Mathematical Logics

12. Soundness and Completeness of tableaux reasoning in first order logic

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Example of tableaux

Example

Consider the following formulas:

$$
\begin{array}{ll}
(a) & \forall xyz(P(x,y) \wedge P(y,z) \supset P(x,z) \\
(b) & \forall xy(P(x,y) \supset P(y,x)) \\
(c) & \forall x \exists y P(x,y) \\
(d) & \forall x P(x,x)
\end{array}
$$

Show that $(a),(b),(c) \models (d)$. and that $(a),(b) \not\models (d)$.

Solution $((a), (b), (c) \models (d))$

- $(a), (b), (c), \neg(d)$
  - $\neg P(a, a)$
    - $\exists y P(a, y)$
      - $P(a, b)$
        - $P(a, b) \rightarrow P(b, a)$
          - $\neg P(a, b)$
            - $\times$
          - $P(b, a)$
            - $P(a, b) \wedge P(b, a) \supset P(a, a)$
              - $\neg(P(a, b) \wedge P(b, a))$
                - $\neg P(a, b)$
                  - $\times$
                - $\neg P(b, a)$
                  - $\times$
              - $P(a, a)$
                - $\times$

Solution $((a), (b), \not\models (d))$

- $(a), (b), \neg(d)$
  - $\neg P(a, a)$
    - $P(a, a) \to P(a, a)$
      - $\neg P(a, a)$
        - $P(a, a) \wedge P(a, a) \supset P(a, a)$
          - $\neg(P(a, a) \wedge P(a, a))$
            - $\neg P(a, a)$
            - $\neg P(a, a)$
          - $P(a, a)$
            - $\times$
      - $P(a, a)$
        - $\times$

The tableaux is complete, i.e., no other rules can be applied, and it contains at least an open branch (the one on the left). From this open branch we can construct an interpretation $\mathcal{I}$ *with* $\Delta^{\mathcal{I}} = \{a\}$ *(the constant that appear in the branch), and* $P^{\mathcal{I}} = \emptyset$*, since* $\neg P(a, a)$ *occurs in the branch. Notice that* $\mathcal{I} \models (a), (b), \neg(d)$*. Therefore we can conclude that* $(a), (b) \not\models (d)$*.*

Soundness and Completeness

Definition (Derivation relation via tableaux)

Let ϕ be a first-order formula and Γ a set of such formulas.

$$\Gamma \vdash \phi$$

means that there exists a closed tableau for $\Gamma \cup \{\neg\phi\}$.

Theorem (Soundness)

If $\Gamma \vdash \phi$ *then* $\Gamma \models \phi$*.*

Theorem (Completeness)

If $\Gamma \models \phi$ *then* $\Gamma \vdash \phi$*.*

Important note

The mere existence of a closed tableau does not mean that we have an effective method of finding it! Concretely: we dont know how often we need to apply the γ rule and what terms to use for the substitutions.

Proof of Soundness

- Soundness means that what you infer via syntactic manipulation ($\vdash$) is correct from the semantic point of view ($\models$). I.e., if you are able to infer ϕ from Γ ($\Gamma \vdash \phi$), then ϕ is a logical consequence of Γ, ($\Gamma \models \phi$)
- We have to show that $\Gamma \vdash \phi \Longrightarrow \Gamma \models \phi$
- which is equivalent to show that $\Gamma \not\models \phi \Longrightarrow \Gamma \not\vdash \phi$
- which is equivalent to show that $\Gamma \cup \{\neg\phi\}$ is consistent $\Longrightarrow$ the saturated tableaux for $\gamma_1 \wedge \cdots \wedge \gamma_n \wedge \neg\phi$ is open, i.e., it contains an open branch.
- in practice we show that each of the expansion rules preserves satisfiability:
  - If a non-branching rule is applied to a satisfiable branch, the result is another satisfiable branch.
  - If a branching rule is applied to a satisfiable branch, at least one of the resulting branches is also satisfiable.

