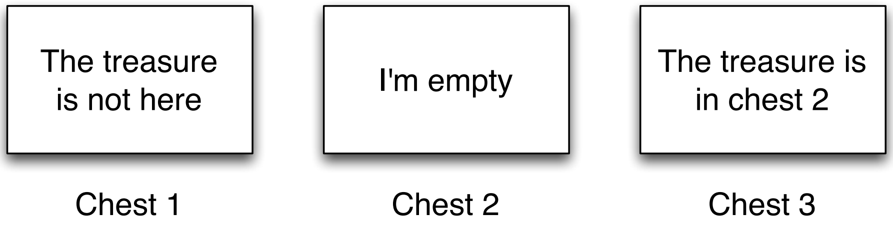
$$\phi = A \to (B \vee C) \qquad \psi = (N \equiv (B \vee C)) \wedge (A \to N) \tag{3}$$

are equi-satisfiable.

Solution. ϕ and ψ are equi-satisfiable, if and only if ϕ is satisfiable iff ψ is satisfiable. Or in other words, there is an interpretation $\mathcal{I}$ that satisfies ϕ if and only if there is an interpretation $\mathcal{J}$ that satisfies ψ.

Let us shows that the formulas in (3) are equisatisfiable. Let $\mathcal{I} \models \phi$. Let's extend $\mathcal{I}$ to $\mathcal{I}'$ setting $\mathcal{I}'(N) = \mathcal{I}(B \vee C)$. We have that $\mathcal{I}' \models \psi$. Viceersa, let $\mathcal{I}$ be an interpretation that satisfies ψ, then $\mathcal{I} \models N \equiv (B \vee C)$ implies that $\mathcal{I}(N) = \mathcal{I}(B \vee C)$. The fact that $\mathcal{I} \models A \to N$ impies that $\mathcal{I} \models A \to (B \vee C)$.

Exercise 7. [6 points] In her travels for treasure hunting, Chiara finds herself in front of three mysterious chests. In one of the chests is a fabulous treasure, all the others are empty. On each chest there is an inscription:



Given the fact that two chests are lying, and one is telling the truth, where is the treasure?

Solution. Let us define the following language:

- $t1$ = the treasure is in chest 1;
- $t2$ = the treasure is in chest 2;
- $t3$ = the treasure is in chest 3;

we can encode the knowledge we have as follows:

(a) "In one of the chests is a fabulous treasure, all the others are empty"

$$(t1 \wedge \neg t2 \wedge \neg t3) \vee (\neg t1 \wedge t2 \wedge \neg t3) \vee (\neg t1 \wedge \neg t2 \wedge t3)$$

(b) the sentence of chest 1: "the treasure is not here"

$$\neg t1$$

(c) the sentence of chest 2: "I'm empty"

$$\neg t2$$

(d) the sentence of chest 3: "the treasure is in chest 2"

$$t2$$

(e) "two chests are lying and one is telling the truth".

$$(\neg t1 \wedge \neg\neg t2 \wedge \neg t2) \vee (\neg\neg t1 \wedge \neg t2 \wedge \neg t2) \vee (\neg\neg t1 \wedge \neg\neg t2 \wedge t2)$$

This sentence can be simplified as follows:

$$(t1 \wedge \neg t2) \vee (t1 \wedge t2)$$

In building the truth tables for $t1$–$t3$ we can consider the combinations in which exactly one is true, to satisfy item (a). We find that row 1 is the only one that satisfies the sentences inscribed on all chests and also the requirement that one chest is telling the truth and two are lying. This row tells us that the treasure is in chest 1.

			(b)	(c)	(d)			(e)
$t1$	$t2$	$t3$	$\neg t1$	$\neg t2$	$t2$	$t1 \wedge \neg t2$	$t1 \wedge t2$	$(t1 \wedge \neg t2) \vee (t1 \wedge t2)$
T	F	F	F	T	F	T	F	T
F	T	F	T	F	T	F	F	F
F	F	T	T	T	F	F	F	F

Mathematical logic

– 2^{nd} assessment – First Order Logic and Modal Logic –
23 October 2013

Instructions

- Answer in English and write in ink unless the question paper gives other instructions.
- Write clearly; illegible answers will not be marked.
- Take care to identify each answer clearly with:
  - the number of the exercise.
  - where appropriate, the part of the exercise you are answering.
- Clearly cross out rough working, or unwanted answers before handing in your answers.

Exercise 1 (FOL syntax). **[6 points]**
Let Σ be the signature that contains

- the constant symbols *alice*, *bob* and *carol*
- the functional symbols $father$ and $firstCommonMaleAncestor$ with arity 1 and 2, respectively
- the predicate symbols *Student* and $Friend$ with arity 1 and 2, respectively

For each of the following expression say:

- if it is a term, a formula, or none of the two
- if it is a formula say if it is closed and if not what are the free variables
- If it is a term say if it is a ground term
- in case it is a term or a formula provide it's intuitive reading

1. $\forall Student.friend(alice, Student)$
2. $firstCommonMaleAncestor(father(alice), father(father(bob))) = carol$

3. $\forall x \forall y (father(x) = father(y) \supset firstCommonMaleAncestor(x, y) = father(x))$
4. $father(alice) = carol \vee bob$
5. $\exists y (friend(x, father(y)) \wedge friend(x, y))$
6. $firstCommonMaleAncestor(father(alice), x)$

Solution. *We summarize the result in the following table*

expr	**term formula or nothing**	**closed or open**	**free variables**	**bound variables**	**ground yes or no**	**intuitive reading**
1	*nothing*					
2	*formula*	*closed*	$\{\}$	$\{\}$	*yes*	*Carol is the first male ancestor of the father of alice and the grandfather of Bob*
3	*formula*	*closed*	$\{\}$	$\{x, y\}$	*no*	*If x and y have the same father, then he is their first common male ancestor*
4	*nothing*					
5	*formula*	*open*	$\{x\}$	$\{y\}$	*no*	*the set of person who are friend of a person and his/her father*
6	*term*				*no*	*the first common male ancestor of x and the father of alice*

Exercise 2 (Semantics). **[6 points]**

For each of the following formulas

- if it is valid prove it via tableaux,
- if it is satisfiable but not valid provide a counter-model, i.e. a model that falsifies it

$$\forall x \forall y (P(x) \wedge P(y) \supset Q(x,y)) \supset \forall x (P(x) \supset \exists y Q(x,y)) \quad (1)$$

$$\forall x \exists y P(x,y) \supset \forall y \exists x P(x,y) \quad (2)$$

$$\neg(P(a,b) \equiv \exists xy P(x,y)) \quad (3)$$

Solution. *The first formula is valid as we can build the following closed tableaux for it's negation.*

$$\neg(\forall x \forall y (P(x) \wedge P(y) \supset Q(x,y)) \supset \forall x (P(x) \supset \exists y Q(x,y)))$$

$$\forall x \forall y (P(x) \wedge P(y) \supset Q(x,y))$$
$$\neg \forall x (P(x) \supset \exists y Q(x,y))$$

$$\neg(P(a) \supset \exists y Q(a,y))$$

$$P(a)$$
$$\neg \exists y Q(a,y)$$

$$\neg Q(a,a)$$

$$\forall y (P(a) \wedge P(y) \supset Q(a,y))$$

$$P(a) \wedge P(a) \supset Q(a,a)$$

- $\neg(P(a) \wedge P(a))$
  - $\neg P(a)$
    - $\times$
  - $\neg P(a)$
    - $\times$
- $Q(a,a)$
  - $\times$

The second formula is not valid. A counter-model is $\mathcal{I} = \left\langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \right\rangle$ *with*

- $\Delta^{\mathcal{I}} = \{1, 2\}$,
- $P^{\mathcal{I}} = \{\langle 1, 2\rangle, \langle 2, 2\rangle\}$

We have that $\mathcal{I} \models \forall x \exists y P(x, y)$ *since for every assignment* $d \in \Delta^{\mathcal{I}}$ *to* x *we have that* $\mathcal{I} \models \exists y P(x, y)[a(x) = d]$. *Indeed we have that*

- $\mathcal{I} \models \exists y P(x, y)[a(x) := 1]$ *since there the assignment to* y *i.e.,2, such that* $\mathcal{I} \models P(x, y)[a(x) := 1, a(y) := 2]$, *and*
- $\mathcal{I} \models \exists y P(x, y)[a(x) := 2]$ *since there the assignment* $a(y) := 2$ *such that* $\mathcal{I} \models P(x, y)[a(x) := 2, a(y) := 2]$

On the other hand we have that $\mathcal{I} \not\models \forall y \exists x P(x, y)$ *because for the assignment* $[y := 1]$ *there is no assignment to* x *such that* $\mathcal{I} \models P(x, y)[y := 1]$. *(Notice that there is no tuple of the form* $\langle \cdot, 1\rangle \in P^{\mathcal{I}}$*).*

Finally the third formula is also not valid and a countermodel is the following

$$\mathcal{I} = \left\langle \Delta^{\mathcal{I}} = \{1\},\ a^{\mathcal{I}} = 1,\ b^{\mathcal{I}} = 1,\ P^{\mathcal{I}} = \{\langle 1, 1\rangle\} \right\rangle$$

Notice that $\mathcal{I} \models P(a, b) \equiv \exists x \exists y P(x, y)$ *since there is an assignment to* x *and* y *such that:* $\mathcal{I} \models P(a, b)$ *if and only if* $\mathcal{I} \models P(x, y)[a(x) := 1, a(y) = 1]$. *This implies that* $\mathcal{I} \not\models \neg(P(a, b) \equiv \exists x \exists y P(x, y))$.

Exercise 3 (Modelling)**. [3 points]**

Transform in FOL the following sentences:

1. Lions are feline and feline are animals
2. Simba is a Lion and there are exactly two animals which Simba cannot eat
3. There is a lion who eats exactly every animal that is not eaten by Simba

Solution.

1. *Lions are feline and feline are animals*

$$\forall x(Lion(x) \supset Feline(x)) \wedge \forall x(Feline(x) \supset Animal(x))$$

2. *Simba is a Lion and there are exactly two animals which Simba cannot eat*

$$Lion(Simba) \wedge \exists xy(x \neq y \wedge \neg Eats(Simba, x) \wedge \neg Eats(Simba, y) \wedge \forall z(z \neq x \wedge z \neq y \supset Eats(Simba, z)))$$

3. *There is a lion who eats exactly every animal that is not eaten by Simba*

$$\exists x(Lion(x) \wedge \forall y(Animal(y) \supset (Eats(Simba, y) \equiv \neg Eats(x, y))))$$

Exercise 4 (Resolution and Unification)**. [6 points]**

Use resolution and unification to solve the problem below.

Given:

$$\forall x(P(x) \supset \exists y Q(y)) \tag{4}$$
$$\neg\exists x(Q(x) \wedge \exists y \neg W(y)) \tag{5}$$
$$\forall x(P(x) \wedge W(x) \supset S(x)) \tag{6}$$
$$P(Mary) \tag{7}$$

Show:

$$S(Mary) \tag{8}$$

Solution. *To show that* (8) *logically follows from* (4)–(7)*, i.e., that* (4)–(7) $\models$ (8) *we have to prove that the set* $S = \{(4), (5), (6), (7), \neg(8)\}$ *is not satisfbiable. I.e., that we can derive the empty clause via resolution from the transformation in clause of* S

First we add the negation of the consequence to be prove to the formulas and transform them in NNF by pushing inside the $\neg$ *symbol obtaining (only the second formula)*

$$\begin{array}{r}\forall x(P(x) \supset \exists y Q(y)) \\ \forall x(\neg Q(x) \vee \forall y W(y)) \\ \forall x(P(x) \wedge W(x)) \supset S(x)) \\ P(Mary)\neg S(Mary)\end{array}$$

Then we transform the formula in prenex normal form

$$\begin{array}{r}\forall x \exists y(P(x) \supset Q(y)) \\ \forall x \forall y(Q(x) \supset W(y)) \\ \forall x(P(x) \wedge W(x)) \supset S(x)) \\ P(Mary) \\ \neg S(Mary)\end{array}$$

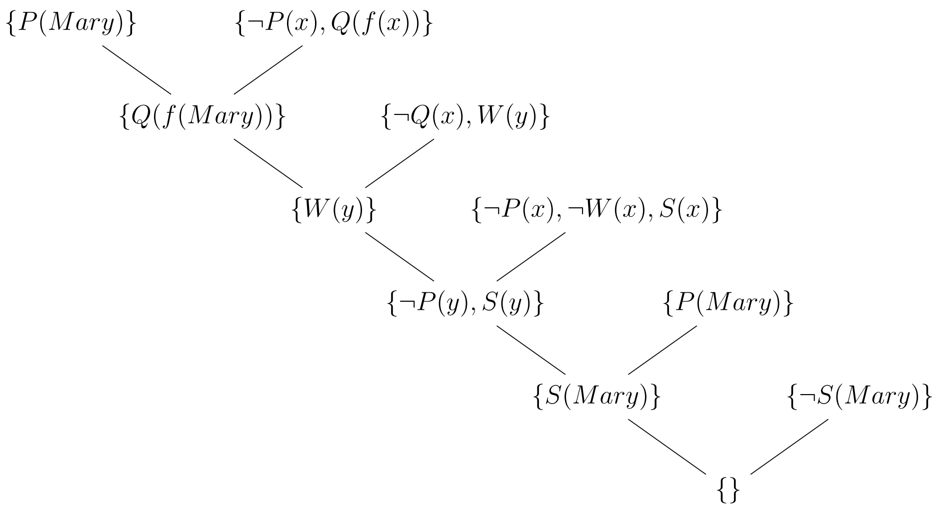
we then skolemize (only the first formula)

$$\begin{array}{r}\forall x(P(x) \supset Q(f(x))) \\ \forall x \forall y(Q(x) \supset W(y)) \\ \forall x(P(x) \wedge W(x)) \supset S(x)) \\ P(Mary) \\ \neg S(Mary)\end{array}$$

then we transform in clausal form

$$\begin{aligned} \{\neg P(x), Q(f(x))\} \\ \{\neg Q(x), W(y)\} \\ \{\neg P(x), \neg W(x), S(x)\} \\ \{P(Mary)\} \\ \{\neg S(Mary)\} \end{aligned}$$

and then we apply resolution:



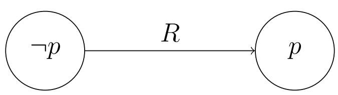
Exercise 5 (Modal logics syntax and semantics)**. [6 points]**

For each of the following formulas, show that it is valid or if not find a countermodel, i.e., a model $\mathcal{M} = \langle \mathcal{F}, \mathcal{I} \rangle$ with $\mathcal{F} = \langle \mathcal{W}, \mathcal{R} \rangle$ and a world $w \in \mathcal{W}$ such that ,$w \not\models \phi$.

1. $\Diamond p \supset p$
2. $\Box p \wedge \neg \Box \bot \supset \Diamond p$
3. $\Diamond q \supset \neg \Diamond \neg q$

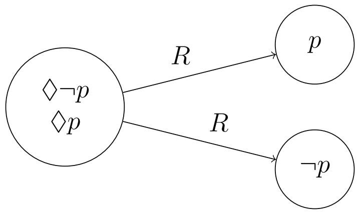
Solution.

