Logics for knowledge representation

A course of the ICT International Doctrorate School

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Modal logic II

Normal Modal logic

A logic is normal if it contains at least the following axiom schemata

A1	$\phi \supset (\psi \supset \phi)$
A2	$(\phi \supset (\psi \supset \theta)) \supset ((\phi \supset \psi) \supset (\phi \supset \theta))$
A3	$(\neg\psi \supset \neg\phi) \supset ((\neg\psi \supset \phi) \supset \phi)$
MP	$\dfrac{\phi \quad \phi \supset \psi}{\psi}$
K	$\Box(\phi \supset \psi) \supset (\Box\phi \supset \Box\psi)$
Nec	$\dfrac{\phi}{\Box\phi}$ the necessitation rule

Basic property of NML

$$\vdash_A \phi \equiv \psi$$

iff

$$\vdash_A \Diamond\phi \equiv \Diamond\psi$$

iff

$$\vdash_A \Box\phi \equiv \Box\psi$$

VIP axiom schema

(4) $\Diamond\Diamond\phi \supset \Diamond\phi$ $\qquad$ $\Box\phi \supset \Box\Box\phi$

(T) $\phi \supset \Diamond\phi$ $\qquad$ $\Box\phi \supset \phi$

(B) $\phi \supset \Box\Diamond\phi$

(D) $\Box\phi \supset \Diamond\phi$

(3) $\Diamond\phi \wedge \Diamond\psi \supset \Diamond(\phi \wedge \Diamond\psi) \vee \Diamond(\phi \wedge \psi) \vee \Diamond(\Diamond\phi \wedge \psi)$

(L) $\Box(\Box\phi \supset \phi) \supset \Box\phi$

Soundness and completeness

K		the class of all frames
K4	**4**	the class of transitive frames
KT	**T**	the class of reflexive frames
KB	**B**	the class of symmetric frames
KD		the class of right unbounded frames
KT4	**S4**	the class of reflexive and transitive frames
KT4B	**S5**	the class of frames with an equivalence relat
K43	**K4.3**	The class of transitive frames with no right branching
KT43	**S4.3**	The class of reflexive and transitive frames with no right branching
KL		The class of finite transitive trees (weakly co

Remind: Soundness and (strong) completene

A set of axioms A is sound w.r.t., a class of frames F

Soundness $\vdash_A \phi$ implies $\models_F \phi$

Weak completeness $\models_F \phi$ implies $\vdash_A \phi$

Strong completeness $\Gamma \models_F \phi$ implies there is a finite (possi-ble empty) set of formulas $\phi_1, \ldots, \phi \in \Gamma$ such that $\vdash_F (\phi_1 \wedge \cdots \wedge \phi_n) \supset \phi$.

Completeness as satisfiability

Proposition A set of axioms A is strongly complete w.r.t., a class of frames F iff for every A-consistent set of formula Γ (i.e., $\Gamma \nvdash_A \bot$), there is a frame $\mathcal{F} = \langle W, R \rangle \in F$, and a world $w \in W$ such that $\mathcal{F}, w \models \Gamma$.

Canonical model

(Strong) completeness theorem for a set of axioms A can be proved by constructing a model for any set of A-consistent formulas Γ.