Proof of Soundness

Definition (Satisfiable branch)

A branch β of a tableaux τ is satisfiable if the set of formulas that occurs in β is satisfiable. I.e., if there is an interpretation $\mathcal{I}$, such that $\mathcal{I} \models \phi$ for all $\phi \in \beta$.

Expansion rules preserve satisfiability

We show that every rule extend a consistent branch β to a branch β' which is consistent.

Proof of Soundness

Propositional α-rules

$$\frac{\phi \wedge \psi}{\begin{array}{c}\phi \\ \psi\end{array}}$$

- let $\mathcal{I}$ be such that $\mathcal{I} \models \beta$
- since $\phi \wedge \psi \in \beta$ then $\mathcal{I} \models \phi \wedge \psi$
- which implies that $\mathcal{I} \models \phi$ and $\mathcal{I} \models \psi$
- which implies that $\mathcal{I} \models \beta \cup \{\phi, \psi\}$.

Proof of Soundness

Propositional β Rules

$$\frac{\phi \vee \psi}{\phi \mid \psi}$$

- let $\mathcal{I}$ be such that $\mathcal{I} \models \beta$
- since $\phi \vee \psi \in \beta$ then $\mathcal{I} \models \phi \vee \psi$
- which implies that $\mathcal{I} \models \phi$ or $\mathcal{I} \models \psi$
- which implies that $\mathcal{I} \models \beta \cup \{\phi\}$ or $\mathcal{I} \models \beta \cup \{\psi\}$.

Proof of Soundness

γ-rules

$$\frac{\forall x \phi(x)}{\phi(a)}$$

- Let β be a that contains the formula $\forall x \phi(x)$. By applying the gamma rule we have that it is extended to $\beta \cup \{\phi(t)\}$ where t is a term occurring in some formula of β.
- If β is satisfiable then there is an interpretation $\mathcal{I} \models \beta$
- This implies that $\mathcal{I} \models \forall x \phi(x)$
- which implies that $\mathcal{I} \models \phi(t)$ for any term t.
- therefore $\mathcal{I}$ satisfies the extended branch $\beta \cup \{\phi(t)\}$.

Similar argument can be done for the second γ-rule. $\frac{\neg \exists x \phi(x)}{\neg \phi(t)}$

Proof of Soundness

δ-rules

$$\frac{\exists x \phi(x)}{\phi(c)} \qquad c \text{ is a fresh constant}$$

- Let $\mathcal{I}$ be such that $\mathcal{I} \models \beta$
- since $\exists x \phi(x) \in \beta$, then $\mathcal{I} \models \exists x \phi(x)$
- this implies that for some $d \in \Delta^{\mathcal{I}}$, $\mathcal{I} \models \phi(x)[a[x/d]]$.
- let $\mathcal{I}'$ be an interpretation obtained by extending $\mathcal{I}$ with $c^{\mathcal{I}'} = d$. Notice that c being fresh, is not interpreted in $\mathcal{I}$ and therefore $\mathcal{I}'$ agrees with $\mathcal{I}$ on the interpretation of every symbol but c.
- The fact that c does not occurs in β, $\mathcal{I}' \models \beta$.
- this implies that $\mathcal{I}' \models \beta \cup \{\phi(c)\}$.
- i.e., $\beta \cup \{\phi(c)\}$ is consistent.

Similar argument can be done for the second γ-rule. $\frac{\neg\forall x \phi(x)}{\neg\phi(c)}$ with c fresh.

Hintikkas Lemma

Definition (Hintikka set)

A set of first-order formulas Γ is called a Hintikka set provided the following hold:

1. not both $P(t_1, \ldots, t_n) \in H$ and $\neg P(t_1, \ldots, t_n) \in H$ for atoms $P(t_1, \ldots, t_n)$;
2. if $\neg\neg\phi \in H$ then $\phi \in H$ for all formulas ϕ ;
3. if $\alpha \in H$ then $\alpha_1 \in H$ and $\alpha_2 \in H$ for alpha formulas α;
4. if $\beta \in H$ then either $\beta_1 \in H$ or $\beta_2 \in H$ for beta formulas β.
5. for all terms t built from function symbols in H (at least one constant symbol): if $\gamma \in H$ then $\gamma_1(t)$ for gamma formulas γ;
6. if $\delta \in H$ then $\delta_1(t) \in H$ for some term t, for delta formulas δ.