1. *$\Diamond p \supset p$ is not valid. The following model does not satisfies it*



2. $\Box p \wedge \neg\Box\bot \supset \Diamond p$ *is valid. Indeed if* $\mathcal{M}, w \models \neg\Box\bot$ *then there must be* w' *with* wRw' *and the fact that* $\mathcal{M}, w \models \Box p$ *implies that* $\mathcal{M}, w' \models p$*, which implies that* $\mathcal{M}, w \models \Diamond p$*.*

3. $\Diamond q \supset \neg\Diamond\neg q$ *is not valid indeed the following is a counter-model:*



Exercise 6 (Modal logics Modal axioms)**. [6 points]**
For <u>one</u> of the following axiom schemata S (choose the one you like), prove that

$$\mathcal{F} \models \mathsf{S} \text{ if and only if } \mathcal{F} \text{ has the property } P$$

you also have to say which is the property P.

(D): $\Box\phi \supset \Diamond\phi$

(T): $\Box\phi \supset \phi$

(B): $\phi \supset \Box\Diamond\phi$

(4): $\Box\phi \supset \Box\Box\phi$

(5): $\Diamond\phi \supset \Box\Diamond\phi$

Mathematical logic

– 1^{st} assessment – Propositional Logic –

23 October 2013

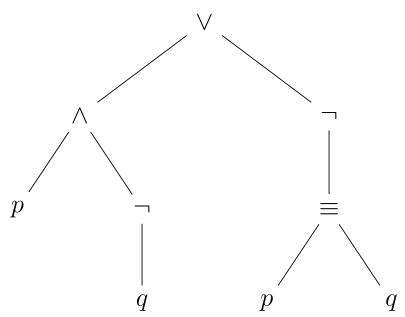
Exercise 1. [3 points] Consider the following formula

$$(p \wedge \neg q) \vee \neg(p \equiv q)$$

1. Write the formula as a tree, and
2. list all its sub-formulae

Solution.

1.



2.
   - $(p \wedge \neg q) \vee \neg(p \equiv q)$
   - $p \wedge \neg q$
   - p
   - $\neg q$
   - q
   - $\neg(p \equiv q)$
   - $p \equiv q$

Exercise 2. [6 points]:

1. Translate the following natural language sentences into propositional logic formulas:
   (a) Claudia gets a pay rise if she acquires a new customer or if she acquires a new project
   (b) Claudia does not acquire a new customer, however she gets a pay rise
   (c) Claudia acquires a new project
2. say whether (c) is a logical consequence of (a) and (b) using the truth tables and motivate your answer.

Solution.

1. Let

   R = Claudia gets a pay rise
   C = Claudia acquires a new customer
   P = Claudia acquires a new project

   A possible formalization is the following:

   (a) $(C \vee P) \supset R$
   (b) $\neg C \wedge R$
   (c) P

2. The truth table for the formulae above is:

		(c)		(b)		(a)
R	C	P	$\neg C$	$\neg C \wedge R$	$C \vee P$	$(C \vee P) \supset R$
T	T	T	F	F	T	T
T	T	F	F	F	T	T
T	F	T	T	T	T	T
T	F	F	T	T	F	T
F	T	T	F	F	T	F
F	T	F	F	F	T	F
F	F	T	T	F	T	F
F	F	F	T	F	F	T

   As we can see from this truth table there are only two assignments that satisfy both premises: the ones in row 3 and 4. One of them (row 4) satisfies both (a) and (b) but does not satisfy (c). Therefore (c) is not a logical consequence of (a) and (b).

Exercise 3. [6 points] Let

(**MyRule1**) $$\frac{\phi \vee \psi \quad \neg\phi}{\psi}$$

and

(**MyRule2**) $$\frac{\phi \supset \psi \quad \neg\phi}{\neg\psi}$$

be two reasoning rules used to build proofs. Say (and prove) whether (**MyRule1**) and (**MyRule2**) are rules that preserve validity (i.e, that transform valid formulae in valid formulae).

Solution.

Proof.

- Let us consider (**MyRule1**).

  We have to prove that if $\phi \vee \psi$ and $\neg\phi$ are valid formulae, then ψ is a valid formula.

  Let us assume that $\phi \vee \psi$ and $\neg\phi$ are valid formulae. Then for each propositional interpretation $\mathcal{I}$ we have that $\mathcal{I} \models \phi \vee \psi$ and $\mathcal{I} \models \neg\phi$. From the definition of satisfiability of $\vee$, we have that $\mathcal{I} \models \phi$ or $\mathcal{I} \models \psi$. Since $\mathcal{I} \models \neg\phi$, and therefore $\mathcal{I} \not\models \phi$, then we can conclude that $\mathcal{I} \models \psi$. Thus ψ is a valid formula and (**MyRule1**) is a sound rule which preserves validity.

- Let us consider (**MyRule2**). Let p, q two propositional atoms, and let

  - $\phi = p \wedge \neg p$
  - $\psi = q$

  It is easy to prove that both $(p \wedge \neg p) \supset q$, and $\neg(p \wedge \neg p)$ are valid formulae. (can be done with the truth tables for instance)

  If we use them with the rule (**MyRule2**) we obtain $\neg q$ which is not a valid formula.

  Indeed let $\mathcal{I}$ be the interpretation $\mathcal{I} = q$. This interpretation does not satisfy $\neg q$.

  Thus rule (**MyRule2**) does not preserve validity.

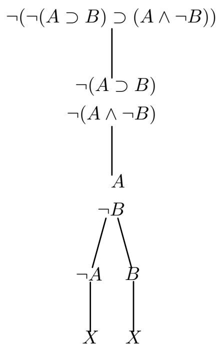
$\square$

Exercise 4. [6 points] For each of the following formulae determine whether they are **valid**, **unsatisfiable**, or **satisfiable** (and not valid) using analytic tableaux. Report the tableau, and use it to justify your answer.

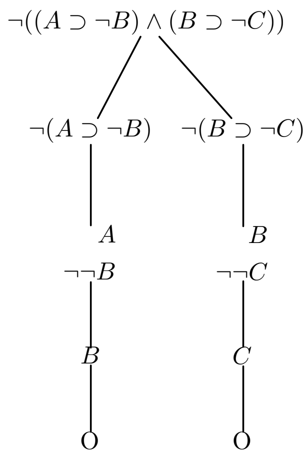
1. $\neg(A \supset B) \supset (A \wedge \neg B)$
2. $(A \supset \neg B) \wedge (B \supset \neg C)$

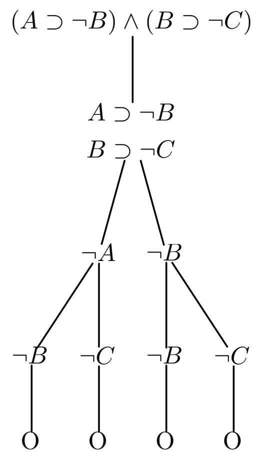
Solution.

1. The formula $\neg(A \supset B) \supset (A \wedge \neg B)$ is Valid. In fact, the tableau for its negated version is the closed tableau reported below



2. $(A \supset \neg B) \wedge (B \supset \neg C)$ is Satisfiable (and not valid). In fact, it is not valid, as shown by the tableau on the left hand side below, which remains open, and it is satisfiable, as shown by the tableau on the right hand side, which has four open branches and therefore shows at least four interpretations that make the formula true.





Exercise 5. [3 points] Apply DPLL procedure to check if the following set of clauses is satisfiable, and if it is so, return a partial assignment that makes the fomula true.

$$\phi = \{\{A, \neg B, \neg D\}, \{\neg A, \neg B, \neg C\}, \{\neg A, C, \neg D\}, \{\neg A, B, C\}\}$$

In the solution you have to specify all the application of unit propagation rule, and all the choices you take when Unit propagation is not applicable.

Solution.

1. ϕ does not contain unit clause, which implies that unit propagation is not applicable.
2. therefore, we select a literal (say A) and set $\mathcal{I}(A) = true$
3. Compute $\phi|_A$:
$$\phi|_A = \{\{\neg B, \neg C\}, \{C, \neg D\}, \{B, C\}\}$$
4. $\phi|_A$ does not contain unit clauses, therefore unit propagation is not applicable.
5. select a second literal, say $\neg B$, and set $\mathcal{I}(B) = false$
6. Compute $(\phi|_A)|_{\neg B}$ (also denoted by $\phi|_{A,\neg B}$).
$$\phi|_{A,\neg B} = \{\{C, \neg D\}, \{C\}\}$$
7. $\phi|_{A,\neg B}$ contain the unit clause $\{C\}$, we therefore extend the partial interpretation with $\mathcal{I}(C) = True$. We then apply unit propagation with $\{C\}$ as unit clause, obtaining $\phi|_{A,\neg B,C} = \{\}$, the empty set of clauses. Which means that the initial formula is satisfiable. The partial assignment is $\mathcal{I}(A) = True$, $\mathcal{I}(B) = false$ and $\mathcal{I}(C) = true$

Exercise 6. [3 points] Say when a formula ϕ is equi-satisfiable of a formula ψ. and show that the two formulas:

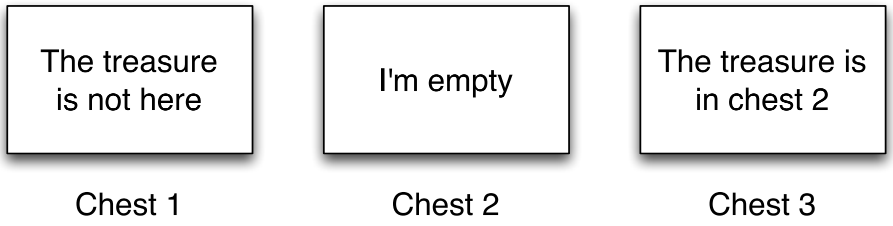
$$\phi = A \rightarrow (B \vee C) \qquad \psi = (N \equiv (B \vee C)) \wedge (A \rightarrow N) \tag{3}$$

are equi-satisfiable.

Solution. ϕ and ψ are equi-satisfiable, if and only if ϕ is satisfiable iff ψ is satisfiable. Or in other words, there is an interpretation $\mathcal{I}$ that satisfies ϕ if and only if there is an interpretation $\mathcal{J}$ that satisfies ψ.

Let us shows that the formulas in (3) are equisatisfiable. Let $\mathcal{I} \models \phi$. Let's extend $\mathcal{I}$ to $\mathcal{I}'$ setting $\mathcal{I}'(N) = \mathcal{I}(B \vee C)$. We have that $\mathcal{I}' \models \psi$. Viceersa, let $\mathcal{I}$ be an interpretation that satisfies ψ, then $\mathcal{I} \models N \equiv (B \vee C)$ implies that $\mathcal{I}(N) = \mathcal{I}(B \vee C)$. The fact that $\mathcal{I} \models A \rightarrow N$ impies that $\mathcal{I} \models A \rightarrow (B \vee C)$.

Exercise 7. [**6 points**] In her travels for treasure hunting, Chiara finds herself in front of three mysterious chests. In one of the chests is a fabulous treasure, all the others are empty. On each chest there is an inscription:



Given the fact that two chests are lying, and one is telling the truth, where is the treasure?

Solution. Let us define the following language:

- $t1$ = the treasure is in chest 1;
- $t2$ = the treasure is in chest 2;
- $t3$ = the treasure is in chest 3;

we can encode the knowledge we have as follows:

(a) "In one of the chests is a fabulous treasure, all the others are empty"

$$(t1 \wedge \neg t2 \wedge \neg t3) \vee (\neg t1 \wedge t2 \wedge \neg t3) \vee (\neg t1 \wedge \neg t2 \wedge t3)$$

(b) the sentence of chest 1: "the treasure is not here"

$$\neg t1$$

(c) the sentence of chest 2: "I'm empty"

$$\neg t2$$

(d) the sentence of chest 3: "the treasure is in chest 2"

$$t2$$

(e) "two chests are lying and one is telling the truth".

$$(\neg t1 \wedge \neg\neg t2 \wedge \neg t2) \vee (\neg\neg t1 \wedge \neg t2 \wedge \neg t2) \vee (\neg\neg t1 \wedge \neg\neg t2 \wedge t2)$$

This sentence can be simplified as follows:

$$(t1 \wedge \neg t2) \vee (t1 \wedge t2)$$

In building the truth tables for $t1$–$t3$ we can consider the combinations in which exactly one is true, to satisfy item (a). We find that row 1 is the only one that satisfies the sentences inscribed on all chests and also the requirement that one chest is telling the truth and two are lying. This row tells us that the treasure is in chest 1.

			(b)	(c)	(d)			(e)
$t1$	$t2$	$t3$	$\neg t1$	$\neg t2$	$t2$	$t1 \wedge \neg t2$	$t1 \wedge t2$	$(t1 \wedge \neg t2) \vee (t1 \wedge t2)$
T	F	F	F	T	F	T	F	T
F	T	F	T	F	T	F	F	F
F	F	T	T	T	F	F	F	F

Mathematical logic

– 2^{nd} assessment – First Order Logic and Modal Logic –

23 October 2013

Instructions

- Answer in English and write in ink unless the question paper gives other instructions.
- Write clearly; illegible answers will not be marked.
- Take care to identify each answer clearly with:
  - the number of the exercise.
  - where appropriate, the part of the exercise you are answering.
- Clearly cross out rough working, or unwanted answers before handing in your answers.

Exercise 1 (FOL syntax)**. [6 points]**
Let Σ be the signature that contains

- the constant symbols *alice*, *bob* and *carol*
- the functional symbols $father$ and $firstCommonMaleAncestor$ with arity 1 and 2, respectively
- the predicate symbols *Student* and $Friend$ with arity 1 and 2, respectively

For each of the following expression say:

- if it is a term, a formula, or none of the two
- if it is a formula say if it is closed and if not what are the free variables
- If it is a term say if it is a ground term
- in case it is a term or a formula provide it's intuitive reading

1. $\forall Student.friend(alice, Student)$
2. $firstCommonMaleAncestor(father(alice), father(father(bob))) = carol$

3. $\forall x \forall y (father(x) = father(y) \supset firstCommonMaleAncestor(x, y) = father(x))$
4. $father(alice) = carol \vee bob$
5. $\exists y (friend(x, father(y)) \wedge friend(x, y))$
6. $firstCommonMaleAncestor(father(alice), x)$

Solution. *We summarize the result in the following table*

expr	**term formula or nothing**	**closed or open**	**free variables**	**bound variables**	**ground yes or no**	**intuitive reading**
1	*nothing*					
2	*formula*	*closed*	$\{\}$	$\{\}$	*yes*	*Carol is the first male ancestor of the father of alice and the grandfather of Bob*
3	*formula*	*closed*	$\{\}$	$\{x, y\}$	*no*	*If x and y have the same father, then he is their first common male ancestor*
4	*nothing*					
5	*formula*	*open*	$\{x\}$	$\{y\}$	*no*	*the set of person who are friend of a person and his/her father*
6	*term*				*no*	*the first common male ancestor of x and the father of alice*