Such a construction is based on the basic and pervasive idea of

CANONICAL MODEL

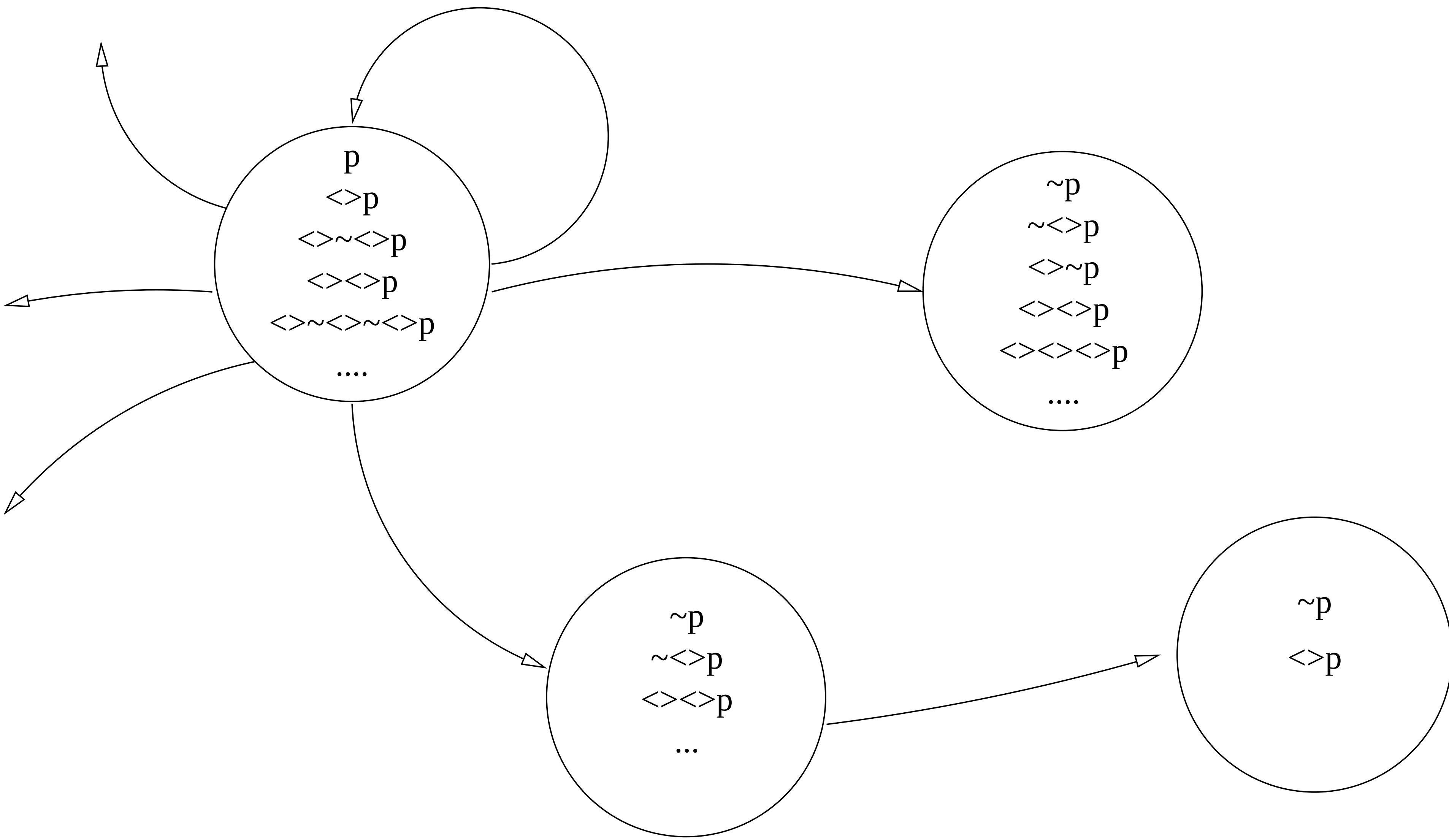
Every completeness result in modal logic is based on canonical models.

Canonical model

Intuitively a canonical model $\mathcal{M}_c = \langle \mathcal{F}_c, \mathcal{I}_c \rangle$ for A, such that

- $\mathcal{F}_c = \langle W_c, R_c \rangle$, such that
  - each $w \in W_c$ is a maximally A-consistent set of formulas;
  - if $\Diamond\phi \in w$ then there is a wRw' such that $\phi \in w'$
- $\mathcal{I}_c(p) = \{w \in W | p \in w\}$.

Canonical model – intuition



A-maximally-consistent-set

A set of formula Γ is A-maximally consistent if it is consistent and any other set Σ, with $\Gamma \subset \Sigma$, is inconsistent.

Lindenbaum's Lemma

Lindenbaum's Lemma Any A-consistent set of formulas Σ can be extended to an A-maximally consistent set of formulas Γ.

Proof.

