Logics for knowledge representation

A course of the ICT International Doctorate School

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Modal logic II
A logic is **normal** if it contains at least the following axiom schemata

```
A1  \( \phi \supset (\psi \supset \phi) \)
A2  \( (\phi \supset (\psi \supset \theta)) \supset ((\phi \supset \psi) \supset (\phi \supset \theta)) \)
A3  \( (\neg \psi \supset \neg \phi) \supset ((\neg \psi \supset \phi) \supset \phi) \)
MP  \[ \frac{\phi \quad \phi \supset \psi}{\psi} \]
K   \[ \Box(\phi \supset \psi) \supset (\Box \phi \supset \Box \psi) \]
Nec \[ \frac{\phi}{\Box \phi} \]
```

the necessitation rule
Basic property of NML

\[
\vdash_A \phi \equiv \psi \\
\text{iff} \\
\vdash_A \Diamond \phi \equiv \Diamond \psi \\
\text{iff} \\
\vdash_A \square \phi \equiv \square \psi
\]


VIP axiom schema

(4)  $\diamond\diamond\phi \supset \diamond\phi$  \hspace{1cm} $\Box\phi \supset \Box\Box\phi$

(T)  $\phi \supset \diamond\phi$  \hspace{1cm} $\Box\phi \supset \phi$

(B)  $\phi \supset \Box\diamond\phi$

(D)  $\Box\phi \supset \diamond\phi$

(3)  $\diamond\phi \wedge \diamond\psi \supset \diamond(\phi \wedge \diamond\psi) \lor \diamond(\phi \wedge \psi) \lor \diamond(\Diamond\phi \wedge \psi)$

(L)  $\Box(\Box\phi \supset \phi) \supset \Box\phi$
Soundness and completeness

- **K** the class of all frames
- **K4** the class of transitive frames
- **KT** the class of reflexive frames
- **KB** the class of symmetric frames
- **KD** the class of right unbounded frames
- **KT4** the class of reflexive and transitive frames
- **KT4B** the class of frames with an equivalence relation
- **K43** The class of transitive frames with no right branching
- **KT43** The class of reflexive and transitive frames with no right branching
- **KL** The class of finite transitive trees (weakly complete)
A set of axioms $A$ is sound w.r.t., a class of frames $F$

**Soundness**  \[ \vdash_A \phi \implies \models_F \phi \]

**Weak completeness**  \[ \models_F \phi \implies \vdash_A \phi \]

**Strong completeness**  \[ \Gamma \models_F \phi \implies \text{there is a finite (possibly empty) set of formulas } \phi_1, \ldots, \phi_n \in \Gamma \text{ such that } \vdash_F (\phi_1 \land \cdots \land \phi_n) \supset \phi. \]
**Proposition**  A set of axioms $A$ is strongly complete w.r.t., a class of frames $F$ iff for every $A$-consistent set of formula $\Gamma$ (i.e., $\Gamma \vDash_A \bot$), there is a frame $\mathcal{F} = \langle W, R \rangle \in F$, and a world $w \in W$ such that $\mathcal{F}, w \models \Gamma$. 
(Strong) completeness theorem for a set of axioms $A$ can be proved by constructing a model for any set of $A$-consistent formulas $\Gamma$.

Such a construction is based on the basic and pervasive idea of

**CANONICAL MODEL**

Every completeness result in modal logic is based on canonical models.
Intuitively a canonical model $M_c = \langle F_c, I_c \rangle$ for $A$, such that

- $F_c = \langle W_c, R_c \rangle$, such that
  - each $w \in W_c$ is a maximally $A$-consistent set of formulas;
  - if $\Diamond \phi \in w$ then there is a $w R w'$ such that $\phi \in w'$
- $I_c(p) = \{ w \in W | p \in w \}$.
Canonical model – intuition
A set of formula $\Gamma$ is $\neg$-maximally consistent if it is consistent and any other set $\Sigma$, with $\Gamma \subset \Sigma$, is inconsistent.
Lindenbaum’s Lemma

Any $\mathcal{A}$-consistent set of formulas $\Sigma$ can be extended to an $\mathcal{A}$-maximally consistent set of formulas $\Gamma$.

Proof.

- Let $\phi_1, \phi_2, \ldots$ an enumeration of all the formulas of the language
- Let $\Sigma = \Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \ldots$, with

$$
\Sigma_{n+1} = \begin{cases}
\Sigma_n \cup \{\phi_n\} & \text{If } \Sigma_n \cup \{\phi_n\} \text{ is consistent} \\
\Sigma_n & \text{otherwise}
\end{cases}
$$

- Let $\Gamma = \bigcup_{n \geq 1} \Sigma_n$
- Each $\Sigma_n$ is consistent!
- $\Gamma$ is maximally consistent!
The canonical model $\mathcal{M}^A$ for a set of axioms $A$ is equal

$$\mathcal{M}^A = \left< \mathcal{F}^A = \left< W^A, R^A \right>, \mathcal{I}^A \right>$$

with:

- $W^A$ is the set of all $A$-maximally consistent set of formulas;
- $R^A$ is such that $wRv$ if and only if $\Diamond v \subseteq w$
  \begin{align*}
  \Diamond X &= \{ \Diamond \phi | \phi \in X \}.
  
