Mathematical Logics

15. Model theory

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Σ-structure

A first order interpretation of the language that contains the signature $\Sigma = \{c_1, c_2, \ldots, f_1, f_2 \ldots, R_1, R_2, \ldots\}$ is called a Σ-structure, to stress the fact that it is relative to a specific vocabulary.

Σ-structure

Given a vocabulary/signature $\Sigma = \langle c_1, c_2, \ldots, f_1, f_2, \ldots, R_1, R_2, \ldots \rangle$ a Σ-structure is $\mathcal{I}$ is composed of a non empty set $\Delta^{\mathcal{I}}$ and an interpretation function such that

- $c_i^{\mathcal{I}} \in \Delta^{\mathcal{I}}$
- $f_i^{\mathcal{I}} \in (\Delta^{\mathcal{I}})^{arity(f_i)} \longrightarrow \Delta^{\mathcal{I}}$: The set of functions from n-tuples of elements of $\Delta^{\mathcal{I}}$ to $\Delta^{\mathcal{I}}$ with $n - arity(f_i)$
- $R_i^{\mathcal{I}} \in (\Delta^{\mathcal{I}})^{arity(R_i)}$ the set of n-tuples of elements of $\Delta^{\mathcal{I}}$ with $n = arity(R_i)$.

Substructures

Substructure

A Σ-structure $\mathcal{I}$ is a *substructure* of a Σ-structure $\mathcal{J}$, in symbols $\mathcal{I} \subseteq \mathcal{J}$ if

- $\Delta^{\mathcal{I}} \subseteq \Delta^{\mathcal{J}}$
- $c^{\mathcal{I}} = c^{\mathcal{J}}$
- $f^{\mathcal{I}}$ is the restriction of $f^{\mathcal{J}}$ to the set $\Delta^{\mathcal{I}}$, i.e., for all $a_1, \dots, a_n \in \Delta^{\mathcal{I}}$, $f^{\mathcal{I}}(a_1, \dots, a_n) = f^{\mathcal{J}}(a_1, \dots, a_n)$.
- $R^{\mathcal{I}} = R^{\mathcal{J}} \cap (\Delta^{\mathcal{I}})^n$

where n is the arity of f and R.

Example

Let $\Sigma = \langle zero, one, plus(\cdot,\cdot), positive(\cdot), negative(\cdot) \rangle$

$\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$	$\mathcal{J} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$
$\Delta^{\mathcal{I}} = \{0, 1, 2, 3, \dots\}$	$\Delta^{\mathcal{J}} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
$zero^{\mathcal{I}} = 0$, $one^{\mathcal{I}} = 1$	$zero^{\mathcal{J}} = 0$, $one^{\mathcal{I}} = 1$
$plus^{\mathcal{I}}(x, y) = x + y$	$plus^{\mathcal{J}}(x, y) = x + y$
$positive^{\mathcal{I}} = \{1, 2, \dots\}$	$positive^{\mathcal{J}} = \{1, 2, \dots\}$
$negative^{\mathcal{I}} = \emptyset$	$negative^{\mathcal{J}} = \{-1, -2, \dots\}$

Proposition

If $\mathcal{I} \subseteq \mathcal{J}$ then for every ground formula ϕ $\mathcal{I} \models \phi$ iff $\mathcal{J} \models \phi$

Proof.

- A ground formula is a formula that does not contain individual variables and quantifiers. So ϕ is ground if it is a boolean combination of atomic formulas of the form $P(t_1, \dots, t_n)$ with t_i's ground terms, i.e., terms that do not contain variables.
- If t is a ground term then $t^{\mathcal{I}} = t^{\mathcal{J}}$ (proof by induction on the construction of t)
  - if t is the constant c, then by definition $c^{\mathcal{I}} = c^{\mathcal{J}}$
  - if t is $f(t_1, \dots, t_n)$, then t is ground implies that each t_i is ground. By induction $t_i^{\mathcal{I}} = t_i^{\mathcal{J}} \in \Delta^{\mathcal{I}} \subseteq \Delta^{\mathcal{J}}$. Since the definitions of $f^{\mathcal{I}}$ and $f^{\mathcal{J}}$ coincide on the elements of $\Delta^{\mathcal{I}} \cap \Delta^{\mathcal{J}}$, we have that $f^{\mathcal{I}}(t_1^{\mathcal{I}}, \dots, t_n^{\mathcal{I}}) = f^{\mathcal{I}}(t_1^{\mathcal{I}}, \dots, t_n^{\mathcal{I}})$ and therefore $(f(t_1, \dots, t_n))^{\mathcal{I}} = (f(t_1, \dots, t_n))^{\mathcal{J}}$
- if ϕ is $P(t_1, \dots, t_n)$ with t_i's ground terms, then, by induction we have that $t_i^{\mathcal{I}} = t_i^{\mathcal{J}} \in \Delta^{\mathcal{I}} \subseteq \Delta^{\mathcal{J}}$ for $1 \leq i \leq n$. The fact that $P^{\mathcal{I}} = P^{J} \cap (\Delta^{\mathcal{I}})^n$ implies that

$$\mathcal{I} \models P(t_1, \dots, t_n) \quad \text{iff} \quad \mathcal{J} \models P(t_1, \dots, t_n)$$

- the fact that $\mathcal{I}$ and $\mathcal{J}$ agree on all the atomic ground formulas implies that they agree also on all the boolean combinations of the ground formulas.