- Let $\phi_1, \phi_2, \ldots$ an enumeration of all the formulas of the language
- Let $\Sigma = \Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \ldots$, with

$$\Sigma_{n+1} = \begin{cases} \Sigma_n \cup \{\phi_n\} & \text{If } \Sigma_n \cup \{\phi_n\} \text{ is consistent} \\ \Sigma_n & \text{otherwise} \end{cases}$$

  Let $\Gamma = \bigcup_{n \geq 1} \Sigma_n$

- Each Σ_n is consistent!
- Γ is maximally consistent!

Canonical model

The canonical model $\mathcal{M}^A$ for a set of axioms A is equal

$$\mathcal{M}^A = \left\langle \mathcal{F}^A = \left\langle W^A, R^A \right\rangle, \mathcal{I}^A \right\rangle$$

with:

- W^A is the set of all A-maximally consistent set of formulas;
- R^A is such that wRv if and only if $\Diamond v \subseteq w$ ($\Diamond X = \{\Diamond\phi | \phi \in X\}$.
- $\mathcal{I}(p) = \{w \in W^A | p \in w\}$.

Properties of the canonical model

1. wRv if and only if for all ϕ, $\Box\phi \in w$ implies $\phi \in v$.
2. $\phi \in w$ implies that there is a $v \in W$, such that wRv and $\phi \in v$
3. $\mathcal{M}^A, w \models \phi$ if and only if $\phi \in w$.

Canonical Model Theorem

Canonical model theorem Any set of axioms A is strongly complete w.r.t., its canonical model

Proof. We have to prove that, Γ A-consistent implies that there is a model $\mathcal{M}$ and a world w such that $\mathcal{M}, w \models \Gamma$. We use the canonical model

Suppose Γ is A-consistent,
by Lindenbaum's lemma there is a A-maximally consistent set Σ, with $\Gamma \in \Sigma$,
Therefore there is a $w \in W^A$, such that $w = \Sigma$, and $\Gamma \subseteq w$.
By the previous properties $\mathcal{M}^A, w \models \Gamma$. $\square$

Completeness via canonical model

To prove strongly completeness of **KX** w.r.t., the class of frames with a property P, it is enough to show that the canonical model $\mathcal{M}^{\mathbf{KX}}$ has the property P

If Γ is **KX**-consistent than it has a model (the canonical model $\mathcal{M}^{\mathbf{KX}}$ which has the property P.

- Suppose that $\Gamma \models_{F_P} \phi$, where F_P is the class of frames with property P.
- Suppose by contradiction that $\Gamma \not\vdash_{\mathbf{KX}} \phi$,
- then $\Gamma \cup \{\neg\phi\}$ is **KX**-consistent
- then there is a model (the canonical model) with property P that satisfies $\Gamma \cup \{\neg\phi\}$
- contradiction with the fact that $\Gamma \models_P \phi$

Completeness via canonical model – Exampl

Prove that the relation $\mathcal{M}^{\textbf{K4}}$ is transitive.

- Suppose that $wR^{\textbf{K4}}v$ and $vR^{\textbf{K4}}u$,
- (remind) $wR^{\textbf{K4}}v$ iff $\Diamond v \subseteq w$.
- suppose that $\phi \in u$, then $\Diamond\phi \in v$, then $\Diamond\Diamond\phi \in w$
- (remind) **K4** $= \Diamond\Diamond\phi \supset \Diamond\phi$.
- by **K4** $\Diamond\phi \in w$,
- which implies that $\Diamond u \in w$
- and therefore $wR^{\textbf{K4}}u$.