  \mathcal{I}(p) &= \{ w \in W^A | p \in w \}.
  
\end{align*}
Properties of the canonical model

1. \( w R v \) if and only if for all \( \phi, \Box \phi \in w \) implies \( \phi \in v \).

2. \( \phi \in w \) implies that there is a \( v \in W \), such that \( w R v \) and \( \phi \in v \).

3. \( M^A, w \models \phi \) if and only if \( \phi \in w \).
Canonical model theorem  Any set of axioms $A$ is strongly complete w.r.t., its canonical model

**Proof.** We have to prove that, $\Gamma$ $A$-consistent implies that there is a model $\mathcal{M}$ and a world $w$ such that $\mathcal{M}, w \models \Gamma$. We use the canonical model

Suppose $\Gamma$ is $A$-consistent, by Lindenbaum’s lemma there is a $A$-maximally consistent set $\Sigma$, with $\Gamma \in \Sigma$,
Therefore there is a $w \in W^A$, such that $w = \Sigma$, and $\Gamma \subseteq w$.
By the previous properties $\mathcal{M}^A, w \models \Gamma$.  \qed
Completeness via canonical model

To prove strongly completeness of $\mathbf{KX}$ w.r.t., the class of frames with a property $P$, it is enough to show that the canonical model $\mathcal{M}^{\mathbf{KX}}$ has the property $P$.

If $\Gamma$ is $\mathbf{KX}$-consistent than it has a model (the canonical model $\mathcal{M}^{\mathbf{KX}}$) which has the property $P$.

- Suppose that $\Gamma \models_{F_P} \phi$, where $F_P$ is the class of frames with property $P$.
- Suppose by contradiction that $\Gamma \not\models_{\mathbf{KX}} \phi$,
- then $\Gamma \cup \{\neg \phi\}$ is $\mathbf{KX}$-consistent
- then there is a model (the canonical model) with property $P$ that satisfies $\Gamma \cup \{\neg \phi\}$
- contradiction with the fact that $\Gamma \models_{P} \phi$
Prove that the relation $M^{K4}$ is transitive.

- Suppose that $wR^{K4}v$ and $vR^{K4}u$,
- (remind) $wR^{K4}v$ iff $\diamond v \subseteq w$.
- suppose that $\phi \in u$, then $\diamond \phi \in v$, then $\diamond \diamond \phi \in w$
- (remind) $K4 = \diamond \diamond \phi \supset \diamond \phi$.
- by $K4$ $\diamond \phi \in w$,
- which implies that $\diamond u \in w$
- and therefore $wR^{K4}u$. 
### Assignment
Prove strong completeness for the following cases

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
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</table>
We study operations on models which preserves some properties.

The invariant property

The most important invariant property we study is $\models$ satisfaction.

Operations on models

- Disjoint union
- Generated submodels
- Bisimulation
Disjoint union $\mathcal{M}_1 \uplus \mathcal{M}_2$ – intuition
Two models $\mathcal{M}_1$ and $\mathcal{M}_2$ are disjoint if $W_1 \cap W_2 = \emptyset$. The disjoint union of $\mathcal{M}_1$ and $\mathcal{M}_2$, $\mathcal{M} = \mathcal{M}_1 \uplus \mathcal{M}_2$ is defined

- $W = W_1 \cup W_2$
- $R = R_2 \cup R_2$
- $\mathcal{I}(p) = \mathcal{I}_1(p) \cup \mathcal{I}_2(p)$

Disjoint union can be generalized to any set of models $\{\mathcal{M}_i\}_{i \in I}$

$$\biguplus_{i \in I} \mathcal{M}_i$$
Invariant property for disjoint union

For any $i \in I$ and $w \in W_i$

$$M_i, w \models \phi \quad \text{iff} \quad \bigcup_{i \in I} M_i, w \models \phi$$

Satisfaction is invariant under disjoint union.
Generated submodels

This is not a generated submodel

This is a generated submodel
\( \mathcal{M}' \) is a generated submodel of \( \mathcal{M} \), in symbols \( \mathcal{M}' \rightarrow \mathcal{M} \) iff the following three conditions hold:

1. \( W' \subseteq W \)
2. \( R' = R \cap W' \times W' \)
3. If \( R(W') \subseteq W' \) (i.e., \( w R v \) and \( w \in W' \) implies \( v \in W' \)).

If conditions 1. and 2. hold then \( \mathcal{M}' \) is a submodel of \( \mathcal{M} \).
If $M' \rightarrow M$ then for each $w \in W'$ and for each $\phi$

$$M, w \models \phi \iff M', w \models \phi$$

Satisfaction is invariant under generated submodel.
Bisimulation

- **Bisimulation** is a very general relation between models, which preserves satisfaction.