Minimal substructure

Smallest Σ-substructure

From the previous property, we have that every substructure of a Σ-structure $\mathcal{J}$, must contain at least enough elements to interpret all the ground terms, i.e., the terms that can be built starting from constants and applying the functions.

- Given a structure $\mathcal{J}$ we can define the smallest Σ-substructure of $\mathcal{J}$ as the structure defined on the domain $\Delta^{\mathcal{I}} \subseteq \Delta^{\mathcal{J}}$ recursively defined as follows:
  - $c_1^{\mathcal{J}}, c_2^{\mathcal{J}}, \dots \in \Delta^{\mathcal{I}}$
  - if $x_1, \dots, x_n \in \Delta^{\mathcal{I}}$ and $f \in \Sigma$ and $arity(f) = n$ then $f^{\mathcal{J}}(x_1, \dots, x_n) \in \Delta^{\mathcal{I}}$
- The minimal Σ-substructure of $\mathcal{J}$ depends from Σ, the larger Σ the larger the minimal Σ-substructure of $\mathcal{J}$
- if Σ contains only a finite number of constants $c_1, \dots, c_n$ and no function symbols, then the minimal Σ-substructure of a Σ-structure $\mathcal{J}$ contains at most n elements. i.e., $\Delta^{\mathcal{I}} = \{c_1^{\mathcal{J}}, \dots, c_n^{\mathcal{J}}\}$.

Minimal substructure

Example

1. Let $\Sigma = \langle a, b, f(\cdot,\cdot), T(\cdot,\cdot)\rangle$.
2. Let $\mathcal{J} = \left\langle \Delta^{\mathcal{J}}, \cdot^{\mathcal{J}} \right\rangle$ be such that
   - $\Delta^{J} = \mathbb{R}$ (the set of real numbers)
   - $a^{\mathcal{J}} = 0$, $b^{\mathcal{J}} = 1$
   - $f^{\mathcal{J}}(x, y) = x + y$.
   - $T^{\mathcal{J}} = \{\langle x, y\rangle \in \mathbb{R}^2 | x \leq y\}$

How does a substructure $\mathcal{I} = \left\langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \right\rangle$ look like?

- If $\Delta^{\mathcal{I}} = \{1, 2, \dots\}$, then $\mathcal{I} \not\subseteq \mathcal{J}$ since $a^{\mathcal{I}} \notin \Delta^{\mathcal{I}}$.
- if $\Delta^{\mathcal{I}} = \{0, 1, 2\}$, then $\mathcal{I} \not\subseteq \mathcal{J}$ as $\Delta^{\mathcal{I}}$ is not closed under $+$ $(1 + 2 \notin \Delta^{\mathcal{I}})$
- $\Delta^{\mathcal{I}} = \mathbb{Z}$ of non negative integers constitue a substructure because:
  - $a^{\mathcal{J}} \in \mathbb{Z}$, $b^{\mathcal{J}} \in \mathbb{Z}$
  - if $x, y \in \mathbb{Z}$ then $f^{\mathcal{J}}(x, y) = x + y \in \mathbb{Z}$.

Smallest Substructure

Let Σ be a countable[1] signature $\langle c_1, c_2, \dots, f_1, f_2, \dots, R_1, R_2, \dots, \rangle$ and $\mathcal{J}$ be a Σ-structure. The minimal Σ-substructure of $\mathcal{J}$ can be defined as follows:

- $\Delta_0^{\mathcal{I}} = \{c_1^{\mathcal{J}}, c_2^{J}, \dots\}$
- $\Delta_{n+1}^{\mathcal{I}} = \{f^{\mathcal{J}}(x_1, \dots, x_{arity(f)}) | x_i \in \Delta_m^{\mathcal{I}},\ m < n,\ f \in \Sigma\}$
- $\Delta^{\mathcal{I}} = \bigcup_{n \geq 0} \Delta_n^{\mathcal{I}}$
- $R_k^{\mathcal{I}} = R^{\mathcal{J}} \cap (\Delta^{\mathcal{I}})^{arity(R_k)}$

Notice that

- if there is no function $\Delta^{\mathcal{I}} = \Delta_0^{\mathcal{I}}$ and it is finite
- if there is at least a function symbol $\Delta^{\mathcal{I}}$ then you can count the elements of $\Delta^{\mathcal{I}}$.
- This implies that the domain of the minimal Σ-structure of a Σ-structure $\mathcal{J}$ is a countable set[1]

[1] A set S is called countable if there exists an injective function $f : S \longrightarrow \mathbb{N}$ from S to the natural numbers $\mathbb{N} = \{0, 1, 2, 3, \dots\}$.