Exercise 2 (Semantics)**. [6 points]**

For each of the following formulas

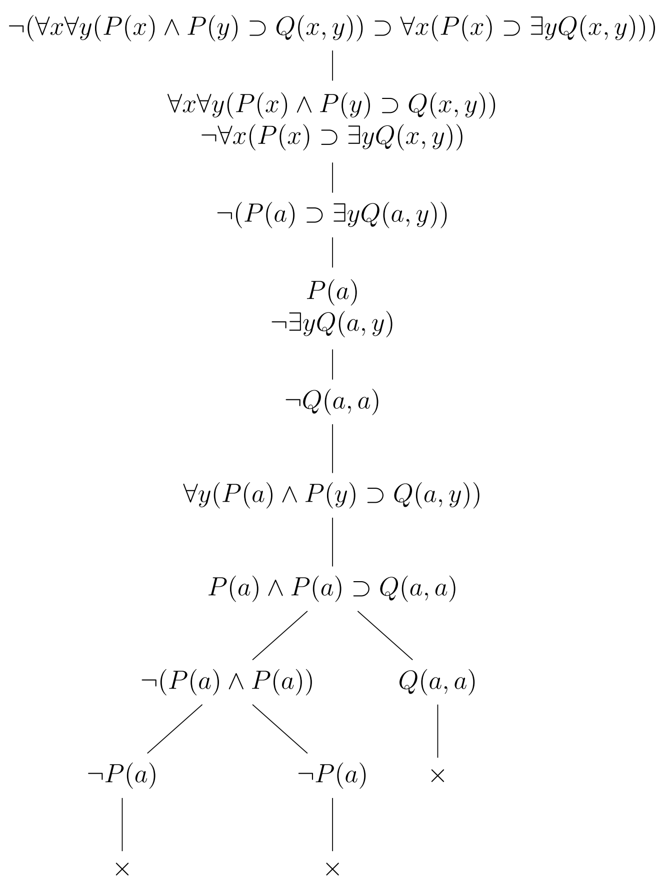
- if it is valid prove it via tableaux,
- if it is satisfiable but not valid provide a counter-model, i.e. a model that falsifies it

$$\forall x \forall y (P(x) \wedge P(y) \supset Q(x,y)) \supset \forall x (P(x) \supset \exists y Q(x,y)) \quad (1)$$

$$\forall x \exists y P(x,y) \supset \forall y \exists x P(x,y) \quad (2)$$

$$\neg (P(a,b) \equiv \exists xy P(x,y)) \quad (3)$$

Solution. *The first formula is valid as we can build the following closed tableaux for it's negation.*



The second formula is not valid. A counter-model is $\mathcal{I} = \left\langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \right\rangle$ *with*

- $\Delta^{\mathcal{I}} = \{1, 2\}$,
- $P^{\mathcal{I}} = \{\langle 1, 2\rangle, \langle 2, 2\rangle\}$

We have that $\mathcal{I} \models \forall x \exists y P(x, y)$ *since for every assignment* $d \in \Delta^{\mathcal{I}}$ *to* x *we have that* $\mathcal{I} \models \exists y P(x, y)[a(x) = d]$. *Indeed we have that*

- $\mathcal{I} \models \exists y P(x, y)[a(x) := 1]$ *since there the assignment to* y *i.e.,2, such that* $\mathcal{I} \models P(x, y)[a(x) := 1, a(y) := 2]$, *and*
- $\mathcal{I} \models \exists y P(x, y)[a(x) := 2]$ *since there the assignment* $a(y) := 2$ *such that* $\mathcal{I} \models P(x, y)[a(x) := 2, a(y) := 2]$

On the other hand we have that $\mathcal{I} \not\models \forall y \exists x P(x, y)$ *because for the assignment* $[y := 1]$ *there is no assignment to* x *such that* $\mathcal{I} \models P(x, y)[y := 1]$. *(Notice that there is no tuple of the form* $\langle \cdot, 1\rangle \in P^{\mathcal{I}}$*).*
Finally the third formula is also not valid and a countermodel is the following

$$\mathcal{I} = \left\langle \Delta^{\mathcal{I}} = \{1\},\ a^{\mathcal{I}} = 1,\ b^{\mathcal{I}} = 1,\ P^{\mathcal{I}} = \{\langle 1, 1\rangle\} \right\rangle$$

Notice that $\mathcal{I} \models P(a, b) \equiv \exists x \exists y P(x, y)$ *since there is an assignment to* x *and* y *such that:* $\mathcal{I} \models P(a, b)$ *if and only if* $\mathcal{I} \models P(x, y)[a(x) := 1, a(y) = 1]$. *This implies that* $\mathcal{I} \not\models \neg(P(a, b) \equiv \exists x \exists y P(x, y))$.

Exercise 3 (Modelling)**. [3 points]**
Transform in FOL the following sentences:

1. Lions are feline and feline are animals
2. Simba is a Lion and there are exactly two animals which Simba cannot eat
3. There is a lion who eats exactly every animal that is not eaten by Simba

Solution.

1. *Lions are feline and feline are animals*

$$\forall x(Lion(x) \supset Feline(x)) \land \forall x(Feline(x) \supset Animal(x))$$

2. *Simba is a Lion and there are exactly two animals which Simba cannot eat*

$$Lion(Simba) \land \exists xy(x \neq y \land \neg Eats(Simba, x) \land \neg Eats(Simba, y) \land \forall z(z \neq x \land z \neq y \supset Eats(Simba, z)))$$

3. *There is a lion who eats exactly every animal that is not eaten by Simba*

$$\exists x(Lion(x) \land \forall y(Animal(y) \supset (Eats(Simba, y) \equiv \neg Eats(x, y))))$$

Exercise 4 (Resolution and Unification)**. [6 points]**

Use resolution and unification to solve the problem below.

Given:

$$\forall x(P(x) \supset \exists y Q(y)) \tag{4}$$

$$\neg\exists x(Q(x) \wedge \exists y \neg W(y)) \tag{5}$$

$$\forall x(P(x) \wedge W(x) \supset S(x)) \tag{6}$$

$$P(Mary) \tag{7}$$

Show:

$$S(Mary) \tag{8}$$

Solution. *To show that* (8) *logically follows from* (4)–(7)*, i.e., that* (4)–(7) $\models$ (8) *we have to prove that the set* $S = \{(4), (5), (6), (7), \neg(8)\}$ *is not satisfbiable. I.e., that we can derive the empty clause via resolution from the transformation in clause of* S

First we add the negation of the consequence to be prove to the formulas and transform them in NNF by pushing inside the $\neg$ *symbol obtaining (only the second formula)*

$$\forall x(P(x) \supset \exists y Q(y))$$
$$\forall x(\neg Q(x) \vee \forall y W(y))$$
$$\forall x(P(x) \wedge W(x)) \supset S(x))$$
$$P(Mary)\neg S(Mary)$$

Then we transform the formula in prenex normal form

$$\forall x \exists y(P(x) \supset Q(y))$$
$$\forall x \forall y(Q(x) \supset W(y))$$
$$\forall x(P(x) \wedge W(x)) \supset S(x))$$
$$P(Mary)$$
$$\neg S(Mary)$$

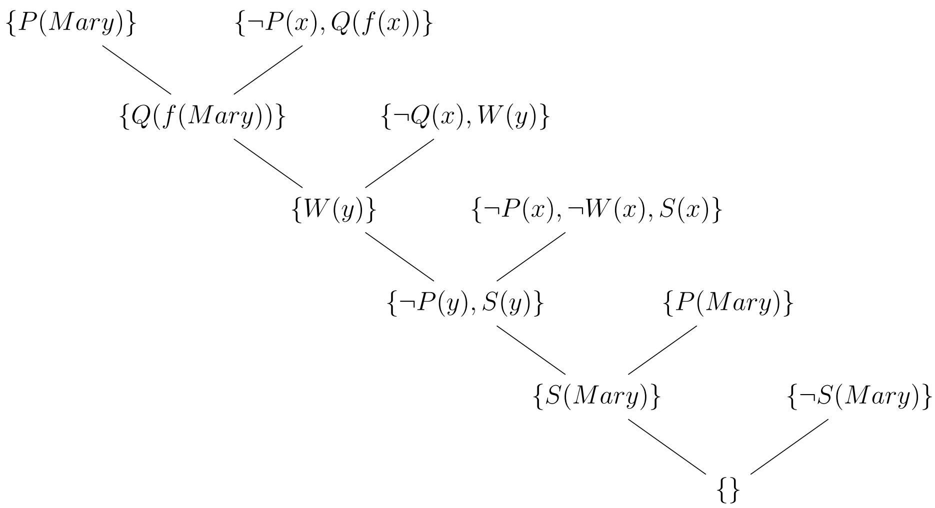
we then skolemize (only the first formula)

$$\forall x(P(x) \supset Q(f(x)))$$
$$\forall x \forall y(Q(x) \supset W(y))$$
$$\forall x(P(x) \wedge W(x)) \supset S(x))$$
$$P(Mary)$$
$$\neg S(Mary)$$

then we transform in clausal form

$$\begin{aligned}
\{\neg P(x), Q(f(x))\} \\
\{\neg Q(x), W(y)\} \\
\{\neg P(x), \neg W(x), S(x)\} \\
\{P(Mary)\} \\
\{\neg S(Mary)\}
\end{aligned}$$

and then we apply resolution:



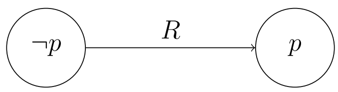
Exercise 5 (Modal logics syntax and semantics)**. [6 points]**

For each of the following formulas, show that it is valid or if not find a countermodel, i.e., a model $\mathcal{M} = \langle \mathcal{F}, \mathcal{I} \rangle$ with $\mathcal{F} = \langle \mathcal{W}, \mathcal{R} \rangle$ and a world $w \in \mathcal{W}$ such that $,w \not\models \phi$.

1. $\Diamond p \supset p$
2. $\Box p \wedge \neg \Box \bot \supset \Diamond p$
3. $\Diamond q \supset \neg \Diamond \neg q$

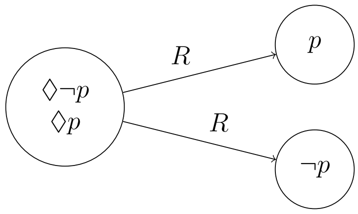
Solution.

1. *$\Diamond p \supset p$ is not valid. The following model does not satisfies it*



2. $\Box p \wedge \neg\Box\bot \supset \Diamond p$ *is valid. Indeed if* $\mathcal{M}, w \models \neg\Box\bot$ *then there must be* w' *with* wRw' *and the fact that* $\mathcal{M}, w \models \Box p$ *implies that* $\mathcal{M}, w' \models p$*, which implies that* $\mathcal{M}, w \models \Diamond p$.

3. $\Diamond q \supset \neg\Diamond\neg q$ *is not valid indeed the following is a counter-model:*



Exercise 6 (Modal logics Modal axioms)**.** [**6 points**]
For <u>one</u> of the following axiom schemata S (choose the one you like), prove that

$$\mathcal{F} \models (\mathsf{S}) \text{ if and only if } \mathcal{F} \text{ has the property } P_{(\mathsf{S})}$$

you also have to say which is the property $P_{(\mathsf{S})}$.

(D): $\Box\phi \supset \Diamond\phi$

(T): $\Box\phi \supset \phi$

(B): $\phi \supset \Box\Diamond\phi$

(4): $\Box\phi \supset \Box\Box\phi$

(5): $\Diamond\phi \supset \Box\Diamond\phi$

Solution. *We analyze each axiom schema separately. For each of axiom schema we prove*

Soundness**:** *If* $\mathcal{F}$ *is a frame that satisfies the property* $P_{(\mathsf{S})}$*, then* (S) *is a valid formula in* $\mathcal{F}$*.*

Completeness**:** *If* (S) *is a valid formula in a frame* $\mathcal{F}$*, then* $\mathcal{F}$ *is a frame that satisfies the property* $P_{(\mathsf{S})}$*. For the completeness we prove the (equivalent) contropositive statement, i.e., that if* $\mathcal{F}$ *does not satisfy the property* $P_{(\mathsf{S})}$ *then* (S) *is not valid in* $\mathcal{F}$*. We do this by building a countermodel* $\mathcal{M} = \langle F, V\rangle$ *for* (S)*, by providing an assignment* V *to propositional variables on* $\mathcal{F}$*, and by selecting a world of* w *in* $\mathcal{F}$ *so that* $\mathcal{M}, w \not\models$ (S)*.*

(D): $\Box\phi \supset \Diamond\phi$ $\quad P_{(\mathsf{S})}$ *is equal to* Seriality*, i.e.,* $\forall w \in W,\ \exists w' \in W$ *s.t.* wRw'*.*

***Soundness*:** *Let* $\mathcal{M}$ *be a model on a serial frame* $\mathcal{F} = \langle W, R\rangle$ *and* w *any world in* W*. We prove that* $\mathcal{M}, w \models \Box\phi \supset \Diamond\phi$*.*

1. *Since* R *is serial there is a world* $w' \in W$ *with* wRw'
2. *Suppose that* $\mathcal{M}, w \models \Box\phi$ *(Hypothesis)*
3. *From the satisfiability condition of* $\Box$*,* $\mathcal{M}, w \models \Box\phi$ *implies that* $\mathcal{M}, w' \models \phi$
4. *Since there is a world* w' *accessible from* w *that satisfies* ϕ*, from the satisfiability conditions of* $\Diamond$ *we have that* $\mathcal{M}, w \models \Diamond\phi$ *(Thesis) .*
5. *Since from (Hypothesis) we have derived (Thesis), we can conclude that* $\mathcal{M}, w \models \Box\phi \supset \Diamond\phi$*.*

***Completeness*:** *Suppose that a frame* $\mathcal{F} = \langle W, R\rangle$ *is not Serial.*

1. *If* R *is not serial then there is a* $w \in W$ *which does not have any accessible world. I.e., for all* w' *it does not hold that* wRw'*.*
2. *Let* $\mathcal{M}$ *be any model on* $\mathcal{F}$*.*
3. *Form the satisfiability condition of* $\Box$ *and from the fact that* w *does not have any accessible world, we have that* $\mathcal{M}, w \models \Box\phi$*.*
4. *Form the satisfiability condition of* $\Diamond$ *and from the fact that* w *does not have any accessible world, we have that* $\mathcal{M}, w \not\models \Diamond\phi$*.*
5. *this implies that* $\mathcal{M}, w \not\models \Box\phi \supset \Diamond\phi$

(T): $\Box\phi \supset \phi$ $\quad P_{(\mathsf{S})}$ *is equal to* Reflexivity*, i.e.,* $\forall w \in W,\ wRw$*.*

***Soundness*:** *Let* $\mathcal{M}$ *be a model on a reflexive frame* $\mathcal{F} = \langle W, R\rangle$ *and* w *any world in* W*. We prove that* $\mathcal{M}, w \models \Box\phi \supset \phi$*.*

1. *Since* R *is reflexive then* wRw
2. *Suppose that* $\mathcal{M}, w \models \Box\phi$ *(Hypothesis)*
3. *From the satisfiability condition of* $\Box$*,* $\mathcal{M}, w \models \Box\phi$*, and* wRw *imply that* $\mathcal{M}, w \models \phi$ *(Thesis)*
4. *Since from (Hypothesis) we have derived (Thesis), we can conclude that* $\mathcal{M}, w \models \Box\phi \supset \phi$*.*

***Completeness*:** *Suppose that a frame* $\mathcal{F} = \langle W, R\rangle$ *is not reflexive.*

1. *If* R *is not reflexive then there is a* $w \in W$ *which does not access to itself. I.e., for some* $w \in W$ *it does not hold that* wRw*.*

2. *Let $\mathcal{M}$ be any model on $\mathcal{F}$, and let ϕ be the propositional formula p. Let V the set p true in all the worlds of W but w where p is set to be false.*
3. *From the fact that w does not access to itself, we have that in all the worlds w accessible from w, p is true, i.e, $\forall w',\ wRw',\ \mathcal{M}, w' \models p$.*
4. *Form the satisfiability condition of $\Box$ we have that $\mathcal{M}, w \models \Box p$.*
5. *since $\mathcal{M}, w \not\models p$, we have that $\mathcal{M}, w \not\models \Box p \supset p$.*

(B): $\phi \supset \Box\Diamond\phi$ $\quad P_{(\mathsf{S})}$ *is equal to* Symmetricity, *i.e.,* $\forall w, w' \in W$, *if* wRw' *then* $w'Rw$.