Stong completeness

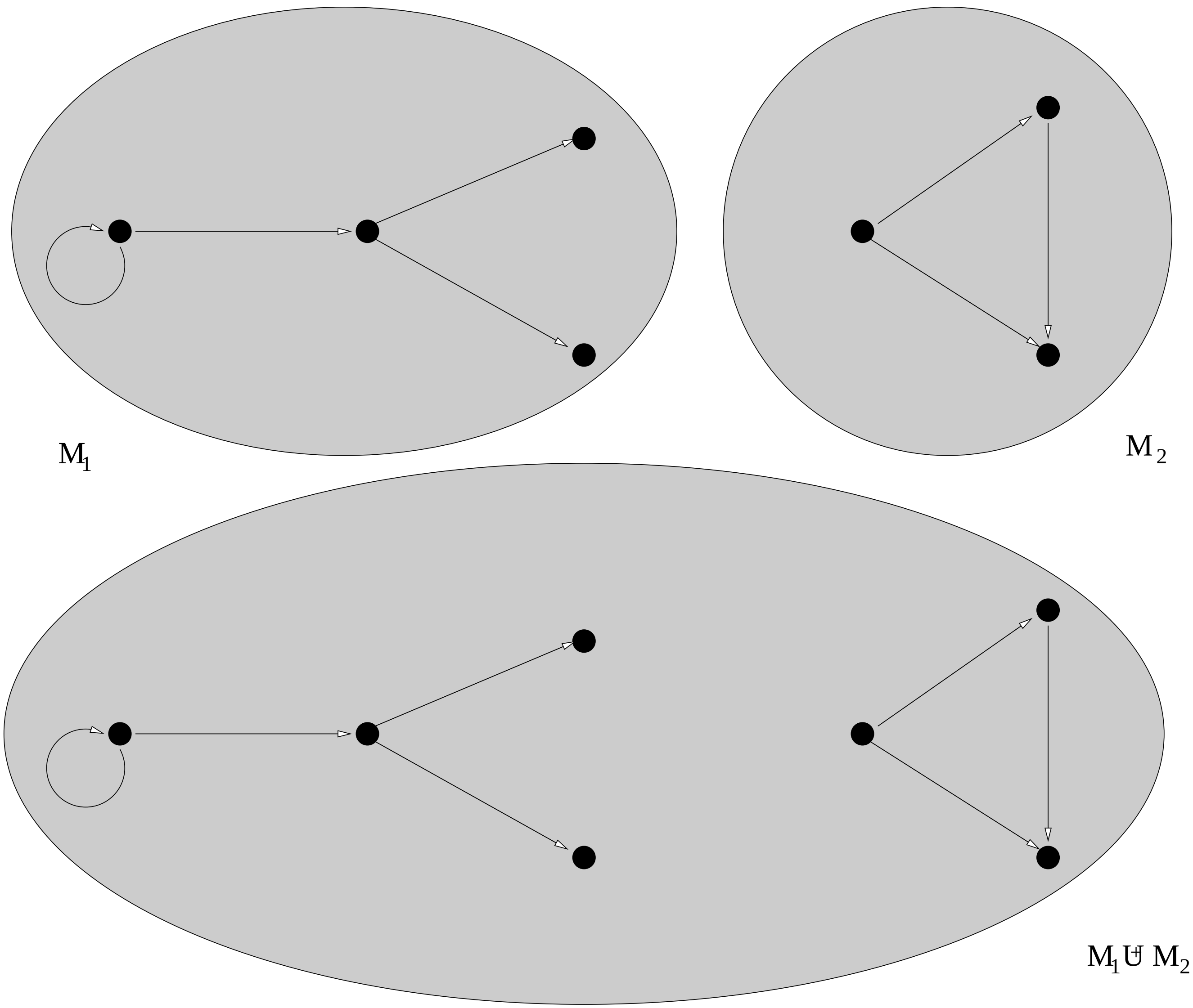
Assignment Prove strong completeness for the following cases

K		the class of all frames
KT	**T**	the class of reflexive frames
KB	**B**	the class of symmetric frames
KD		the class of right unbounded frames
KT4	**S4**	the class of reflexive and transitive frames
KT4B	**S5**	the class of frames with an equivalence relatio

Basic model theory for modal logics

- We study operations on models which preserves some properties.
- The invariant property
- The most important invariant property we study is $\models$ satisfaction.
- Operations on models
  - Disjoint union
  - Generated submodels
  - **Bisimulation**