Universal Formulas stay True in Substructures

Definition (Universal formula)

A universal formula, i.e., a formula with only universal quantifiers (e.g. after Skolemization)

$$\forall x_1, \ldots, x_n . \phi(x_1, \ldots, x_n)$$

where ϕ is a boolean combination of atomic formulas

Property

If ψ is a universal formula and $I \subseteq J$, then

$$\mathcal{J} \models \psi \quad \Longrightarrow \quad \mathcal{I} \models \psi$$

Universal Formulas stay True in Substructures

Proof.

Suppose that ψ is of the form $\forall x_1, \ldots, x_n.\phi(x_1, \ldots, x_n)$ If

$$\mathcal{J} \models \forall x_1, \ldots, x_n.\phi(x_1, \ldots, x_n)$$

then for every assignment a to the variable $x_1, \ldots, x_n$ to the elements of $\Delta^{\mathcal{J}}$ we have that

$$\mathcal{J} \models \phi(x_1, \ldots, x_n)[a] \tag{1}$$

Since $\Delta^{\mathcal{I}} \subseteq \Delta^{\mathcal{J}}$, we have that for all the assignments a' of the variables $x_1, \ldots, x_n$ to the elements of $\Delta^{\mathcal{I}}$,

$$\mathcal{J} \models \phi(x_1, \ldots, x_n)[a'] \tag{2}$$

Since $\mathcal{I}$ and $\mathcal{J}$ coincides on the elements of $\Delta^{\mathcal{I}} \cap \Delta^{\mathcal{J}}$ then

$$\mathcal{I} \models \phi(x_1, \ldots, x_n)[a'] \tag{3}$$

with implies that

$$\mathcal{I} \models \forall x_1, \ldots, x_n \phi(x_1, \ldots, x_n)[a] \tag{4}$$

□

$\exists$-Formulas do not stay true in substructures

Example ($\Sigma = \langle zero, one, plus(\cdot,\cdot), positive(\cdot), negative(\cdot)\rangle$)

$\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$	$\mathcal{J} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$
$\Delta^{\mathcal{I}} = \{0, 1, 2, 3, \dots\}$	$\Delta^{\mathcal{J}} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
$zero^{\mathcal{I}} = 0, one^{\mathcal{I}} = 1$	$zero^{\mathcal{J}} = 0, one^{\mathcal{I}} = 1$
$plus^{\mathcal{I}}(x, y) = x + y$	$plus^{\mathcal{J}}(x, y) = x + y$
$positive^{\mathcal{I}} = \{1, 2, \dots\}$	$positive^{\mathcal{J}} = \{1, 2, \dots\}$
$negative^{\mathcal{I}} = \emptyset$	$negative^{\mathcal{J}} = \{-1, -2, \dots\}$

Consider the formulas:

$$\exists x.negative(x) \qquad \exists x.x + one = zero \qquad \forall x.\exists y(x + y = zero)$$

They are satisfiable in $\mathcal{J}$ but not in $\mathcal{I}$. In all cases, the existential quantified variable is instantiated to a negative integer, and in $\mathcal{I}$ there is no negative integers, while $\mathcal{J}$ domain contains also negative integers

- $\mathcal{I} \not\models \exists x.negative(x)$ since there is no element in $negative^{\mathcal{I}}$
- $\mathcal{I} \not\models \exists x.x + one = zero$ since $x + 1 > 0$ for every positive integer x
- $\mathcal{I} \not\models \forall x.\exists y(x + y = zero)$ since if we take $x > 0$ then for all $y \geq 0$, $x + y > 0$.

How can we get rid of $\exists$-quantifiers?

Removing $\exists x$ in front of a formula

From previous classes we know that the formula $\exists x P(x)$ is satisfiable if the formula $P(c)$ for some "fresh" constant c is satisfiable. We can extend this trick: ...

Removing $\exists x$ after $\forall$

- Consider the formula $\forall x \exists y Friend(x, y)$, which means: everybody has at least a friend.
- Therefore for every person p, we can find another person p' which is his/her friend.
- p' depends from p. in the sens that for two person p and q, p' and q' might be different.
- So we cannot replace the existential variable with a constant obtaining $\forall x.Friend(x, c)$.
- we have represent this "pic up" action as a function $f(\cdot)$, and the above formula can be rewritten as

$$\forall x.Friend(x, f(x))$$

Skolemization

Property

Let $\phi(x_1, \ldots, x_n, y)$ be a formula with no $\exists$-quantifiers and with free variables $x_1, \ldots, x_n$ and y.

$$\forall x_1, \ldots, x_n \exists y. \phi(x_1, \ldots, x_n, y) \tag{5}$$

is satisfiable if and only if

$$\forall x_1, \ldots, x_n. \phi(x_1, \ldots, x_n, f(x_1, \ldots, x_n)) \tag{6}$$

is satisfiable.
(6) is called the Skolemization of (5).