Soundness*: *Let $\mathcal{M}$ be a model on a symmetric frame $\mathcal{F} = \langle W, R\rangle$ and w any world in W. We prove that $\mathcal{M}, w \models \phi \supset \Box\Diamond\phi$.

1. *Suppose that $\mathcal{M}, w \models \phi$ (Hypothesis)*
2. *we want to show that $\mathcal{M}, w \models \Box\Diamond\phi$ (Thesis)*
3. *Form the satisfiability conditions of $\Box$, we need to prove that for every world w' accessible from w, $\mathcal{M}, w' \models \Diamond\phi$.*
4. *Let w', be any world accessible from w, i.e., wRw'*
5. *from the fact that R is symmetric, we have that $w'Rw$*
6. *From the satisfiability condition of $\Diamond$, from the fact that $w'Rw$ and that $\mathcal{M}, w \models \phi$, we have that $\mathcal{M}, w' \models \Diamond\phi$.*
7. *so for every world w' accessible from w, we have that $\mathcal{M}, w' \models \Diamond\phi$.*
8. *From the satisfiability condition of $\Box$, $\mathcal{M}, w \models \Box\Diamond\phi$ (Thesis)*
9. *Since from (Hypothesis) we have derived (Thesis), we can conclude that $\mathcal{M}, w \models \phi \supset \Box\Diamond\phi$.*

***Completeness*:** *Suppose that a frame $\mathcal{F} = \langle W, R\rangle$ is not Symmetric.*

1. *If R is not symmetric then there are two worlds $w, w' \in W$ such that wRw' and not $w'Rw$*
2. *Let $\mathcal{M}$ be any model on $\mathcal{F}$, and let ϕ be the propositional formula p. Let V the set p true in all the worlds of W but w where p is set to be false.*
3. *From the fact that w' does not access to w, it means that in all the worlds accessible from w', p is false,*
4. *i.e. there is no world w'' accessible from w' wuch that $\mathcal{M}, w'' \models p$.*
5. *by the satisfiability conditions of $\Diamond$, we have that $\mathcal{M}, w' \not\models \Diamond p$.*
6. *Since there is a world w' accessible from w, with $\mathcal{M}, w \not\models \Diamond p$, form the satisfiability condition of $\Box$ we have that $\mathcal{M}, w \not\models \Box\Diamond p$.*
7. *since $\mathcal{M}, w \models p$, and $\mathcal{M}, w \not\models \Box\Diamond p$. we have that $\mathcal{M}, w \not\models p \supset \Box\Diamond p$.*

(4): $\Box\phi \supset \Box\Box\phi$ $P_{(\mathsf{S})}$ *is equal to* Transitivity, *i.e.,* $\forall w, w', w'' \in W$, wRw' *and* $w'Rw''$ *implies that* wRw''

***Soundness*:** *Let* $\mathcal{M}$ *be a model on a transitive frame* $\mathcal{F} = \langle W, R\rangle$ *and* w *any world in* W. *We prove that* $\mathcal{M}, w \models \Box\phi \supset \Box\Box\phi$.

1. *Suppose that* $\mathcal{M}, w \models \Box\phi$ *(Hypothesis).*
2. *We have to prove that* $\mathcal{M}, w \models \Box\Box\phi$ *(Thesis)*
3. *From the satisfiability condition of* $\Box$, *this is equivalent to prove that for all world* w' *accessible from* w $\mathcal{M}, w' \models \Box\phi$.
4. *Let* w' *be any world accessible from* w. *To prove that* $\mathcal{M}, w' \models \Box\phi$ *we have to prove that for all the world* w'' *accessible from* w', $\mathcal{M}, w'' \models \phi$.
5. *Let* w'' *be a world accessible from* w', *i.e.,* $w'Rw''$.
6. *From the facts* wRw' *and* $w'Rw''$ *and the fact that* R *is transitive, we have that* wRw''.
7. *Since* $\mathcal{M}, w \models \Box\phi$, *from the satisfiability conditions of* $\Box$ *we have that* $\mathcal{M}, w'' \models \phi$.
8. *Since* $\mathcal{M}, w'' \models \phi$ *for every world* w'' *accessible from* w', *then* $\mathcal{M}, w' \models \Box\phi$.
9. *and therefore* $\mathcal{M}, w \models \Box\Box\phi$. *(Thesis)*
10. *Since from (Hypothesis) we have derived (Thesis), we can conclude that* $\mathcal{M}, w \models \Box\phi \supset \Box\Box\phi$.

***Completeness*:** *Suppose that a frame* $\mathcal{F} = \langle W, R\rangle$ *is not transitive.*

1. *If* R *is not transitive then there are three worlds* $w, w', w'' \in W$, *such that* wRw', $w'Rw''$ *but not* wRw''.
2. *Let* $\mathcal{M}$ *be any model on* $\mathcal{F}$, *and let* ϕ *be the propositional formula* p. *Let* V *the set* p *true in all the worlds of* W *but* w'' *where* p *is set to be false.*
3. *From the fact that* w *does not access to* w'', *and that* w'' *is the only world where* p *is false, we have that in all the worlds accessible from* w, p *is true.*
4. *This implies that* $\mathcal{M}, w \models \Box p$.
5. *On the other hand, we have that* $w'Rw''$, *and* $w'' \not\models p$ *implies that* $\mathcal{M}, w' \not\models \Box\phi$.
6. *and since* wRw', *we have that* $\mathcal{M}, w \not\models \Box\Box p$.
7. *In summary:* $\mathcal{M}, w \not\models \Box\Box p$, *and* $\mathcal{M}, w \models \Box P$; *from which we have that* $\mathcal{M}, w \not\models \Box p \supset \Box\Box p$.

(5): $\Diamond\phi \supset \Box\Diamond\phi$ $\quad P_{(\mathsf{S})}$ *is equal to* Euclideanity, *i.e.,* $\forall w, w', w'' \in W$, wRw' *and* wRw'' *implies that* $w'Rw''$

***Soundness*: *Let* $\mathcal{M}$ *be a model on a euclidean frame* $\mathcal{F} = \langle W, R\rangle$ *and* w *any world in* W*. We prove that* $\mathcal{M}, w \models \Diamond\phi \supset \Box\Diamond\phi$.

1. *Suppose that* $\mathcal{M}, w \models \Diamond\phi$ *(Hypothesis).*
2. *The satisfiability condition of* $\Diamond$ *implies that there is a world* w' *accessible from* w *such that* $\mathcal{M}, w' \models \phi$.
3. *We have to prove that* $\mathcal{M}, w \models \Box\Diamond\phi$ *(Thesis)*
4. *From the satisfiability condition of* $\Box$*, this is equivalent to prove that for all world* w'' *accessible from* w $\mathcal{M}, w'' \models \Diamond\phi$,
5. *let* w'' *be any world accessible from* w*. The fact that* R *is euclidean, the fact that* wRw' *implies that* $w''Rw'$.
6. *Since* $\mathcal{M}, w' \models \phi$*, the satisfiability condition of* $\Diamond$ *implies that* $\mathcal{M}, w'' \models \Diamond\phi$.
7. *and therefore* $\mathcal{M}, w \models \Box\Diamond\phi$*. (Thesis)*
8. *Since from (Hypothesis) we have derived (Thesis), we can conclude that* $\mathcal{M}, w \models \Box\phi \supset \Box\Diamond\phi$.

***Completeness*: *Suppose that a frame* $\mathcal{F} = \langle W, R\rangle$ *is not euclidean.*

1. *If* R *is not euclidean then there are three worlds* $w, w', w'' \in W$*, such that* wRw'*,* wRw'' *but not* $w'Rw''$.
2. *Let* $\mathcal{M}$ *be any model on* $\mathcal{F}$*, and let* ϕ *be the propositional formula* p*. Let* V *the set* p *false in all the worlds of* W *but* w' *where* p *is set to be true.*
3. *From the fact that* w'' *does not access to* w'*, and in all the other worlds* p *is false, we have that* $w'' \not\models \Diamond p$
4. *this implies that* $\mathcal{M}, w \not\models \Box\Diamond p$.
5. *On the other hand, we have that* wRw'*, and* $w' \models p$*, and therefore* $\mathcal{M}, w \models \Diamond p$. $\mathcal{M}, w \not\models \Box p \supset \Box\Box p$.
6. *In summary:* $\mathcal{M}, w \not\models \Box\Diamond p$*, and* $\mathcal{M}, w \models \Diamond P$*; from which we have that* $\mathcal{M}, w \not\models \Diamond p \supset \Box\Diamond p$.

Mathematical Logic Exam

22 January 2014

Instructions

- Answer in English and write in ink unless the question paper gives other instructions.
- Write clearly; illegible answers will not be marked.
- Take care to identify each answer clearly with:
  - the number of the exercise.
  - where appropriate, the part of the exercise you are answering.
- Clearly cross out rough working, or unwanted answers before handing in your answers.
- If you take the exam to recover one of the midterms, Please state clearly which part (Propositional Logic or First Order + Modal Logic) you intend to re-do. If you do not state this in an explicit manner, we will assume that you are taking the entire exam, and the midterm marks will not be taken into account anymore.

Propositional Logic

Exercise 1 (Theory)**. [5 points]**

Let A, B and C be propositional formulas. Show the following equivalence:

$$A, B \models C \text{ if and only if } A \models B \supset C$$

Solution. *We show here a semantic based solution.*

$A, B \models C$ **implies** $A \models B \supset C$ *Let, us assume that* $A, B \models C$ *as an hypothesis and show that* $A \models B \supset C$.

Let I *be an interpretation such that* $I \models A$. *There are two cases: either (i)* $I \models B$ *or (ii)* $I \models \neg B$. *If* $I \models B$, *then, from the hypothesis that* $A, B \models C$ *we can conclude that* $I \models C$, *and from the fact that If* $I \models B$ *and If* $I \models C$, *that* $I \models B \supset C$. *This ends the proof.*

$A \models B \supset C$ **implies** $A, B \models C$ *Let, us assume that* $A \models B \supset C$ *as an hypothesis and show that* $A, B \models C$.

Let I *be an interpretation such that* $I \models A$ *and* $I \models B$. *From the hypothesis that* $A \models B \supset C$ *we can infer that* $I \models B \supset C$, *and since* $I \models B$ *we can infer that* $I \models C$ *from the definition of satisfiability of* $\supset$. *This ends the proof.*

Exercise 2 (Tableaux). **[6 points]**

For each of the following formulae determine whether it is **valid**, **unsatisfiable**, or **satisfiable** (and not valid) using analytic tableaux. For each formula you have to provide the answer for all the options. Furthermore,

- if the formula is valid or unsatisfiable, provide a proper closed tableaux;
- if the formula is not valid but satisfiable, provide a model and a countermodel for the formula derived from the tableaux

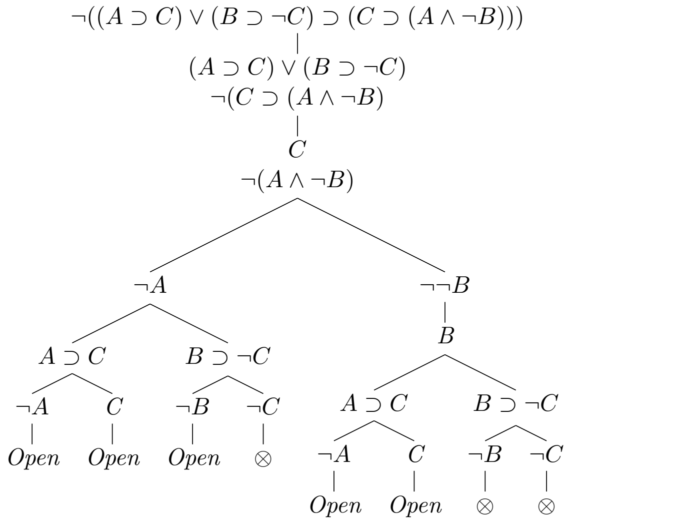
1. $(A \supset B) \supset (\neg A \vee B)$
2. $(A \supset C) \vee (B \supset \neg C) \supset (C \supset (A \wedge \neg B))$

Solution. *1. The formula* $(A \supset B) \supset (\neg A \vee B)$ *is valid. The closed tableaux is shown below*

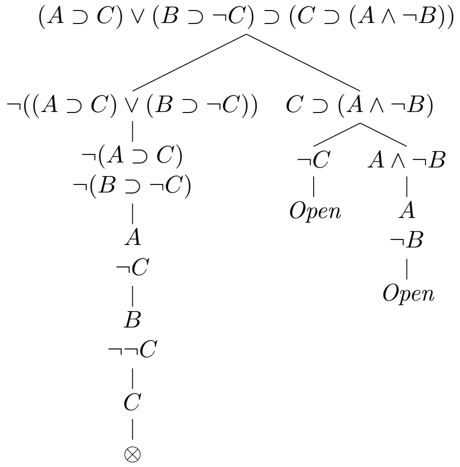
$$\neg((A \supset B) \supset (\neg A \vee B))$$
$$|$$
$$A \supset B$$
$$\neg(\neg A \vee B)$$
$$|$$
$$\neg\neg A$$
$$\neg B$$
$$|$$
$$A$$
$$\neg A \qquad B$$
$$| \qquad |$$
$$\otimes \qquad \otimes$$

2. The formula $(A \supset C) \vee (B \supset C) \supset (C \supset (A \wedge \neg B))$ *is not valid but satisfiable.*

A countermodel can be derived from the open tableau reported below. An example is $I = \{A, B, C\}$



A model that satisfies $(A \supset C) \vee (B \supset \neg C) \supset (C \supset (A \wedge \neg B))$ can be derived from one of the open branches of the tableau for this formula shown below.



An example is the model $I = \{A\}$.

Exercise 3 (Modeling). **[5 points]**

The Treasure Guardian

You are walking in a labyrinth and all of a sudden you find yourself in front of three mysterious chests: a gold chest, a marble chest, and a stones chest. Each chest is protected by a guardian. You talk to the guardians and this is what they tell you:

- The guardian of the gold chest: "This chest contains a treasure. Moreover, if the stones chest contains a treasure, then also the marble chest contains a treasure."

- The guardian of the marble chest: "Neither the gold nor the stones chests will lead you to the treasure."
- The guardian of the marble chest: "Follow the gold and youll reach the treasure, follow the marble and you will be lost."

Given that you know that all the guardians are liars, can you choose a chest being sure that you will find a treasure? If this is the case, which chest will you choose? Provide a propositional language and a set of axioms that formalize the problem and show whether you can chest a road being sure it will lead to the treasure.

Solution. *Variante banale di un esercizio risolto nel booklet di esercizi. La "trappola" che ci sono due modelli possibili, ma tutti e due dicono che stones ha un tesoro.*

First Order and Modal Logic

Exercise 4 (Modelling). **[6 points]**
Transform in FOL the following sentences:

1. The fathers of dogs are dogs.
2. There are at least two students enrolled in every course.
3. No region is part of each of two disjoint regions

Transform in Natural Language the following sentences:

1. $\forall x(Bag(x) \supset \exists y(Coin(y) \wedge Contains(x, y)))$
2. $\exists x(Telephone(x) \wedge \forall y(Secretary(y) \supset \neg Uses(x, y)))$
3. $\exists x(Buyer(x) \wedge Bought(x, TheScream) \wedge \forall y(Buyer(y) \wedge Bought(y, TheScream) \supset x = y))$ [1].