Disjoint union $\mathcal{M}_1 \uplus \mathcal{M}_2$ – intuition



Disjoint union $\mathcal{M}_1 \uplus \mathcal{M}_2$

Two models $\mathcal{M}_1$ and $\mathcal{M}_2$ are disjoint if $W_1 \cap W_2 = \emptyset$.
The disjoint union of $\mathcal{M}_1$ and $\mathcal{M}_2$, $\mathcal{M} = \mathcal{M}_1 \uplus \mathcal{M}_2$ is defined

- $W = W_1 \cup W_2$
- $R = R_2 \cup R_2$
- $\mathcal{I}(p) = \mathcal{I}_1(p) \cup \mathcal{I}_2(p)$

Disjoint union can be generalized to any set of models $\{\mathcal{M}_i\}_{i \in I}$

$$\biguplus_{i \in I} \mathcal{M}_i$$

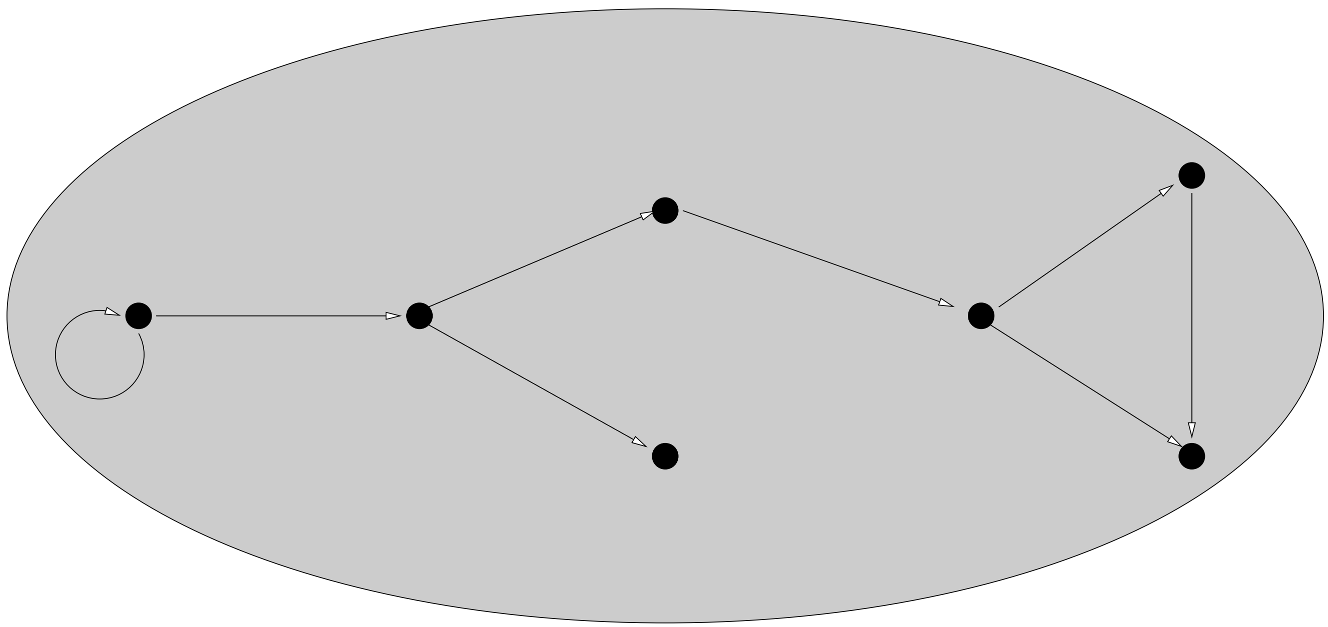
Invariant property for disjoint union

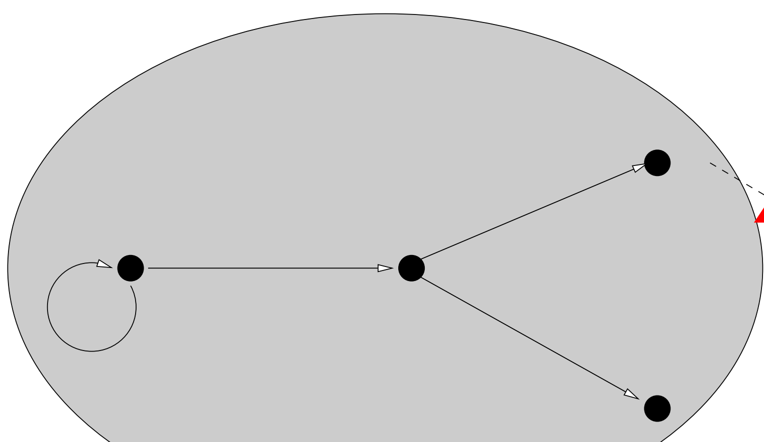
For any $i \in I$ and $w \in W_i$

$$\mathcal{M}_i, w \models \phi \quad \text{iff} \quad \biguplus_{i \in I} \mathcal{M}_i, w \models \phi$$

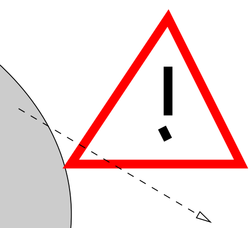
Satisfaction is invariant under disjoint union.

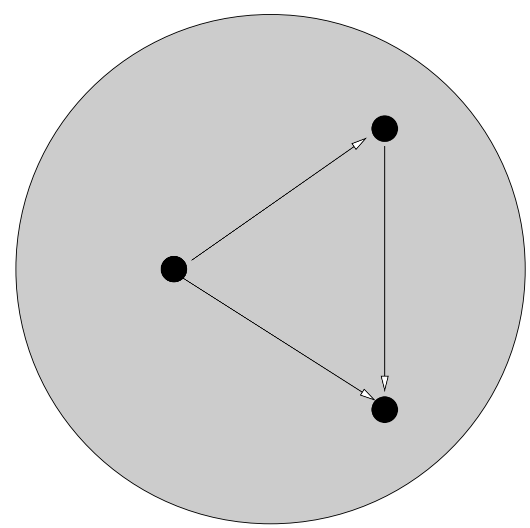
Generated submodels





This is not a generated submodel





This is a generated submodel

Generated submodel

$\mathcal{M}'$ is a generated submodel of $\mathcal{M}$, in symbols $\mathcal{M}' \rightarrowtail \mathcal{M}$ iff the following three condition holds

1. $W' \subseteq W$
2. $R' = R \cap W' \times W'$
3. if $R(W') \subseteq W'$ (i.e., wRv and $w \in W'$ implies $v \in W'$).

If conditions 1. and 2. hold then $\mathcal{M}'$ is a submodel of $\mathcal{M}$.

Invariant property for generated submodels

If $\mathcal{M}' \rightarrowtail \mathcal{M}$ then for each $w \in W'$ and for each ϕ