Skolemization

Proof.

- $\forall x_1, \ldots, x_n \exists y.\phi(x_1, \ldots, x_n, y)$ satisfiable implies that
- there is an $\mathcal{I}$, $\mathcal{I} \models \forall x_1, \ldots, x_n \exists y.\phi(x_1, \ldots, x_n, y)$. This implies that
- for all assignments a to $x_1, \ldots, x_n$, $\mathcal{I} \models \exists y.\phi(x_1, \ldots, x_n, y)[a]$
- which implies that every assignment a for $x_1, \ldots, x_n$ can be extended to an assignment a' for y, such that $\mathcal{I} \models \phi(x_1, \ldots, x_n, y)[a']$
- let $\mathcal{I}'$ be the interpretation that coincides with $\mathcal{I}$ in all symbols and that interpret a new n-ary function symbol f, as the function returns for every assignment $a(x_1), \ldots, a(x_n)$ the value $a'(y)$.
- $\mathcal{I}' \models \phi(x_1, \ldots, x_n, f(x_1, \ldots, x_n))[a]$ for all assignment a, and therefore
- $\mathcal{I}' \models \forall x_1, \ldots, x_n.\phi(x_1, \ldots, x_n, f(x_1, \ldots, x_n))$
- $\forall x_1, \ldots, x_n.\phi(x_1, \ldots, x_n, f(x_1, \ldots, x_n))$ is satisfiable

$\square$

Prenex Normal Form

Definition (Prenex Normal Form)

A formula is in prenex normal form if it is in the form

$$Q_1 x_1 \dots Q_n x_n \phi(x_1, \dots, x_n)$$

where $\phi(x_1, \dots, x_n)$ is a quantifier free formula, called matrix, and $Q_i \in \{\forall, \exists\}$ for $1 \leq i \leq n$.

Property

Every formula ϕ can be translated in formula $pnf(\phi)$ which is in prenex normal form and such that

$$\models \phi \equiv pnf(\phi)$$

Prenex Normal Form

Proof.

Rename quantified variable, so that each quantifier $\forall x$ and $\exists x$ is defined on a separated variable

$$\forall x P(x) \wedge \exists x P(x) \quad \Longrightarrow \quad \forall x_1 P(x_1) \wedge \exists x_2 P(x_2)$$

Convert to Negation Normal Form using the propositional rewriting rules plus the additional rules

$$\neg(\forall x A) \Longrightarrow \exists x \neg A$$
$$\neg(\exists x A) \Longrightarrow \forall x \neg A$$

Move quantifiers to the front using (provided x is not free in B)

$$(\forall x A) \wedge B \equiv \forall x (A \wedge B)$$
$$(\forall x A) \vee B \equiv \forall x (A \vee B)$$

Skolemization of a PNF formula

Definition

The Skolemization of a pnf formula ϕ, denoted by $sk(\phi)$ is defined as follows:

- if ϕ is $\forall x_1 \dots \forall x_n \psi$, and ψ is a quantifier free formula then

$$sk(\phi) = \phi$$

- if ϕ is $\forall x_1 \dots \forall x_n \exists x_{n+1} \psi(x_1, \dots, x_n, x_{n+1})$, then

$$sk(\phi) = \forall x_1 \dots \forall x_n sk(\psi(x_1, \dots, x_n, f(x_1, \dots, x_n)))$$

  for a "fresh" n-ary functional symbol f.

Property

If ϕ is satisfiable then $sk(\phi)$ is also satisfiable.

Countable Model Theorem

Lemma

A set of universal first-order formulas Γ has a model if and only if it has a countable model.

Proof.

Let $\mathcal{J}$ be a model. Then $\mathcal{J}$ induces a countable sub-structure $\mathcal{I}$. Because all formulas in Γ are universal, $\mathcal{J} \models \Gamma$ implies that $\mathcal{I} \models \Gamma$. $\square$

Theorem

A set of first-order formulas has a model if and only if it has a countable model.

Proof.

Let the set of formulas have a model. Transform the formulas into prenex normal form and skolemize them to eliminate existential quantifiers, which introduces a countable number of skolem

Ground term

A ground term of a signature Σ is a term of Σ that does not contain any variable.