Solution.

1. *Tom is a car or a truck but cannot be both of them*

$$Car(Tom) \vee Truck(Tom) \wedge \neg(Car(Tom) \wedge Truck(Tom))$$

2. *The fathers of dogs are dogs*

$$\forall x(Dog(x) \supset Dog(father(x))$$

[1] TheScream is a famous painting

3. *There are at least two students enrolled in every course.*

   $\forall z \exists x \exists y (Course(z) \supset (Student(x) \wedge Student(y) \wedge \neg(x = y) \wedge Enrolled(x, z) \wedge Enrolled(y, z)))$

4. *No region is part of each of two disjoint regions.*

   $\neg \exists x \exists y \exists z (Reg(x) \wedge Reg(y) \wedge Reg(z) \wedge P(x, y) \wedge P(x, z) \wedge Disjoint(y, z))$

1. $\forall x (Bag(x) \supset \exists y (Coin(y) \wedge Contains(x, y)))$

   Every bag contains at least one coin.

2. $\exists x (Telephone(x) \wedge \forall y (Secretary(y) \supset \neg Uses(x, y)))$

   There is a telephone that is not used by any secretary

3. $\exists x (Buyer(x) \wedge Bought(x, TheScream) \wedge \forall y (Buyer(y) \wedge Bought(y, TheScream) \supset x = y))$

4. *Only one buyer bought The Scream.*

Exercise 5 (Resolution and Unification). **[5 points]**

Give the definition of *sound inference rule* and use Resolution to prove the soundness of the following rule

$$\frac{\begin{array}{c}\forall x(\exists y P(x, y) \vee \forall y \exists z Q(x, y, z)) \\ \forall xy(P(x, y) \rightarrow R(x) \vee R(y)) \\ \forall xyz(Q(x, y, z) \rightarrow R(x) \vee R(y) \vee R(z))\end{array}}{\exists x R(x)}$$

Solution. *An inference rule with n premises:*

$$\frac{\phi_1, \ldots, \phi_n}{\phi}$$

is sound if the consequence ϕ is a logical consequence of the premises $\{\phi_1, \ldots, \phi_n\}$, i.e., if $\{\phi_1, \ldots, \phi_n\} \models \phi$

Therefore we have to prove that $\exists x R(x)$ is a logical consequence of the set of formulas:

$$\left\{ \begin{array}{c} \forall x(\exists y P(x, y) \vee \forall y \exists z Q(x, y, z)) \\ \forall xy(P(x, y) \rightarrow R(x) \vee R(y)) \\ \forall xyz(Q(x, y, z) \rightarrow R(x) \vee R(y) \vee R(z)) \end{array} \right\}$$

Which is equivalent to show that the above set extended with the negation of $\exists x R(x)$ is unsatisfiable, i.e. that the set:

$$\left\{ \begin{array}{c} \forall xy(P(x, y) \rightarrow R(x) \vee R(y)) \\ \forall xyz(Q(x, y, z) \rightarrow R(x) \vee R(y) \vee R(z)) \\ \neg \exists x R(x) \end{array} \right\}$$

is unsatisfiable. To use the resolution method we have first to transform this set of formulas in set of clauses.

$$
\begin{array}{rl}
 & \forall x(\exists y P(x,y) \vee \forall y \exists z Q(x,y,z)) \\
\textit{Rename variables} & \forall x(\exists y P(x,y) \vee \forall w \exists z Q(x,w,z)) \\
\textit{Prenex form} & \forall x \exists y \forall w \exists z (P(x,y) \vee Q(x,w,z)) \\
\textit{Skolemization} & \forall x \forall w (P(x,f(x)) \vee Q(x,w,g(x,w))) \\
\textit{Clausal form} & \{P(x,f(x)),\ Q(x,w,g(x,w))\}
\end{array}
$$

$$
\begin{array}{rl}
 & \forall xy(P(x,y) \rightarrow R(x) \vee R(y)) \\
\textit{CNF} & \forall xy(\neg P(x,y) \vee R(x) \vee R(y)) \\
\textit{Clausal form} & \{\neg P(x,y),\ R(x),\ R(y)\}
\end{array}
$$

$$
\begin{array}{rl}
 & \forall xyz(Q(x,y,z) \rightarrow R(x) \vee R(y) \vee R(z)) \\
\textit{CNF} & \forall xyz(Q(x,y,z) \vee R(x) \vee R(y) \vee R(z)) \\
\textit{Clausal form} & \{Q(x,y,z),\ R(x),\ R(y),\ R(z)\}
\end{array}
$$

$$
\begin{array}{rl}
 & \exists x R(x) \\
\textit{Nagate} & \neg \exists x R(x) \\
\textit{CNF} & \forall x \neg R(x) \\
\textit{Clausal form} & \{\neg R(x)\}
\end{array}
$$

Now we can apply Resolution and unification to the set of clauses

$$
\left\{
\begin{array}{c}
\{P(x,f(x)),\ Q(x,w,g(x,w))\} \\
\{\neg P(x,y),\ R(x),\ R(y)\} \\
\{Q(x,y,z),\ R(x),\ R(y),\ R(z)\} \\
\{\neg R(x)\}
\end{array}
\right\}
$$

$$
\begin{array}{rrr}
(1) & \{P(x,f(x)),\ Q(x,w,g(x,w))\} & \\
(2) & \{\neg P(x,y),\ R(x),\ R(y)\} & \\
(3) & \{Q(x,y,z),\ R(x),\ R(y),\ R(z)\} & \\
(4) & \{\neg R(x)\} & \\
(5) & \{R(x),\ R(f(x)),\ Q(x,w,g(x,w))\} & \textit{from (1) and (2), } \sigma = [f(x)/y] \\
(6) & \{R(x),\ R(f(x)),\ R(w),\ R(g(x,w))\} & \textit{from (5) and (3), } \sigma = [w/y, g(x,w)/z] \\
(7) & \{R(f(x)),\ R(w),\ R(g(x,w))\} & \textit{from (6) and (4), } \sigma = [] \\
(8) & \{R(f(w)),\ R(g(w,w))\} & \textit{from (7) and (4), } \sigma = [w/x] \\
(8) & \{R(g(w,w))\} & \textit{from (8) and (4), } \sigma = [w/x] \\
(10) & \{\} & \textit{from (9) and (4), } \sigma = [g(w,w)]
\end{array}
$$

Exercise 6 (Modal logic axioms). **[5 points]**

Show that if $\Box\Box\phi \supset \Box\phi$ is valid in a frame $\mathcal{F} = \langle W, R\rangle$ then R is transitive.

Solution. *Suppose that a frame $\mathcal{F} = \langle W, R\rangle$ is not transitive. We show that the formula $\Box\phi \supset \Box\Box\phi$ is not valid for some ϕ.*

1. *If R is not transitive then there are three worlds $w, w', w'' \in W$, such that wRw', $w'Rw''$ but not wRw''.*
2. *Let $\mathcal{M}$ be any model on $\mathcal{F}$, and let ϕ be the propositional formula p. Let V the set p true in all the worlds of W but w'' where p is set to be false.*
3. *From the fact that w does not access to w'', and that w'' is the only world where p is false, we have that in all the worlds accessible from w, p is true.*
4. *This implies that $\mathcal{M}, w \models \Box p$.*
5. *On the other hand, we have that $w'Rw''$, and $w'' \not\models p$ implies that $\mathcal{M}, w' \not\models \Box\phi$.*
6. *and since wRw', we have that $\mathcal{M}, w \not\models \Box\Box p$.*
7. *In summary: $\mathcal{M}, w \not\models \Box\Box p$, and $\mathcal{M}, w \models \Box P$; from which we have that $\mathcal{M}, w \not\models \Box p \supset \Box\Box p$.*

Mathematical Logic Exam

13 February 2014

Instructions

- Answer in English and write in ink unless the question paper gives other instructions.
- Write clearly; illegible answers will not be marked.
- Take care to identify each answer clearly with:
  - the number of the exercise.
  - where appropriate, the part of the exercise you are answering.
- Clearly cross out rough working, or unwanted answers before handing in your answers.
- If you take the exam to recover one of the midterms, Please state clearly which part (Propositional Logic or First Order + Modal Logic) you intend to re-do. If you do not state this in an explicit manner, we will assume that you are taking the entire exam, and the midterm marks will not be taken into account anymore.

Propositional Logic

Exercise 1 (PL Theory)**.** **[6 points]** Let Γ a set of formulas and Σ a maximally consistent set of formulas. Show that either $\Gamma \subseteq \Sigma$ or $\Gamma \cup \Sigma \models \bot$.

Solution. *Suppose that* $\Gamma \not\subseteq \Sigma$ *then there is a formula* ϕ *such that* $\phi \in \Gamma$ *and* $\phi \notin \Sigma$*. The fact that* Σ *is maximally consistent implies that* $\neg\phi \in \Sigma$*, and therefore* $\{\phi, \neg\phi\} \subseteq \Gamma \cup \Sigma$ *From the fact that* $\{\phi, \neg\phi\} \models \bot$ *we can infer that* $\Gamma \cup \Sigma \models \bot$.

Exercise 2 (PL Modelling)**.** **[6 points]** Let $T = \langle V, E\rangle$ be a town, that contains a set $V = \{v_1, \dots, v_n\}$ of points of interests and a set $E \subseteq V \times V$ of streets that connect points of interests. The pair $\langle v, w\rangle \in E$ if and only if there is a street that connects point v with point w. Let v_1 and v_n be tow points of interests: write a set of formulas Φ such that, from every assignment $\mathcal{I}$ that satisfies Φ you can extract a single path starting from v_1 and ending in v_n. A path is a sequence of adjacent streets (a street $\langle v, w\rangle$ is adjacent to a street $\langle v', w'\rangle$ if $w = v'$).

Suggestion: use the following set of propositional variables:

- e_{ij} that means that the path goes through the street $\langle v_i, v_j \rangle$.
- p_{ij} that means that the path pass through v_i and then through v_j

Solution. *The set Φ contains the following formulas:*

- p_{1n}
- $e_{ij} \rightarrow \neg e_{ij'}$ *for $j \neq j'$, if in v_i you take the street $\langle v_i, v_j \rangle$ then you don't go in all the other streets starting from v_i.*
- $p_{ij} \equiv e_{ij} \vee \bigvee_{k=1}^{n}(e_{ik} \wedge p_{kj})$ *a path from v_i to v_j is either a street that directly connects v_i with v_j, or a street that connect v_i to some other point v_k and a path from that point to v_j.*
- $\bigwedge_{\langle v_i, v_j \rangle \notin E} \neg e_{ij}$, *you can only go on streets, i.e., if there is no street from v_i to v_j then you cannot take it.*

Exercise 3 (PL Reasoning). **[6 points]** Apply DPLL procedure to check if the following set of clauses is satisfiable, and if it is so, return a partial assignment that makes the fomula true.

$$\phi = \{\{A, B, D\}, \{\neg A, B, \neg C\}, \{\neg A, C, D\}, \{\neg A, \neg B, C\}\}$$

In the solution you have to specify all the application of unit propagation rule, and all the choices you take when Unit propagation is not applicable.

Solution.

1. *ϕ does not contain unit clause, which implies that unit propagation is not applicable.*
2. *therefore, we select a literal (say A) and set $\mathcal{I}(A) = true$*
3. *Compute $\phi|_A$:*
$$\phi|_A = \{\{B, \neg C\}, \{C, D\}, \{\neg B, C\}\}$$
4. *$\phi|_A$ does not contain unit clauses, therefore unit propagation is not applicable.*
5. *select a second literal, say B, and set $\mathcal{I}(B) = True$*
6. *Compute $(\phi|_A)|_B$ (also denoted by $\phi|_{A,B}$).*
$$\phi|_{A,B} = \{\{C, D\}, \{C\}\}$$
7. *$\phi|_{A,B}$ contain the unit clause $\{C\}$, we therefore extend the partial interpretation with $\mathcal{I}(C) = True$. We then apply unit propagation with $\{C\}$ as unit clause, obtaining $\phi|_{A,B,C} = \{\}$, the empty set of clauses. Which means that the initial formula is satisfiable. The partial assignment is $\mathcal{I}(A) = True$, $\mathcal{I}(B) = True$ and $\mathcal{I}(C) = True$*

Exercise 4 (FOL Theory). **[6 points]** Suppose that a first order language L contains only the set of constants $\{a, b, c\}$ and no functional symbols, and and the unary predicate symbol P.

Say if the following formula is valid, i.e., true in all interpretations. If it is valid give a proof of it's validity, you can choose any method; if it is not valid provide a counter-model.

$$P(a) \wedge P(b) \wedge P(c) \supset \forall x P(x)$$

Solution. *The formula is not valid, just consider the interpretation* $\mathcal{I} = \langle \Delta^I, \cdot^{\mathcal{I}} \rangle$, *with*

- $\Delta^{\mathcal{I}} = \{1, 2, 3, 4\}$, $a^{\mathcal{I}} = 1$, $b^{\mathcal{I}} = 2$, $c^{\mathcal{I}} = 3$, *and* $P^{\mathcal{I}} = \{1, 2, 3\}$.

We have that $\mathcal{I} \models P(a) \wedge P(b) \wedge P(c)$ *but* $\mathcal{I} \not\models \forall x P(x)$ *since* $\mathcal{I} \not\models P(x)[x := 4]$.

Exercise 5 (FOL tableaux). **[6 points]** Show by means of tableaux that the following formula is valid:

$$\forall xyz(R(x, y) \wedge R(x, z) \supset R(y, z)) \supset \forall x(\exists w R(w, x) \supset R(x, x))$$

Solution.

$$\neg(\forall xyz(R(x, y) \wedge R(x, z) \supset R(y, z)) \supset \forall x(\exists w R(w, x) \supset R(x, x)))$$

$$\forall xyz(R(x, y) \wedge R(x, z) \supset R(y, z))$$
$$\neg\forall x(\exists w R(w, x) \supset R(x, x))$$

$$\neg(\exists w R(w, a) \supset R(a, a))$$

$$\exists w R(w, a)$$
$$\neg R(a, a)$$

$$R(b, a)$$

$$R(b, a) \wedge R(b, a) \supset R(a, a)$$

Left branch: $\neg(R(b, a) \wedge R(b, a))$, which branches into $\neg R(b, a)$ — $\times$ and $\neg R(b, a)$ — $\times$

Right branch: $R(a, a)$ — $\times$

Exercise 6 (Modal logics). **[6 points]** For the following formula either prove that it is valid or find a Model $\langle \mathcal{F}, \mathcal{I} \rangle$ on a frame $\mathcal{F} = \langle W, R \rangle$, and a world a $w \in W$ that does not satisfy it.

$$\Diamond A \wedge (\Box B \vee \Box C) \rightarrow \Diamond (A \wedge (B \vee C))$$

Solution. *This formula is valid as,* $w_0 \models \Diamond A \wedge (\Box B \vee \Box C)$ *implies that there is a world* w_1 *accessible from* w_0 *such that* $w_1 \models A$. *Suppose* $w_0 \models \Box B$ *then* $w_1 \models B$ *and therefore* $w_1 \models A \wedge B$. *If, instead* $w_0 \models \Box C$, *then* $w_1 \models C$ *and therefore* $w_1 \models A \wedge C$. *In both cases* $w_1 \models A \wedge (B \vee C)$. *Which implies that* $w_0 \models \Diamond (A \wedge (B \vee C))$.