$$\mathcal{M}, w \models \phi \quad \text{iff} \quad \mathcal{M}', w \models \phi$$

Satisfaction is invariant under generated submodel.

Bsimulation

- bisimulation is a very general relation between models, which preserves satisfaction.
- a bisimulation between $\mathcal{M}$ and $\mathcal{M}'$ is a relation $Z \subseteq W \times W'$
- wZw' intuitively means that any computation starting from w can be simulated by a computation starting from w' and viceversa.

Bisimulation – intuition

- Models describes the possible evolution of a finite state machine;
- Two models $\mathcal{M}$ and $\mathcal{M}'$ bisimulate, if any computation described in $\mathcal{M}$ can be simulated in $\mathcal{M}'$ and viceversa.

Bisimulation – formal definition

Given two models $\mathcal{M}$ and $\mathcal{M}'$, a relation $Z \subseteq W \times W'$ is a bisimulation if and only if the following conditions hold:

1. wZw' implies that $w \in \mathcal{I}(p)$ iff $w' \in \mathcal{I}(p)$ for all primitive propositions $p \in P$ (i.e., w and w' agree on the interpretations of all the propositional formulas).
2. wZw' and wRv then there is a $v' \in W'$ such that vZv'
3. (the converse of 2) wZw' and $w'Rv'$ implies that there is a $v \in W$ such that wZv

Bisimulation

- Disjoint union is a special case of bisimulation.

$$Z = \{\langle w, w\rangle \, | w \in W_i\}$$

  Z is a bisimulation between $\mathcal{M}_i$ and $\biguplus_{i \in I} \mathcal{M}_i$

- Generated submodel is a special case of bisimulation too.

$$Z = \{\langle w, w\rangle \, | w \in W'\}$$

  Z is a bisimulation betwen the model $\mathcal{M}'$ generated from $\mathcal{M}$.