The set of ground terms of a signature Σ can be recursively defined as follows:

- every constant a of Σ is a ground term
- if $t_1, \ldots, t_n$ are ground terms, and f a function symbols of Σ with $arity(f) = n$, then $f(t_1, \ldots, t_n)$ is a ground term
- nothing else is a ground term

The set of ground terms on a signature Σ is known as the

Herbrand Universe on Σ

Herbrand Model: A Generic Countable Model

- Observe that if $\mathcal{J}$ is Σ-structure that satisfies a formulas ϕ in PNF, the domain $\Delta^{\mathcal{I}}$ of the minimal Σ-substructure $\mathcal{I}$ of $\mathcal{J}$, is such that:
  - $\Delta^{\mathcal{I}}$ contains the interpretations of all the constants in Σ, i.e., $a^{\mathcal{J}} \in \Delta^{\mathcal{I}}$
  - $\Delta^{\mathcal{I}}$ is closed under the application of $f^{\mathcal{J}}$ for every function symbol $f \in \Sigma$. i.e., if $x_1, \dots, x_n \in \Delta^{\mathcal{I}}$ then $f^{\mathcal{J}}(x_1, \dots, x_n) \in \Delta^{\mathcal{I}}$, where $k = arity(f)$.
- This implies that all the minimal Σ-substructures of any interpretation that satisfies a PNF formula ϕ, are "similar" to some interpretation defined on the domain of ground terms.
- Instead of looking at arbitrary countable domains and functions on them, we show we can consider a more special class of structures: called ground term models
- In these models the domain the set of expressions built from constants and function symbols, i.e., the Herbrand universe

Herbrand Interpretation

Definition (Herbrand interpretation)

A Herbrand interpretation on Σ is a Σ-structure $\mathcal{H}$ defined on the Herbrand universe $\Delta^{\mathcal{H}}$ such that the following holds:

- $a^{\mathcal{H}} = a$ for every constant a
- for every $t_1, \ldots, t_n \in \Delta^{\mathcal{H}}$, $f^H(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$ for $f \in \Sigma$ function symbol with $arity(f) = n$,

Herbrand interpretation associated to another interpretation

Starting from any interpretation $\mathcal{I}$ we can define the associated Herbrand interpretation $\mathcal{H}(\mathcal{I})$ on the Herbrand Universe as follows:

- $P^{\mathcal{H}(\mathcal{I})}$ as the set of tuples of terms $\langle t_1, \ldots, t_n \rangle$ such that $\mathcal{I} \models P(t_1, \ldots, t_n)$.

Herbrand's Theorem

Lemma

Let $\mathcal{I}$ be a Σ-structure and $\mathcal{H}(\mathcal{I})$ it's associated Herbrand interpretation. For every quantifier free formula $\phi(x_1, \dots, x_n)$

$$\mathcal{I} \models \phi(x_1, \dots, x_n)[a] \quad \textit{if and only if} \quad \mathcal{H}(\mathcal{I}) \models \phi(x_1, \dots, x_n)[a']$$

where

- *a is an assignment to variables on $\Delta^{\mathcal{I}}$, with $a(x_k) = t_k^{\mathcal{I}}$, for $1 \leq k \leq n$*
- *$a'(x_i)$ is an assignment on $\Delta^{\mathcal{H}(\mathcal{I})}$, with $a'(x_k) = t_k$ for $1 \leq k \leq n$.*

Herbrand's Theorem

Proof of Lemma.

We start by showing that $t(x_1, \dots, x_n)^{\mathcal{I}}[a] = t(t_1, \dots, t_n)^{\mathcal{I}}$ by induction on the complexity of $t(x_1, \dots, x_n)$[a]

- **Base case 1:** $t(x_1, \dots, x_n)$ is the constant c, then $c^{\mathcal{I}}[a] = c^{\mathcal{I}}$ by definition
- **Base case 2:** If $t(x_1, \dots, x_n)$ is the variable x_i, then $x_i^{\mathcal{I}}[a] = a(x_i) = t^{\mathcal{I}}$
- **Step case:** if $t(x_1, \dots, x_n)$ is $f(u_1(x_1, \dots, x_n), \dots, u_k(x_1, \dots, x_n))$,
  By definition
  $f(u_1(x_1, \dots, x_n), \dots, u_k(x_1, \dots, x_n))^{\mathcal{I}}[a] =$
  $f^{\mathcal{I}}(u_1(x_1, \dots, x_n)^{\mathcal{I}}[a], \dots, u_k(x_1, \dots, x_n)^{\mathcal{I}}[a])$
  By induction for each $1 \leq h \leq k$,
  $u_h(x_1, \dots, x_n)^{\mathcal{I}}[a] = u_h(t_1, \dots, t_n)^{\mathcal{I}}$,
  and therefore
  $f(u_1(x_1, \dots, x_n), \dots, u_k(x_1, \dots, x_n))^{\mathcal{I}}[a] =$
  $f^{\mathcal{I}}(u_1(t_1, \dots, t_n)^{\mathcal{I}}, \dots, u_k(t_1, \dots, t_n)^{\mathcal{I}})$
  and therefore
  $f(u_1(x_1, \dots, x_n), \dots, u_k(x_1, \dots, x_n))^{\mathcal{I}}[a] =$
  $f(u_1(t_1, \dots, t_n), \dots, u_k(t_1, \dots, t_n))^{\mathcal{I}}$

Herbrand's Theorem

Proof of Lemma (cont'd).