Mathematical Logic Exam

27 February 2014

Instructions

- Answer in English and write in ink unless the question paper gives other instructions.
- Write clearly; illegible answers will not be marked.
- Take care to identify each answer clearly with:
  - the number of the exercise.
  - where appropriate, the part of the exercise you are answering.
- Clearly cross out rough working, or unwanted answers before handing in your answers.
- If you take the exam to recover one of the midterms, Please state clearly which part (Propositional Logic or First Order + Modal Logic) you intend to re-do. If you do not state this in an explicit manner, we will assume that you are taking the entire exam, and the midterm marks will not be taken into account anymore.

Propositional Logic

Exercise 1 (PL Theory). **[6 points]** Let ϕ and ψ be two formulas which are built starting from two sets P and Q of primitive propositions, respectively.

- Show that when $P \cap Q = \emptyset$,

  $\phi \wedge \psi$ is satisfiable iff ϕ is satisfiable and ψ is satisfiable

  You have to prove both the directions of the implication
- Show that if $P \cap Q \neq \emptyset$, then it is possible that

  $\phi \wedge \psi$ is mot satisfiable and ϕ is satisfiable and ψ is satisfiable

  Suggestion: Provide a specific example of ϕ and ψ.

Solution.

- *If ϕ is satisfiable there there is an assignment $\mathcal{I}$ to the propositional variables contained in ϕ (i.e., to the elements of P) such that $\mathcal{I} \models \phi$.*

  If ψ is satisfiable there there is an assignment $\mathcal{J}$ to the propositional variables contained in ψ (i.e., to the elements of Q) such that $\mathcal{J} \models \phi$.

  Since P and Q are disjoin then the assignments $\mathcal{I}$ and $\mathcal{J}$, never assigns a truth value to the same variable, and therefore we can define the assugment $\mathcal{I} \cup \mathcal{J}$ that assigns the varibles in $P \cup Q$.

  Clearly we have that $\mathcal{I} \cup \mathcal{J} \models \phi$ and $\mathcal{I} \cup \mathcal{J} \models \psi$, since $\mathcal{I}$ agrees with $\mathcal{I} \cup \mathcal{J}$ on the variables in P and $\mathcal{J}$ agrees with $\mathcal{I} \cup \mathcal{J}$ on the variables in Q.

  This implies that $\mathcal{I} \cup \mathcal{J} \phi \wedge \psi$ and therefore that $\phi \wedge \psi$ is satisfiable.

- *Suppose that $p \in P \cap Q$, then if ϕ is p and ψ is $\neg p$, we have that ϕ is satisfiable by $\mathcal{I}(p) = true$ and ψ is also satisfiable by $\mathcal{J}(p) = false$ but $\phi \wedge \psi$ is equal $p \wedge \neg p$ which is not satisfiable.*

Exercise 2 (PL modeling)**. [6 points]** Brown, Jones, and Smith are three friends. They say the following:

- **Brown**: *"Jones is happy and Smith is sad".*
- **Jones**: *"If Brown is happy then so is Smith".*
- **Smith**: *"I'm sad, but at least one of the others is happy".*

Let B, J, and S be the statements "*Brown is happy*", "*Jones is happy*", and "*Smith is happy*", respectively, and consider being sad as the negation of being happy. Do the following:

1. Express the sentence of each friend as a PL formula.
2. Write a truth table for the three sentences.
3. Use the truth table to answer the following questions:
   (a) Are the three sentences satisfiable (together)?
   (b) The sentence of one of the friends follows from that of another. Which from which?
   (c) Assuming that all sentences are true, who is sad and who is happy?
   (d) Assuming that the sad friends told the truth and the happy friends told lies, who is sad and who is happy?

Solution.

1. *The three statements can be expressed as $J \wedge \neg S$, $B \supset S$, and $\neg S \wedge (B \vee J)$.*

2.

	B	J	S	$J \wedge \neg S$	$B \supset S$	$\neg S \wedge (B \vee J)$
(1)	T	T	T	F	T	F
(2)	T	T	F	T	F	T
(3)	T	F	T	F	T	F
(4)	T	F	F	F	F	T
(5)	F	T	T	F	T	F
(6)	F	T	F	T	T	T
(7)	F	F	T	F	T	F
(8)	F	F	F	F	T	F

3. (a) *Yes, assigment (6) makes them all true*

(b) $J \wedge \neg S \models \neg S \wedge (B \vee J)$

(c) *Assuming that all sentences are true corresponds to assignment (6). In this case Jones is happy and the others are sad.*

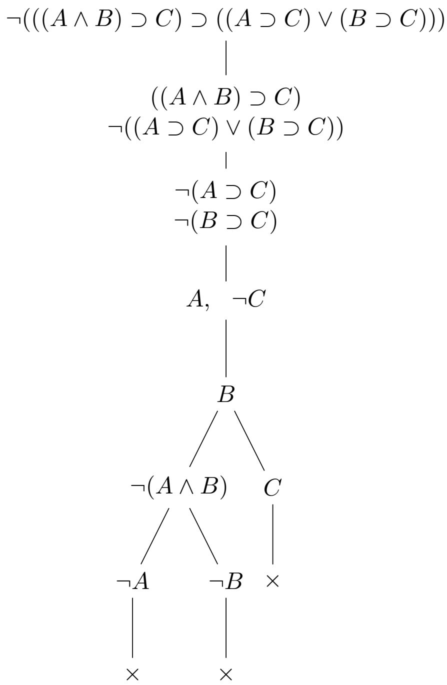
(d) *We have to search for an assignment such that if B (resp. J and S) is false then the sentence of B (resp. J and S) is true and that if B (resp. J and S) is true, then the sentence of B (resp. J and S) is false. The only assignment satisfying this restriction is assignment (3) in which Jones is sad and Brown and Smith are happy.*

Exercise 3 (PL Reasoning)**. [6 points]** For each of the following formula either prove via tableaux that it is valid, or construct a counter-model, i.e., an assignment that does not satisfy the formula.

1. $((A \wedge B) \supset C) \supset ((A \supset C) \vee (B \supset C))$
2. $((A \wedge B) \supset C) \supset (A \supset C)$
3. $((A \supset B) \supset A) \supset A$

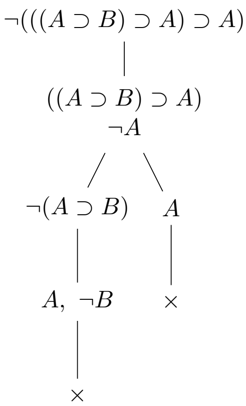
Solution.

1. *This formula is valide as the tableaux for its negation is closed*



2. $((A \wedge B) \supset C) \supset (A \supset C)$ *is not valide and the assignment* $\mathcal{I}(A) =$ *True*, $\mathcal{I}(B) =$ *False*, $\mathcal{I}(C) =$ *False is such that* $\mathcal{I} \not\models ((A \wedge B) \supset C) \supset (A \supset C)$

3. *This formula is valide as the tableaux for its negation is closed*



Exercise 4 (FOL Sentences). **[3 points]** Let Σ be the signature that contains

- the constant symbols *alice*, *bob* and *carol*
- the functional symbol $father$ with arity 1
- the predicate symbols *Student* and *Friend* with arity 1 and 2, respectively

For each of the following expression say:

- if it is a term, a formula, or none of the two
- if it is a formula say if it is closed and if not what are the free variables
- If it is a term say if it is a ground term
- in case it is a term or a formula provide it's intuitive reading

1. $father(alice \wedge bob) = carol$
2. $\forall x(Student(x) \supset friends(y, x))$
3. $Friend(alice, carol) \equiv \neg Friend(alice, bob)$

Solution.

1. *nothing*
2. *formula, open, free variable y. Intuitive reading: the set of elements y such that all students are their friends.*
3. *formula, closed. Intuitive reading: alice is friend of either carol or bob but not bo*

Exercise 5 (FOL Sentences)**. [3 points]** Formalize the following statements, by using only the following first order predicates:

$P(x)$	x is a person
$hates(x, y)$	x hates y

1. No person hates John but at least one person hates Peter.
2. John hates all persons who do not hate themselves.
3. Only one person hates John

Solution.

1. $\neg\exists x.(P(x) \wedge hates(x, John)) \wedge \exists x.(P(x) \wedge hates(x, Peter))$
2. $\forall x.P(x) \wedge \neg hates(x, x) \supset hates(John, x)$

3. $\exists x.(P(x) \wedge hates(x, John) \wedge \forall y.(P(y) \wedge hates(y, John) \supset x = y))$

Exercise 6 (FOL reasoning). **[6 points]** Show by means of resolution that if P satisfies the following properties, than it cannot be a transitive relation.

$$\forall x \exists y (P(x,y) \wedge \exists z (P(y,z) \wedge P(z,x))) \tag{1}$$

$$\forall xy (P(x,y) \supset \neg P(y,x)) \tag{2}$$

Solution. *First we have to formalize the fact that P is transitive in terms of first order formula.*

$$\forall x, y, z (P(x,y) \wedge P(y,z) \supset P(x,z)) \tag{3}$$

and then prove that the set of formulas (1), (2), and (3) is inconsistent.

We first start transforming the in Prenex normal form

$$\forall x \exists y \exists z (P(x,y) \wedge P(y,z) \wedge P(z,x))$$
$$\forall xy (\neg P(x,y) \vee \neg P(y,x))$$
$$\forall x, y, z (\neg P(x,y) \vee P(y,z) \vee P(x,z))$$

We eliminate existential quantifiers via skolemization

$$\forall x (P(x, f(x)) \wedge P(f(x), g(x)) \wedge P(g(x), x))$$
$$\forall xy (\neg P(x,y) \vee \neg P(y,x))$$
$$\forall x, y, z (\neg P(x,y) \vee \neg P(y,z) \vee P(x,z))$$

and finally we put the formulas in Clausal form

$$\{P(x, f(x))\} \tag{4}$$
$$\{P(f(x), g(x))\} \tag{5}$$
$$\{P(g(x), x)\} \tag{6}$$
$$\{\neg P(x,y),\ \neg P(y,x)\} \tag{7}$$
$$\{\neg P(x,y),\ \neg P(y,z),\ P(x,z)\} \tag{8}$$

Then we apply resolution as follows:

(9)	$\{\neg P(f(x), z),\ P(x,z)\}$	*From* (4) *and* (8) $\sigma = [f(x)/y]$
(10)	$\{P(x, g(x))\}$	*From* (5) *and and* (9) $\sigma = [g(x)/z]$
(11)	$\{\neg P(g(x), x)\}$	*From* (10) *and* (7) $\sigma = [g(x)/y]$
(12)	$\{\}$	*From* (11) *and* (6), $\sigma = []$

Exercise 7 (Modal logics). **[6 points]** Let $\mathcal{F}$ be a Kripke frame such that the following formulas are valid in $\mathcal{F}$

(T) $\Box\phi \supset \phi$
(5) $\Diamond\phi \supset \Box\Diamond\phi$

Show that also the formula

(4) $\Box\phi \supset \Box\Box\phi$

is valid in $\mathcal{F}$.

[Suggestion: find the corresponding propoerties formalized by each of the formulas, and try to prove implications among them]

Solution. *If* $\mathcal{F} \models$ *(T) then* R *is reflexive, i.e., forall* $w \in W,\ R(w,w)$.

If $\mathcal{F} \models$ *(5) then* R *is euclidean, i.e., forall* w, v, u, $R(w,v)$ *and* $R(w,u)$ *implies* $R(u,v)$.

Let us prove that R *is transitive and therefore satisfies (4).*

Suppose that $R(w,v)$ *and* $R(v,u)$*, then, by reflexivity of* R *we have that* $R(w,w)$.

Since R *is euclidean, then* $R(w,w)$ *and* $R(w,v)$ *implies that* $R(v,w)$.

Again from he fact that R *is euclidean, we have that* $R(v,w)$ *and* $R(v,u)$ *implies that* $R(w,u)$*, and therefore that* R *is transitive.*

We know that if $\mathcal{F}$ *is transitive then* (4) *is valid in* $\mathcal{F}$.

Mathematical Logic Exam

10 June 2014

Instructions

- Answer in English and write in ink unless the question paper gives other instructions.
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  - the number of the exercise.
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Propositional Logic

Exercise 1 (PL Theory)**. [6 points]** Show that the propositional α-rule

$$R_\wedge \ \frac{\phi \wedge \psi}{\begin{array}{c}\phi \\ \psi\end{array}}$$

preserves the satisfiability of the tableau (that is, $R_\wedge$ extends a satisfiable branch β to a branch β' that is also satisfiable)

Solution.

- *let $\mathcal{I}$ be an interpretation that satisfies β, i.e., $\mathcal{I} \models \beta$*
- *since $\phi \wedge \psi \in \beta$ then $\mathcal{I} \models \phi \wedge \psi$*
- *which implies that $\mathcal{I} \models \phi$ and $\mathcal{I} \models \psi$*

- *which implies that $\mathcal{I} \models \beta'$ with $\beta' = \beta \cup \{\phi, \psi\}$.*

Exercise 2 (PL modeling)**. [6 points]** Brown, Jones, and Smith are three friends. They say the following:

- **Brown**: *"Jones is drunk and Smith is sober".*
- **Jones**: *"If Brown is drunk then so is Smith".*
- **Smith**: *"I'm sober, but at least one of the others is drunk".*

Let B, J, and S be the statements "*Brown is drunk*", "*Jones is drunk*", and "*Smith is drunk*", respectively, and consider being sober as the negation of being drunk. Do the following:

1. Express the sentence of each friend as a PL formula.
2. Write a truth table for the three sentences.
3. Use the truth table to answer the following questions:
   (a) Are the three sentences satisfiable (together)?
   (b) The sentence of one of the friends follows from that of another. Which from which?
   (c) Assuming that all sentences are true, who is sober and who is drunk?
   (d) Assuming that the sober friends told the truth and the drunk friends told lies, who is sober and who is drunk?

Solution.

1. *The three statements can be expressed as $J \wedge \neg S$, $B \supset S$, and $\neg S \wedge (B \vee J)$.*

2.

	B	J	S	$J \wedge \neg S$	$B \supset S$	$\neg S \wedge (B \vee J)$
(1)	T	T	T	F	T	F
(2)	T	T	F	T	F	T
(3)	T	F	T	F	T	F
(4)	T	F	F	F	F	T
(5)	F	T	T	F	T	F
(6)	F	T	F	T	T	T
(7)	F	F	T	F	T	F
(8)	F	F	F	F	T	F

3. *(a) Yes, assigment (6) makes them all true*
   (b) $J \wedge \neg S \models \neg S \wedge (B \vee J)$
   (c) Assuming that all sentences are true corresponds to assignment (6). In this case Jones is drunk and the others are sober.

(d) *We have to search for an assignment such that if B (resp. J and S) is false then the sentence of B (resp. J and S) is true and that if B (resp. J and S) is true, then the sentence of B (resp. J and S) is false. The only assignment satisfying this restriction is assignment (3) in which Jones is sober and Brown and Smith are drunk.*

Exercise 3 (PL Reasoning)**. [6 points]** Apply DPLL procedure to check if the following set of clauses is satisfiable, and if it is so, return a partial assignment that makes all the fomulas true.