- A bisimulation between $\mathcal{M}$ and $\mathcal{M}'$ is a relation $Z \subseteq W \times W'$

- $wZw'$ intuitively means that any computation starting from $w$ can be simulated by a computation starting from $w'$ and vice versa.
Bisimulation – intuition

- Models describes the possible evolution of a finite state machine;
- Two models $\mathcal{M}$ and $\mathcal{M}'$ bisimulate, if any computation described in $\mathcal{M}$ can be simulated in $\mathcal{M}'$ and viceversa.
Given two models $\mathcal{M}$ and $\mathcal{M}'$, a relation $Z \subseteq W \times W'$ is a **bisimulation** if and only if the following conditions hold:

1. $wZw'$ implies that $w \in \mathcal{I}(p)$ iff $w' \in \mathcal{I}(p)$ for all primitive propositions $p \in P$ (i.e., $w$ and $w'$ agree on the interpretations of all the propositional formulas).

2. $wZw'$ and $wRv$ then there is a $v' \in W'$ such that $vZv'$

3. (the converse of 2) $wZw'$ and $w'Rv'$ implies that there is a $v \in W$ such that $wZv$
Bisimulation

- Disjoint union is a special case of bisimulation.

\[ Z = \{ \langle w, w \rangle \mid w \in W_i \} \]

\( Z \) is a bisimulation between \( M_i \) and \( \bigcup_{i \in I} M_i \)

- Generated submodel is a special case of bisimulation too.

\[ Z = \{ \langle w, w \rangle \mid w \in W' \} \]

\( Z \) is a bisimulation between the model \( M' \) generated from \( M \).
Invariant property for bisimulation

If $Z$ is a bisimulation between $M$ and $M'$ then

$$w Z w' \quad \text{implies that for all } \phi \ M, w \models \phi \iff M' w' \models \phi$$

Proof. By induction on $\phi$.

$\phi$ is $p$. $M, w \models p \iff w \in I(p)$ iff (by condition 1 of bisimulation)

$$w' \in I(p) \iff M' w' \models p$$

$\phi$ is $\phi_1 \land \phi_2$ . . .

$\phi$ is $\lozenge \psi$. $M, w \models \lozenge \psi \iff$ there is a $v$ with $w R v$ and $M, v \models \phi$. By condition 2 of bisimulation there is a $v'$ with $w' R v'$ and $v Z v'$. By induction $v' \models \psi$. This implies that $M' w' \models \lozenge \psi$.

For the vice-versa we reason similarly, using the converse condition on the definition of bisimulation (condition 3)
What about the converse?

For all $\phi M, w \models \phi$ implies $wZw'$

NO!!
Finite model property tells us that a formula $\phi$ is satisfiable by any (possibly infinite) model, iff it is satisfiable by a finite model

Finite model property is very important in order to define a decision procedure for satisfiability in modal logics
Build a model that satisfies the following formula

\( \lozenge(p \land \lozenge(p \land \lozenge q) \land \neg \lozenge r) \)
Build a model that satisfies the following formula

$$\Diamond (p \land \Diamond (p \land \Diamond q) \land \neg \Diamond r)$$
Finite models — formal definition

**Finite Model Property**  A class of frames $F$ has the finite model property iff for every formula $\phi$ is satisfiability in $F$ if and only if there is a finite $\mathcal{F} \in F$ such that $\phi$ is satisfiable in $\mathcal{F}$. 
Finite model property via filtration

Large model with property $P$ \rightarrow \text{filtration} \rightarrow \text{finite model with property } P

Given a set of formulas $\Sigma$ and a model $\mathcal{M}$ and two worlds $w, v \in W$

\[ w \leftrightarrow \Sigma v \]

if and only if for all $\phi \in \Sigma$, $\mathcal{M}, w \models \phi$ iff $\mathcal{M}', v \models \phi$. 
The filtration of $\mathcal{M}$ w.r.t. $\Sigma$, denoted with $\mathcal{M}_\Sigma^f = \langle W^f_\Sigma, R^f_\Sigma, \mathcal{I}^f_\Sigma \rangle$

where

- $W^f_\Sigma = W/ \leftrightarrow_\Sigma$
- if $wRv$ then $[w]R^f_\Sigma[v]$
- $\mathcal{I}^f_\Sigma(p) = \{[w]| w \in \mathcal{I}(p)\}$. 
**Filtration theorem**

**Proposition on finiteness**  If $\Sigma$ is finite and closed under subformula, then $\mathcal{M}_\Sigma^f$ has at most $2^{|\Sigma|}$ nodes

**Filtration theorem**  IF $\Sigma$ is closed under subformula then, for all $\phi \in \Sigma$

$$\mathcal{M}, w \models \phi \text{ iff } \mathcal{M}_\Sigma^f, [w] \models \phi$$
If $\phi$ is satisfiable then it is satisfiable in a model which has at most $2^{|\phi|}$, where $|\phi|$ is the number of subformulas of $\phi$. 
Grazie a tutti e buon week end!!!!!!