Then we show by induction on the complexity of $\phi(x_1, \dots, x_n)$ that

$$\mathcal{I} \models \phi(x_1, \dots, x_n)[a] \quad \text{if and only if} \quad \mathcal{H}(\mathcal{I}) \models \phi(x_1, \dots, x_n)[a']$$

- **Base case:** If $\phi(x_1, \dots, x_n)$ is atomic, i.e, it is $P(u_1(x_1, \dots, x_n), \dots, u_k(x_1, \dots, x_n))$. Then

  $$\mathcal{I} \models P(u_1(x_1, \dots, x_n), \dots, u_k(x_1, \dots, x_n))[a]$$

  if and only if

  $$\left\langle u_1(x_1, \dots, x_n)^{\mathcal{I}}[a], \dots, u_k(x_1, \dots, x_n)^{\mathcal{I}}[a] \right\rangle \in P^{\mathcal{I}}$$

  if and only if (by previous part of the proof)

  $$\left\langle u_1(t_1, \dots, t_n)^{\mathcal{I}}, \dots, u_k(t_1, \dots, t_n)^{\mathcal{I}} \right\rangle \in P^{\mathcal{I}}$$

  if and only if (by definition of $\mathcal{H}(\mathcal{I})$)

  $$\langle u_1(t_1, \dots, t_n), \dots, u_k(t_1, \dots, t_n) \rangle \in P^{\mathcal{H}(\mathcal{I})}$$

  if and only if

  $$\mathcal{H}(\mathcal{I}) \models P(u_1(t_1, \dots, t_n), \dots, u_k(t_1, \dots, t_n))$$

  if and only if (from the fact that $a'[x_i] = t_i$)

  $$\mathcal{H}(\mathcal{I}) \models P(u_1(x_1, \dots, x_n), \dots, u_k(x_1, \dots, x_n))[a']$$

Proof of Lemma (cont'd).

- **Step case $\wedge$:** if $\phi(x_1, \dots, x_n)$ is of the form $\phi_1(x_1, \dots, x_n) \wedge \phi_2(x_1, \dots, x_n)$ then

  $$\mathcal{I} \models \phi_1(x_1, \dots, x_n) \wedge \phi_2(x_1, \dots, x_n)[a]$$

  if and only if (by definition of satisfiability of $\wedge$)

  $$\mathcal{I} \models \phi_1(x_1, \dots, x_n)[a] \text{ and } \mathcal{I} \models \phi_2(x_1, \dots, x_n)[a]$$

  if and only if (by induction)

  $$\mathcal{H}(\mathcal{I}) \models \phi_1(x_1, \dots, x_n)[a'] \text{ and } \mathcal{H}(\mathcal{I}) \models \phi_2(x_1, \dots, x_n)[a']$$

  if and only if (by definition of satisfiability of $\wedge$)

  $$\mathcal{H}(\mathcal{I}) \models \phi_1(x_1, \dots, x_n) \wedge \phi_2(x_1, \dots, x_n)[a']$$

- **Step case $\vee$:** if $\phi(x_1, \dots, x_n)$ is of the form $\phi_1(x_1, \dots, x_n) \vee \phi_2(x_1, \dots, x_n)$ then ... reason in analogous way ...

□

Herbrand's Theorem

Herbrand's theorem is one of the fundamental theorems of mathematical logic and allows a certain type of reduction of first-order logic to propositional logic. In its simplest form it states:

Definition (Ground instance)

A ground instance of the universally quantified formula $\forall x_1, \ldots, x_n \phi(x_1, \ldots, x_n)$ is a ground formula $\phi(t_1, \ldots, t_n)$ obtained by replacing $x_1, \ldots, x_n$ with an n-tuple of ground terms $t_1, \ldots, t_n$.

Theorem (Herbrand)

A set Γ of universally quantified formulas (i.e., formulas of the form $\forall x_1, \ldots x_n \phi(x_1, \ldots, x_n)$ with $\phi(x_1, \ldots, x_n)$ quantified free formula) is unsatisfiable if and only if there is finite set of ground instances of Γ which is unsatisfiable.

Herbrand's theorem

Proof.

Let Γ' be the set of all grounding formula of the formulas in Γ. Γ' is a set of propositional formulas, and it is unsatisfiable if and only if there is a finite subset of Γ' which is unsatisfiable. (By compactness theorem for propositional logic). We therefore prove that

$$\Gamma \text{ is unsat if and only if } \Gamma' \text{ is unsat}$$

□

Herbrand's theorem

Proof of the $\Rightarrow$ direction.