1. $p \vee u$
2. $\neg u \vee \neg v$
3. $q \vee \neg v$
4. $\neg q \vee s$
5. $\neg s \vee \neg u \vee m$
6. $\neg m \vee u \vee \neg s$

In the solution you have to specify all the applications of unit propagation rule, and all the choices you take when Unit propagation is not applicable.

Solution.

1. *Let ϕ the CNF of the conjunction of 1–6. ϕ does not contain unit clause, which implies that unit propagation is not applicable.*
2. *therefore, we select a literal (say $\neg u$) and set $\mathcal{I}(u) = false$*
3. *Compute $\phi|_{\neg u}$:*

$$\phi|_{\neg u} = \{\{p\},\ \{q, \neg v\},\ \{\neg q, s\},\ \{\neg m, \neg s\}\}$$

4. *$\phi|_{\neg u}$ contains the unit clause $\{p\}$, we therefore extend the partial interpretation with $\mathcal{I}(p)$ = True. We then apply unit propagation with $\{p\}$ as unit clause, obtaining*

$$\phi|_{\neg u,p} = \{\{q, \neg v\},\ \{s\},\ \{\neg m, \neg s\}\}$$

5. *$\phi|_{\neg u,p}$ contains the unit clause $\{s\}$, we therefore extend the partial interpretation with $\mathcal{I}(s)$ = True. We then apply unit propagation with $\{s\}$ as unit clause, obtaining*

$$\phi|_{\neg u,p,s} = \{\{q, \neg v\},\ \{\neg m\}\}$$

6. *$\phi|_{\neg u,p,s}$ contains the unit clause $\{\neg m\}$, we therefore extend the partial interpretation with $\mathcal{I}(m)$ = False. We then apply unit propagation with $\{\neg m\}$ as unit clause, obtaining*

$$\phi|_{\neg u,p,s,\neg m} = \{\{q, \neg v\},\ \}$$

7. *$\phi|_{\neg u,p,s,\neg m}$ does not contain unit clause, which implies that unit propagation is not applicable. We, therefore, select a literal (say q) and set $\mathcal{I}(q) = True$. We then compute $\phi|_{\neg u,p,s,\neg m,q} = \{\}$. Which implies that the initial formula is satisfiable, by the partial assignment:*

$$\begin{array}{lll} I(u) = \mathit{False} & I(p) = \mathit{True} & I(s) = \mathit{True} \\ I(m) = \mathit{False} & I(q) = \mathit{True} & \end{array}$$

Exercise 4 (FOL Theory)**. [6 points]** Let $\mathcal{L}$ be a first order language on a signatore containing

- the constant symbols a and b,
- the binary function symbol f, and
- the binary predicate symbol P.

Answer to the following questions:

1. What is the Herbrand Universe for $\mathcal{L}$ (2 point)
2. Does $\mathcal{L}$ have a finite model? If yes define it, if not explain why. (2 point)
3. Let $\mathcal{T}$ be a theory containing the following axioms
   (a) $\forall y.\neg P(x, x)$ (P is irreflexive)
   (b) $\forall xyz.(P(x, y) \wedge P(y, z) \supset P(x, z))$ (P is transitive)
   (c) $\forall xy.(P(x, f(x, y)) \wedge P(y, f(x, y))$

   Is $\mathcal{T}$ satisfiable?. If yes can you provide a model for $\mathcal{T}$ (2 points)

Solution.

1. *The Herbrand Universe for $\mathcal{L}$ is the set of ground terms that can be built starting from the constants by applying the function symbols. In this case it is the following infinite set of terms.*

$$\begin{array}{l} \{a, b, f(a, a), f(a, b), f(b, a), f(b, b), \\ f(a, f(a, a)), f(a, f(a, b)), f(a, f(b, a)), f(a, f(b, b)), \\ f(b, f(a, a)), f(b, f(a, b)), f(b, f(b, a)), f(b, f(b, b)) \ldots\} \end{array}$$

2. *$\mathcal{L}$ has a finite model. For instance $\mathcal{I} = \left\langle \Delta^{\mathcal{I}} = \{0\}, f^{\mathcal{I}}(0, 0) = 0, P^{\mathcal{I}} = \emptyset \right\rangle$ is a model of $\mathcal{L}$, and it is finite since $|\Delta^{\mathcal{I}}| = 1$ i.e., the cardinality of the domain of $\mathcal{I}$ is a finite number. namely 1.*

3. *$\mathcal{T}$ is satisfiable. Consider the herbrand interpretation $\mathcal{H}$ defined on the domain which is the herbrand universe, where P is interpreted in the following binary relation:*

$$\langle t, t' \rangle \in P^{\mathcal{H}} \quad \textit{if and only if} \quad t \textit{ is a substring of } t'$$

Where t is a substring of t' means that when t' is of the form $f(\ldots t \ldots)$ It's easy to check that the three axioms of $\mathcal{T}$ are all satisfied by $\mathcal{B}$

Exercise 5 (FOL tableaux)**. [6 points]** Show by means of tableaux that the following formula is valid:

$$\forall xyz(P(x,y) \wedge P(x,z) \supset P(y,z)) \supset \forall x(\exists w P(w,x) \supset P(x,x))$$

Solution.

$$\neg(\forall xyz(P(x,y) \wedge P(x,z) \supset P(y,z)) \supset \forall x(\exists w P(w,x) \supset P(x,x)))$$

$$\forall xyz(P(x,y) \wedge P(x,z) \supset P(y,z))$$
$$\neg\forall x(\exists w P(w,x) \supset P(x,x))$$

$$\neg(\exists w P(w,a) \supset P(a,a))$$

$$\exists w P(w,a)$$
$$\neg P(a,a)$$

$$P(b,a)$$

$$P(b,a) \wedge P(b,a) \supset P(a,a)$$

$$\neg(P(b,a) \wedge P(b,a)) \qquad\qquad P(a,a)$$

$$\neg P(b,a) \qquad\qquad \neg P(b,a) \qquad\qquad \times$$

$$\times \qquad\qquad \times$$

Exercise 6 (Modal logics Modal axioms)**. [6 points]** Consider the axiom schema $\Box\phi \supset \phi$. Say which is the property P such that (1) holds.

$$\mathcal{F} \models \Box\phi \supset \phi \text{ if and only if } \mathcal{F} \text{ has the property } P \tag{1}$$

Prove (1).

Solution. *We have to prove*

Soundness: *If $\mathcal{F}$ is a frame that satisfies the property P, then $\Box\phi \supset \phi$ is a valid formula in $\mathcal{F}$.*

Completeness: *If $\Box\phi \supset \phi$ is a valid formula in a frame $\mathcal{F}$, then $\mathcal{F}$ is a frame that satisfies the property P. For the completeness we prove the (equivalent) contropositive statement, i.e., that if $\mathcal{F}$ does not satisfy the property P then $\Box\phi \supset \phi$ is not valid in $\mathcal{F}$. We do this by building a countermodel $\mathcal{M} = \langle F, V\rangle$ for $\Box\phi \supset \phi$, by providing an assignment V to propositional variables on $\mathcal{F}$, and by selecting a world of w in $\mathcal{F}$ so that $\mathcal{M}, w \not\models \Box\phi \supset \phi$.*

(T): $\Box\phi \supset \phi$ *P is equal to* Reflexivity, *i.e.,* $\forall w \in W,\ wRw$.

Soundness: *Let $\mathcal{M}$ be a model on a reflexive frame $\mathcal{F} = \langle W, R\rangle$ and w any world in W. We prove that $\mathcal{M}, w \models \Box\phi \supset \phi$.*

1. *Since R is reflexive then wRw*
2. *Suppose that $\mathcal{M}, w \models \Box\phi$ (Hypothesis)*
3. *From the satisfiability condition of $\Box$, $\mathcal{M}, w \models \Box\phi$, and wRw imply that $\mathcal{M}, w \models \phi$ (Thesis)*
4. *Since from (Hypothesis) we have derived (Thesis), we can conclude that $\mathcal{M}, w \models \Box\phi \supset \phi$.*

Completeness: *Suppose that a frame $\mathcal{F} = \langle W, R\rangle$ is not reflexive.*

1. *If R is not reflexive then there is a $w \in W$ which does not access to itself. I.e., for some $w \in W$ it does not hold that wRw.*
2. *Let $\mathcal{M}$ be any model on $\mathcal{F}$, and let ϕ be the propositional formula p. Let V the set p true in all the worlds of W but w where p is set to be false.*
3. *From the fact that w does not access to itself, we have that in all the worlds w accessible from w, p is true, i.e, $\forall w',\ wRw',\ \mathcal{M}, w' \models p$.*
4. *Form the satisfiability condition of $\Box$ we have that $\mathcal{M}, w \models \Box p$.*
5. *since $\mathcal{M}, w \not\models p$, we have that $\mathcal{M}, w \not\models \Box p \supset p$.*

Mathematical Logic Exam

10 June 2014

Instructions

- Answer in English and write in ink unless the question paper gives other instructions.
- Write clearly; illegible answers will not be marked.
- Take care to identify each answer clearly with:
  - the number of the exercise.
  - where appropriate, the part of the exercise you are answering.
- Clearly cross out rough working, or unwanted answers before handing in your answers.

Propositional Logic

Exercise 1 (PL Theory)**.** [**6 points**] Show that if both $\Gamma \cup \{\neg\phi\}$ and $\Gamma \cup \{\phi\}$ are not satisfiable then Γ is also not satisfiable.

Solution. *By contradiction, suppose that* Γ *is satisfiable, then there is an interpretation* $\mathcal{I}$ *that satisfies* Γ*, i.e.,* $\mathcal{I} \models \Gamma$*. By definition of satisfiability in classical propositional logic, either* $\mathcal{I} \models \phi$ *or* $\mathcal{I} \models \neg\phi$*, i.e., that one of the two sets is satisfiable. But this contraddicts the fact that both sets* $\Gamma \cup \{\phi\}$ *and* $\Gamma \cup \{\neg\phi\}$ *are unsatisfiable.*

Exercise 2 (PL modeling)**.** [**6 points**] Alice and Bob are playing with a two face coin. In a first round each of them tosses the coin obtaining the same result. In a second round, the result of Alice toss is different from that of Bob. Show by means of truth tables that either Alice or Bob has obtained the same result in the two rounds.

Sugestion: Use the propositional letters A_1, A_2, B_1 and B_2 to represent the outcome of Alice and Bob tosses in the first and second round.

Solution.

- *Result first toss:* $A_1 \equiv B_1$
- *Result second toss:* $A_2 \equiv \neg B_2$

- *Alice same toss:* $A_1 \equiv A_2$
- *Bob same toss:* $B_1 \equiv B_2$

Show that the formula

$$A_1 \equiv B_1 \wedge A_2 \equiv \neg B_2 \supset A_1 \equiv A_1 \vee B_1 \equiv B_2$$

is valid

A_1	B_1	A_2	B_2	$(A_1 \equiv B_1$	$\wedge$	$A_2 \equiv \neg B_2)$	$\supset$	$(A_1 \equiv A_2$	$\vee$	$B_1 \equiv B_2)$
0	0	0	0	1	0	0	1	1	1	1
0	0	0	1	1	1	1	1	1	1	0
0	0	1	0	1	1	1	1	0	1	1
0	0	1	1	1	0	0	1	0	0	0
0	1	0	0	0	0	0	1	1	1	0
0	1	0	1	0	0	1	1	1	1	1
0	1	1	0	0	0	1	1	0	0	0
0	1	1	1	0	0	0	1	0	1	1
1	0	0	0	0	0	1	1	0	1	1
1	0	0	1	0	0	1	1	0	0	0
1	0	1	0	0	0	1	1	1	1	1
1	0	1	1	0	0	0	1	1	1	0
1	1	0	0	1	0	0	1	0	0	0
1	1	0	1	1	1	1	1	0	1	1
1	1	1	0	1	1	1	1	1	1	0
1	1	1	1	1	0	0	1	1	1	1

Exercise 3 (PL Reasoning)**.** [**6 points**] Prove by resolution that the formula $\neg Q \supset \neg R$ is not a logical consequence of the set of formulas $\{P \supset Q,\ \neg P \supset R\}$

Solution. *1. To prove that a formula ϕ is not a logica consequence of a set of formulas Γ, we have to find a model for Γ and $\neg\phi$. I.e., we can check that $\Gamma \cup \{\neg\phi\}$ is satisfiable.*

2. We therefore consider the three formulas

$$P \supset Q$$
$$\neg P \supset R$$
$$\neg(Q \supset \neg R)$$

and check via resolution if they are satisfiable

3. We first transform the previous formulas in clausal normal form, obtaining:

$$\{\neg P, Q\}$$
$$\{P, R\}$$
$$\{Q\}$$
$$\{R\}$$

4. *by applying resolution to the above formulas we can derive only the clause*

$$\{Q, R\}$$

and no other rules are applicable. This implies that from the initial set of formulas it is not possible to derive the empty clause. Which implies that the initial set of formulas are satisfiable.

Exercise 4 (FOL Theory)**. [6 points]** Show that the tableaux rule is sound

$$\frac{\begin{array}{c}\vdots\\ \exists x\phi(x)\end{array}}{\phi(c)}$$

when c is a new constant, not appearing in the branch above $\exists x.\phi(x)$. Suggestion: you have to prove that if $\Gamma \cup \{\exists x.\phi(x)\}$ is satisfiable, then $\Gamma \cup \{\exists x.\phi(x),\ \phi(c)\}$ is also satisfiable. Explain why c must be new, i.e., that if it appears in Γ is it possible that the rule is not sound.

Solution.

A tableaux rule

$$\frac{\begin{array}{c}\vdots\\ \phi\end{array}}{\psi}$$

is sound whenever, if Γ is the set of formulas of in the branch above ϕ, and if $\Gamma \cup \{\phi\}$ is satisfiable, then $\Gamma \cup \{\phi, \psi\}$ is also satisfiable.

In this case, let Γ be the set of formulas occurring in the branch β above $\exists x.\phi(x)$. Suppose that $\Gamma \cup \{\exists x.\phi(x)\}$ is satisfiable. This means that there is an interpretation $\mathcal{I}$ such that $\mathcal{I} \models \Gamma \cup \{\exists x.\phi(x)\}$. Therefore $\mathcal{I} \models \exists x.\phi(x)$. From the definition of satisfiability of $\exists x.\phi(x)$, we know that there is a $d \in \Delta^{\mathcal{I}}$ such that $\mathcal{I} \models \phi(x)[x := d]$. Let $\mathcal{I}'$ be the extension of $\mathcal{I}$, with $c^{\mathcal{I}'} = d$. This choice implies that $\mathcal{I}' \models \phi(c)$. Since c does not occur in any formulas of $\Gamma \cup \exists x.\phi(x)$, and $\mathcal{I}'$ coincides with $\mathcal{I}$ on the interpretation of all the other symbols, we have that $\mathcal{I}' \models \Gamma \cup \{\exists x.\phi(x)\}$. And therefore, we have that the formulas in the branch of $\phi(c)$, i.e., $\Gamma \cup \{\exists x.\phi(x), \phi(c)\}$ is satisfiable. Therefore we can conclude that the rule is sound.

Exercise 5 (FOL resolution)**. [6 points]** Prove by resolution the validity of the following formula.

$$(\exists x \forall y.Q(x, y) \wedge \forall x.(Q(x, x) \supset \exists y.R(y, x))) \supset \exists y.\exists x.R(x, y)$$

Solution.