Lemma (Hintikka)

Every Hintikka set is satisfiable

Proof of Hintikkas Lemma

Construct a model $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ from a given Hintikka set H:

- $\Delta^{\mathcal{I}}$ is the set of terms constructible from function symbols appearing in H (add one constant symbol in case there are none). Namely. if H contains the constants, $c_1, c_2, \ldots$ and the function symbols $f_1, f_2, \ldots$ with arity, then $\Delta^{\mathcal{I}}$ is the set of strings recursively defined as follows:
  - $c_1, c_2, \cdots \in \Delta^{\mathcal{I}}$
  - if $x_1, \ldots x_{arity(f_i)} \in \Delta^{\mathcal{I}}$ then $f_i(x_1, \ldots, x_{arity(f_i)}) \in \Delta^{\mathcal{I}}$
- $\cdot^{\mathcal{I}}$ is defined as follows:
  1. $c^{\mathcal{I}} =$ "c"
  2. function symbols are interpreted as themselves: $f^{\mathcal{I}}(d_1, \ldots, d_n) = f(d_1, \ldots, d_n)$
  3. predicate symbols: $P^{\mathcal{I}} = \{\langle d_1, \ldots, d_n \rangle \in \Delta^{\mathcal{I}} \mid P(d_1, \ldots, d_n) \in H\}$

Claim: $\phi \in H$ implies $\mathcal{I} \models \phi$

Proof: By structural induction on ϕ.

Proof of Hintikkas Lemma - example

Example

Consider the following Hintikka set

$$H = \left\{ \begin{array}{c} P(a), \neg P(f(a)), Q(a,b), Q(g(a,b),a), \\ P(b) \supset \exists x Q(x,b), \exists x Q(x,b) \end{array} \right\}$$

Then the interpretation $\mathcal{I}$ associated to H is the following:

- $\Delta^{\mathcal{I}} = \left\{ \begin{array}{c} a, b, f(a), f(b), g(a,a), g(a,b), g(b,a), g(b,b) \\ f(f(a)), f(f(b)), f(g(a,a)), f(g(a,b)), f(g(b,a)), f(g(b,b)) \\ g(a, f(a)), g(a, f(b)), g(b, f(a)), g(b, f(b)) \\ g(f(a), a), g(f(a), b), g(f(b), a), g(f(b), b) \\ g(f(a), f(a)), g(f(a), f(b)), g(f(b), f(a)), g(f(b), f(b)), \dots \end{array} \right\}$
- $f^{\mathcal{I}}(x) = f(x)$ for every $x \in \Delta^{\mathcal{I}}$
- $P^{\mathcal{I}} = \{a\}$,
- $Q^{\mathcal{I}} = \{\langle a, b\rangle, \langle g(a,b), a\rangle\}$.

Proof of Completeness

Definition (Fairness)

We call a tableau fair if every non-literal of a branch gets eventually analysed on this branch and, additionally, every γ-formula gets eventually instantiated with every term constructible from the function symbols appearing on a branch.

Proof of Completeness

Completeness proof (sketch).

- We show that $\Gamma \nvdash \phi$ implies $\Gamma \not\models \phi$.
- Suppose that there is no proof for $\Gamma \cup \{\neg\phi\}$
- Let τ a fair tableaux that start with $\Gamma \cup \{\neg\phi\}$,
- The fact that $\Gamma \nvdash \phi$ implies that there is at least an open branch β.
- fairness condition implies that the set of formulas in β constitute an Hintikka set H_β
- From Hintikka lemma we have that there is an interpretation $\mathcal{I}_\beta$ that satisfies β.
- since every branch of τ contains its root we have that $\Gamma \cup \{\neg\phi\} \subseteq \beta$ and therefore $\mathcal{I}_\beta \models \Gamma \cup \{\neg\phi\}$.
- which implies that $\Gamma \not\models \phi$.

$\square$