- We prove the converse i.e.,

$$\text{if } \Gamma' \text{ is satisfiable, then } \Gamma \text{ is satisfiable.}$$

- If Γ' is satisfiable, then there is an Herbrand Interpretation $\mathcal{H}$ that satisfies Γ'. Indeed if Γ' is satisfiable then there is an interpretation $\mathcal{I} \models \Gamma'$. We can taket $\mathcal{H} = \mathcal{H}(\mathcal{I})$. And by the previous lemma we have that $\mathcal{H}(\mathcal{I}) \models \Gamma'$.
- We show that $\mathcal{H} \models \Gamma$. Let $\forall x_1, \dots, x_n.\phi(x_1, \dots, x_n) \in \Gamma$
  We have that, for all n-tuple $t_1, \dots, t_n$ of elements in $\Delta^{\mathcal{H}}$
  $\mathcal{H} \models \phi(t_1, \dots, t_n)$ since $\phi(t_1, \dots, t_n)$ is a ground instance of $\forall x_1, \dots, x_n.\phi(x_1, \dots, x_n)$ and it belongs to Γ' and $\mathcal{H} \models \Gamma'$
  This implies that for all assignments a to $x_1, \dots, x_n$ of elements of $\Delta^{\mathcal{H}}$ (i.e., ground terms $t_1, \dots, t_n$) $\mathcal{H} \models \phi(x_1, \dots, x_n)[a]$, which implies that, $\mathcal{H} \models \forall x_1, \dots, x_n.\phi(x_1, \dots, x_n)$.

$\square$

Herbrand's theorem

Proof of the $\Leftarrow$ direction.

Also in this case we prove the converse. I.e., that if Γ is satisfiable then Γ' (the set of groundings of Γ) is also satisfiable:

- Let $\mathcal{I} \models \Gamma$, and let $\phi(t_1, \dots, t_n) \in \Gamma'$.
- $\phi(t_1, \dots, t_n) \in \Gamma'$ implies that there is a formula $\forall x_1, \dots, x_n.\phi(x_1, \dots, x_n) \in \Gamma$, and the fact that $\mathcal{I} \models \Gamma$ implies that

$$\mathcal{I} \models \forall x_1, \dots, x_n.\phi(x_1, \dots, x_n)$$

- This implies that all assignment a, and in particular for those with $a(x_i) = t_i$ for any ground term $t_i \in \Delta^{\mathcal{H}(\mathcal{I})}$

$$\mathcal{I} \models \phi(x_1, \dots, x_n)[a]$$

- by the previous Lemma we have that

$$\mathcal{H}(\mathcal{I}) \models \phi(x_1, \dots, x_n)[a']$$

  where $a'(x_i) = t_i$, and therefore that

$$\mathcal{H}(\mathcal{I}) \models \phi(t_1, \dots, t_n)$$

Herbrand's Theorem - Example of usage

Exercize

Check if the formula ϕ equal to $\exists y \forall x P(x, y) \supset \forall x \exists y P(x, y)$ is VALID.

solution

- We check if the negation of ϕ is UNSATISFIABLE

$$\neg\phi = \neg(\exists y \forall x P(x, y) \supset \forall x \exists y P(x, y))$$

- We first rename the variables of $\neg\phi$ so that every quantifier quantifies a different variable.

$$\neg(\exists y \forall x P(x, y) \supset \forall v \exists w P(v, w))$$

Herbrand's Theorem - Example of usage

solution (cont'd)

- We transform $\neg\phi$ in prenex normal form obtaining as follows

$$\begin{aligned}\neg\phi = \neg(\exists y\forall x P(x,y) \supset \forall v\exists w P(v,w)) &\equiv \\ \exists y\forall x P(x,y) \wedge \neg\forall v\exists w P(v,w)) &\equiv \\ \exists y\forall x P(x,y) \wedge \exists v\forall w\neg P(v,w) &\equiv \\ \exists y\exists v\forall x\forall w(P(x,y) \wedge \neg P(v,w)) &= pnf(\neg\phi)\end{aligned}$$

- we can apply Skolemization to $pnf(\neg\phi)$ eliminating $\exists y\exists v$ introducing two new Skolem constants a and b obtaining

$$sk(pnf(\neg\phi) = \forall x\forall w(P(x,a) \wedge \neg P(b,y))$$

- $sk(pnf(\neg\phi)$ is a universally quantified formulas. So we can apply Herbrand's Theorem. In orer to prove that it is unsatisfiable we have to provide a grounding of $sk(pnf(\neg\phi)$ which is unsatisfiable.
- If we ground $sk(pnf(\neg\phi)$ with $x \rightarrow b$ and $y \rightarrow a$, we obtaine the grounded formula

$$(P(b,a) \wedge \neg P(b,a))$$

which is not satisfiable. We therefore conclude that $\neg\phi$ is unsatisfiable

Definability

We can consider the expressiveness of first order logic by observing which are the mathematical objects (actually the relations) that can be defined.

For example we can define the unit circle as the binary relation $\{\langle x, y\rangle \,|x^2 + y^2 = 1\}$ on $\mathbb{R}$. We can also define the symmetry property for a binary relation R as $\forall x \forall y (xRy \leftrightarrow yRx)$ which is satisfied by all symmetric binary relations including the circle relations.