1. *negate the formula:*

$$\neg((\exists x \forall y.Q(x, y) \wedge \forall x.(Q(x, x) \supset \exists y.R(y, x))) \supset \exists y.\exists x.R(x, y))$$

2. *rename variables:*

$$\neg((\exists x \forall y.Q(x,y) \wedge \forall z.(Q(z,z) \supset \exists w.R(w,z))) \supset \exists v.\exists t.R(t,v))$$

3. *transform it in prenex normal form:*

$$(\exists x \forall y.Q(x,y) \wedge \forall z.(Q(z,z) \supset \exists w.R(w,z))) \wedge \neg\exists v.\exists t.R(t,v)$$
$$\exists x \forall y.Q(x,y) \wedge (\forall z.(\neg Q(z,z) \vee \exists w.R(w,z)) \wedge \forall v.\forall t.\neg R(t,v)$$
$$\exists x \forall y.Q(x,y) \wedge (\forall z.(\neg Q(z,z) \vee \exists w.R(w,z)) \wedge \forall v.\forall t.\neg R(t,v)$$
$$\exists x \forall y \forall z \exists w \forall v \forall t.(Q(x,y) \wedge (\neg Q(z,z) \vee R(w,z)) \wedge \neg R(t,v))$$

4. *Skolemize:*

$$\forall y \forall z \forall v \forall t.(Q(a,y) \wedge (\neg Q(z,z) \vee R(f(z),z)) \wedge \neg R(t,v))$$

5. *put in clausal form:*

$$\{Q(a,y)\},\ \{\neg Q(z,z), R(f(z),z)\}, \{\neg R(t,v)\}$$

6. *apply resolution and unification algorithm:*

(1)	$\{Q(a,y)\}$	*input clause*
(2)	$\{\neg Q(z,z), R(f(z),z)\}$	*input clause*
(3)	$\{\neg R(t,v)\}$	*input clause*
(4)	$\{R(f(a),a)\}$	*from* (1) *and* (2) *with* $\sigma = [a/z, z/y]$
(5)	$\{\}$	*from* (3) *and* (4) *with* $\sigma = [f(a)/t, a/v]$

Exercise 6 (Modal logics modelling)**. [6 points]** Suppose you want to represent the preferences of Ana by the modal operator $\Box_{\text{Ana}}$ The formula $\Box_{\text{Ana}}\phi$ states that, in a certain situation, Ana prefers ϕ being true to ϕ being false.

For instance if the propositional variables R and H formalize the two propositions "it's raining" and "Ana stays at home", the formula $\Box_{\text{Ana}}H$ means that Alice prefers to stay at Home, while $\Box_{\text{Ana}}\neg R$ means that Alice prefers that it is not raining. The formula $\neg\Box_{\text{Ana}}R$ means that Alice does not prefer that it is raining. Notice that "non preferring something" is different from "preferring not something".

1. Using R and H and the modal operator $\Box_{\text{Ana}}$ formulate the following statements:
   (a) when it is raining Ana prefers to stay home
   (b) when it is not raining Ana has no preference between going out or staing at home.
2. Give a Kripke model that satisfies the formula $\neg\Box\phi \wedge \neg\Box\neg\phi$.

3. Use modal schemas to encode the following assumptions:

   (a) Ana prefers something that can be true. In other words, if a formula ϕ is always false then it cannot be preferred to $\neg\phi$ by Ana.

   (b) If Ana prefers $\phi \vee \psi$ to $\neg(\phi \vee \psi)$, then she either prefers ϕ over $\neg\phi$ or ψ over $\neg\psi$.

Solution.

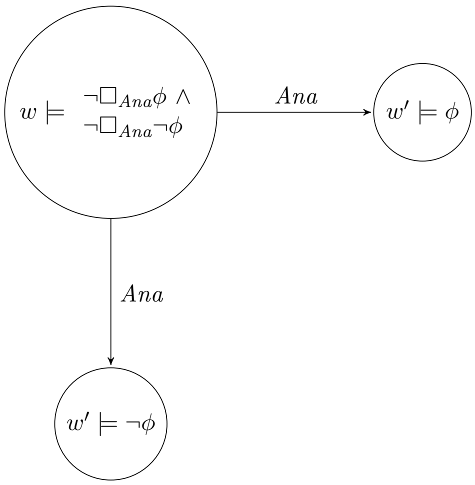
1. (a) *when it is raining Ana prefers to stay home*

$$R \supset \Box_{Ana} H$$

   (b) *when it is not raining Ana has no preference between going out or staing at home.*

$$\neg R \supset \neg\Box_{Ana} H \wedge \neg\Box_{Ana}\neg H$$

2. *A Kripke model that satisfies the formula $\neg\Box_{Ana}\phi \wedge \neg\Box_{Ana}\neg\phi$, should be a model that contains a world w where $\Box_{Ana}\phi$ and $\Box_{Ana}\neg\phi$ are both false. To force this ne need to have an accessible world where ϕ is false and one in which ϕ is true. The fact that $\Box_{Ana}\phi$ (resp. $\Box_{Ana}\neg\phi$) is true in w if and only if ϕ (res. $\neg\phi$) is true in all the worlds accessible from w, implies that if we have one world where ϕ is true and one where ϕ is false, the formulas $\Box_{Ana}\phi$ and $\Box_{Ana}\neg\phi$ both false.*



3. *Use modal schemas to encode the following assumptions:*

   (a) *To represent the fact that if ϕ is always false then it cannot be preferred to $\neg\phi$ by Ana, we use the formula which is always false (i.e., $\bot$) and state that it is never preferred by Ana. As follows:*

$$\neg\Box_{Ana}\bot$$

(b) *If Ana prefers $\phi \vee \psi$ to $\neg(\phi \vee \psi)$, then she either prefers ϕ over $\neg\phi$ or ψ over $\neg\psi$.*

$$\Box_{Ana}(\phi \vee \psi) \supset \Box_{Ana}\phi \vee \Box_{Ana}\psi$$

Mathematical Logic Exam

4 September 2014

Instructions

- Answer in English and write in ink unless the question paper gives other instructions.
- Write clearly; illegible answers will not be marked.
- Take care to identify each answer clearly with:
  - the number of the exercise.
  - where appropriate, the part of the exercise you are answering.
- Clearly cross out rough working, or unwanted answers before handing in your answers.

Propositional Logic

Exercise 1 (PL Theory)**. [6 points]** Show that the propositional α-rule

$$R_\wedge \ \frac{\phi \wedge \psi}{\begin{array}{c}\phi\\ \psi\end{array}}$$

preserves the satisfiability of the tableau (that is, $R_\wedge$ extends a satisfiable branch β to a branch β' that is also satisfiable)

Solution.

- *let $\mathcal{I}$ be an interpretation that satisfies β, i.e., $\mathcal{I} \models \beta$*
- *since $\phi \wedge \psi \in \beta$ then $\mathcal{I} \models \phi \wedge \psi$*
- *which implies that $\mathcal{I} \models \phi$ and $\mathcal{I} \models \psi$*
- *which implies that $\mathcal{I} \models \beta'$ with $\beta' = \beta \cup \{\phi, \psi\}$.*

Exercise 2 (PL modeling). **[6 points]** Alice and Bob are playing with a two face coin. In the first round each of them tosses the coin and the result of Alice toss is different from that of Bob. In the second round they toss the coin obtaining the same result. Show by means of truth tables that either Alice or Bob has obtained different results in the two rounds.

Sugestion: Use the propositional letters A_1, A_2, B_1 and B_2 to represent the outcome of Alice and Bob tosses in the first and second round.

Solution.

- *Result first toss:* $A_1 \equiv \neg B_1$
- *Result second toss:* $A_2 \equiv B_2$
- *Alice different toss:* $A_1 \equiv \neg A_2$
- *Bob different toss:* $B_1 \equiv \neg B_2$

Show that the formula

$$A_1 \equiv \neg B_1 \wedge A_2 \equiv B_2 \supset A_1 \equiv \neg A_2 \vee B_1 \equiv \neg B_2$$

is valid

A_1	B_1	A_2	B_2	$(A_1 \equiv \neg B_1$	$\wedge$	$A_2 \equiv B_2)$	$\supset$	$(A_1 \equiv \neg A_2$	$\vee$	$B_1 \equiv \neg B_2)$
0	0	0	0	0	0	1	1	0	0	0
0	0	0	1	0	0	0	1	0	1	1
0	0	1	0	0	0	1	1	1	1	0
0	0	1	1	0	0	1	1	1	1	1
0	1	0	0	1	1	1	1	0	1	1
0	1	0	1	1	0	0	1	0	0	0
0	1	1	0	1	0	0	1	1	1	1
0	1	1	1	1	1	1	1	1	1	0
1	0	0	0	1	0	0	1	1	1	0
1	0	0	1	1	0	0	1	1	1	1
1	0	1	0	1	0	0	1	0	0	0
1	0	1	1	1	1	1	1	0	1	1
1	1	0	0	0	0	1	1	1	1	1
1	1	0	1	0	0	0	1	1	1	0
1	1	1	0	0	0	0	1	0	1	1
1	1	1	1	0	0	1	1	0	0	0

Exercise 3 (PL Reasoning). **[6 points]**

Prove by resolution that the formula $\neg A \supset \neg B$ is not a logical consequence of the set of formulas $\{C \supset A,\ \neg C \supset B\}$

Solution. *1. To prove that a formula ϕ is <u>not</u> a logica consequence of a set of formulas Γ, we have to find a model for Γ and $\neg\phi$. I.e., we can check that $\Gamma \cup \{\neg\phi\}$ is satisfiable.*

2. We therefore consider the three formulas

$$C \supset A$$
$$\neg C \supset B$$
$$\neg(A \supset \neg B)$$

and check via resolution if they are satisfiable

3. We first transform the previous formulas in clausal normal form, obtaining:

$$\{\neg C, A\}$$
$$\{C, B\}$$
$$\{A\}$$
$$\{B\}$$

4. by applying resolution to the above formulas we can derive only the clause

$$\{A, B\}$$

and no other rules are applicable. This implies that from the initial set of formulas it is not possible to derive the empty clause. Which implies that the initial set of formulas are satisfiable.

Exercise 4 (FOL Theory). **[6 points]** Suppose that a first order language L contains only the set of constants $\{a\}$, no functional symbols, and and the unary predicate symbol R.

Say if the following formula is valid, i.e., true in all interpretations. If it is valid give a proof of it's validity, you can choose any method; if it is not valid provide a counter-model.

$$\exists x P(x) \supset P(a) \supset$$

Solution. *The formula is not valid, just consider the interpretation $\mathcal{I} = \langle \Delta^I, \cdot^{\mathcal{I}} \rangle$, with*

- $\Delta^{\mathcal{I}} = \{1, 2\}$, $a^{\mathcal{I}} = 1$, *and* $P^{\mathcal{I}} = \{2\}$.

We have that $\mathcal{I} \models \exists x P(x)$ since $\mathcal{I} \models P(x)[x := 2]$, but $\mathcal{I} \not\models P(a)$ since $a^{\mathcal{I}} \notin P^{\mathcal{I}}$.

Exercise 5 (FOL tableaux). **[6 points]** Show by means of tableaux that the following formula is valid:

$$\forall xyz(P(x, y) \wedge P(x, z) \supset P(y, z)) \supset \forall x(\exists w P(w, x) \supset P(x, x))$$

Solution.

$$\neg(\forall xyz(P(x,y) \land P(x,z) \supset P(y,z)) \supset \forall x(\exists w P(w,x) \supset P(x,x)))$$

$$\forall xyz(P(x,y) \land P(x,z) \supset P(y,z))$$
$$\neg\forall x(\exists w P(w,x) \supset P(x,x))$$

$$\neg(\exists w P(w,a) \supset P(a,a))$$

$$\exists w P(w,a)$$
$$\neg P(a,a)$$

$$P(b,a)$$

$$P(b,a) \land P(b,a) \supset P(a,a)$$

$$\neg(P(b,a) \land P(b,a)) \qquad P(a,a)$$

$$\neg P(b,a) \qquad \neg P(b,a) \qquad \times$$

$$\times \qquad \times$$

Exercise 6 (Modal logics modelling)**.** [**6 points**] Suppose you want to represent the beliefs of Ana by the modal operator $\Box_{\text{Ana}}$ The formula $\Box_{\text{Ana}}\phi$ states that, in a certain situation, Ana believes that ϕ holds.

For instance if the propositional variables R and U formalize the two propositions "it's raining" and "Ana takes the umbrella", the formula $\Box_{\text{Ana}}R$ means that Alice believes that it is raining, while $\Box_{\text{Ana}}\neg R$ means that Alice believes that it is not raining. Instead the formula $\neg\Box_{\text{Ana}}R$ means that Alice does not believe that it is raining. Notice that "non believing something" is different from "believing not something".

1. Using R and U and the modal operator $\Box_{\text{Ana}}$ formulate the following statements:

   (a) when it is raining Ana believes she takes the umbrella

   (b) when it is not raining Ana has no belief between taking or not taking the umbrella.

2. Give a Kripke model that satisfies the formula $\neg\Box_{\text{Ana}}\phi \wedge \neg\Box_{\text{Ana}}\neg\phi$.
3. Use modal schemas to encode the following assumptions:
   (a) Ana believes something that can be true. In other words, if a formula ϕ is always false then it cannot be believed by Ana.
   (b) If Ana believes $\phi \vee \psi$ then she either believes ϕ or ψ.

Solution.

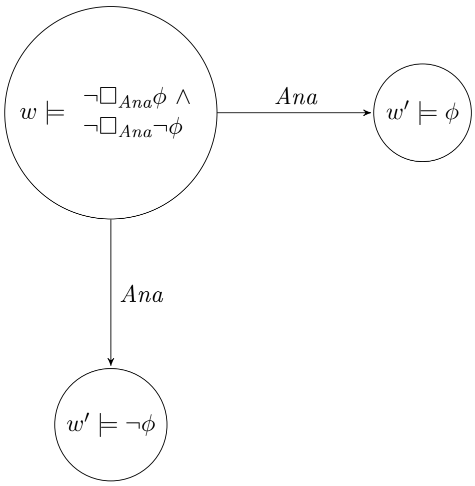
1. (a) when it is raining Ana believes she takes the umbrella

$$R \supset \Box_{Ana} U$$

(b) when it is not raining Ana has no belief between taking or not taking the umbrella.

$$\neg R \supset \neg\Box_{Ana} U \wedge \neg\Box_{Ana}\neg U$$

2. A Kripke model that satisfies the formula $\neg\Box_{Ana}\phi \wedge \neg\Box_{Ana}\neg\phi$, should be a model that contains a world w where $\Box_{Ana}\phi$ and $\Box_{Ana}\neg\phi$ are both false. To force this ne need to have an accessible world where ϕ is false and one in which ϕ is true. The fact that $\Box_{Ana}\phi$ (resp. $\Box_{Ana}\neg\phi$) is true in w if and only if ϕ (res. $\neg\phi$) is true in <u>all</u> the worlds accessible from w, implies that if we have one world where ϕ is true and one where ϕ is false, the formulas $\Box_{Ana}\phi$ and $\Box_{Ana}\neg\phi$ both false.



3. Use modal schemas to encode the following assumptions:

(a) To represent the fact that if ϕ is always false then it cannot be believed by Ana, we use the formula which is always false (i.e., $\bot$) and state that it is never believed by Ana. As follows:

$$\neg\Box_{Ana}\bot$$

(b) If Ana believes $\phi \vee \psi$ *then she either believes* ϕ *or* ψ.

$$\Box_{Ana}(\phi \vee \psi) \supset \Box_{Ana}\phi \vee \Box_{Ana}\psi$$