Invariant property for bisimulation

if Z is a bisimulation between $\mathcal{M}$ and $\mathcal{M}'$ then

$$wZw' \quad \text{implies that for all } \phi \quad \mathcal{M}, w \models \phi \text{ iff } \mathcal{M}'w' \models \phi$$

Proof. By induction on ϕ.

ϕ is p . $\mathcal{M}, w \models p$ iff $w \in \mathcal{I}(p)$ iff (by condition 1 of bisimulation) $w' \in \mathcal{I}(p)$ iff $\mathcal{M}'w' \models p$

ϕ is $\phi_1 \wedge \phi_2$ ...

ϕ is $\Diamond\psi$ $\mathcal{M}, w \models \Diamond\psi$ iff there is a v with wRv and $\mathcal{M}, v \models \phi$. By condition 2 of bisimulation there is a v' with $w'Rv'$ and vZv'. By induction $v' \models \psi$. This implies that $\mathcal{M}'w' \models \Diamond\psi$.
For the vice-versa we reason similarly, using the converse condition on the defi nition of bisimulation (condition 3)

Invariant property for bisimulation

What about the converse?

For all $\phi \mathcal{M}, w \models \phi$ implies wZw'

NO!!

Finite models

- Finite model property tells us that a formula ϕ is satisfiable by any (possibly infinite) model, iff it is satisfiable by a finite model
- Finite model property is very important in order to define a decision procedure for satisfiability in modal logics

Finite models – the intuition

Build a model that satisfies the following formula

$$\Diamond(p \wedge \Diamond(p \wedge \Diamond q) \wedge \neg\Diamond r)$$

Finite models – the intuition

Build a model that satisfies the following formula

$$\Diamond(p \wedge \Diamond(p \wedge \Diamond q) \wedge \neg\Diamond r)$$

Finite models — formal definition

Finite Model Property A class of frames F has the finite model property iff for every formula ϕ is satisfiability in F if and only if there is a finite $\mathcal{F} \in F$ such that ϕ is satisfiabile in $\mathcal{F}$.

Finite model property via filtration

Large model with property P $\longrightarrow$ $\boxed{\text{filtration}}$ $\longrightarrow$ finite model with property P

Given a set of formulas Σ and a model $\mathcal{M}$ and two worlds $w, v \in W$

$$w \leftrightsquigarrow_{\Sigma} v$$

if and only if for all $\phi \in \Sigma$, $\mathcal{M}, w \models \phi$ iff $\mathcal{M}', v \models \phi$.

Filtration – formal definition

The filtration of $\mathcal{M}$ w.r.t, Σ, denoted with $\mathcal{M}^f_\Sigma = \left\langle W^f_\Sigma, R^f_\Sigma, \mathcal{I}^f_\Sigma \right\rangle$ where

- $W^f_\Sigma = W / \leftrightsquigarrow_\Sigma$
- if wRv then $[w]R^f_\Sigma[v]$
- $\mathcal{I}^f_\Sigma(p) = \{[w] | w \in \mathcal{I}(p)\}$.

Filtration theorem

Proposition on finiteness If Σ is finite and closed under subformula, then $\mathcal{M}^f_\Sigma$ has at most $2^{|\Sigma|}$ nodes

Filtration theorem IF Σ is closed under subformula then, for all $\phi \in \Sigma$

$$\mathcal{M}, w \models \phi \quad \text{iff} \quad \mathcal{M}^f_\Sigma, [w] \models \phi$$

Finite model property via Filtration

If ϕ is satisfiable then it is satisfiable in a model which has at most $2^{|\phi|}$, where $|\phi|$ is the number of subformulas of ϕ.

Grazie a tutti e buon week end!!!!!