- definability within a fixed Σ-Structure
- definability within a class of Σ-Structure.

Definability within a structure

Definability of a relation w.r.t. a structure

An n-ary relation R defined over the domain $\Delta^{\mathcal{I}}$ of a Σ-structure $\mathcal{I}$ is definable in $\mathcal{I}$ if there is a formula φ that contains n free variables (in symbols $\phi(x_1, \dots, x_n)$) such that for every n-tuple of elements $a_1, \dots, a_n \in \Delta^{\mathcal{I}}$

$$\langle a_1, \dots, a_n \rangle \in R \quad \textit{iff} \quad \mathcal{I} \models \varphi(x_1, \dots, x_n)[a_1, \dots a_n]$$

i

Definability within a structure (cont'd)

Example (Definition of 0 in different structures)

- In the structure of ordered natural numbers $\langle \mathbb{N}, < \rangle$, the singleton set (= unary relation containing only one element) $\{0\}$ is defined by the following formula

$$\forall y (y \neq x \rightarrow x < y)$$

- In the structure of ordered real numbers $\langle \mathbb{R}, < \rangle$, $\{0\}$ has no special property that distinguish it from the other real numbers, and therefore it cannot be defined.
- In the structure of real numbers with sum $\langle \mathbb{R}, + \rangle$, $\{0\}$ can be defined in two alternatives way:

$$\forall y (x + y = y) \qquad x + x = x$$

- In the structure of real numbers with product $\langle \mathbb{R}, \cdot \rangle$, $\{0\}$ can be defined by the following formula:

$$\forall y (x \cdot y = x)$$

  Notice that unlike the previous case $\{0\}$ cannot be defined by $x \cdot x = x$ since also $\{1\}$ satisfies this property $(1 \cdot 1 = 1)$

(un)Definability of transitive closure in FOL

Definability within a structure (cont'd)

Example (Definition of reachability relation in a graph)

Consider a graph structure $G = \langle V, E\rangle$, we would like to define the reachability relation between two nodes. I.e., the relation

$$Reach = \{\langle x, y\rangle \in V^2 | \text{there is a path from } x \text{ to } y \text{ in } G\}$$

We can scompose *Reach* in the following relations
"y is reachable from x in 1 step" or
"y is reachable from x in 2 steps" or ....
And define each single relation for all $n \geq 0$ as follows:

$$reach_1(x, y) \equiv E(x, y) \tag{7}$$

$$reach_{n+1}(x, y) \equiv \exists z(reach_n(x, z) \wedge E(z, y)) \tag{8}$$

If V is finite, then the relation Reach can be defined by the formula

$$reach_0(x, y) \vee reach_1(x, y) \vee \cdots \vee reach_n(x, y)$$

Where n is the number of vertexes of the graph.

Examples on definability in a structure

Example

Let Σ the signature $\langle 0, s, + \rangle$ and $\mathcal{I}$ the standard Σ-structure for arithmetic, i.e., $\Delta^{\mathcal{I}} = \mathbb{N}$ the set of natural numbers $\{0, 1, 2, 3, \dots\}$, $0^{\mathcal{I}} = 0$, $s^{\mathcal{I}}(x) = x + 1$ and $+^{\mathcal{I}}(x, y) = x + y$.
Define the following predicates:

- x is an Even number $\exists y.x = y + y$
- x is an odd number $\exists y.x = s(y + y)$
- x is greater than y $\exists z, x = s(y + z)$

Definability within a class of structures

Class of structures defined by a (set of) formula(s)

Given a formula φ of the alphabet Σ we define $mod(\phi)$ as the class of Σ-structures that satisfies φ. i.e.,

$$mod(\varphi) = \{\mathcal{I} \mid \mathcal{I} \text{ is a } \Sigma\text{-structures and } \mathcal{I} \models \varphi\}$$

Given a set of formulas T, $mod(T)$ is the class of Σ structures that satisfies each formula in T.

Example

$$mod(\forall xy\ x = y) = \{\mathcal{I} \mid \Delta^{\mathcal{I}} = 1\}$$

The question we would like to answer is: What classes of Σ-structures can we describe using first order sentences? For instance can we describe the class of all connected graphs?

Definability within a class of structures (cont'd)

Example (Classes definable with a single formula)

- The class of undirected graphs

$$\varphi_{UG} = \forall x\, \neg E(x,x) \wedge \forall xy\, (E(x,y) \equiv E(y,x))$$

- the class of partial orders:

$$\begin{aligned}\varphi_{PO} =& \forall x R(x,x) \wedge \\ & \forall xy(R(x,y) \wedge R(y,x) \rightarrow x = y) \wedge \\ & \forall xyz(R(x,y) \wedge R(y,z) \rightarrow R(x,z))\end{aligned}$$

- the class of total orders:

$$\varphi_{TO} = \varphi_{PO} \wedge \forall xy(R(x,y) \vee R(y,x))$$

Definability within a class of structures (cont'd)

Example (Classes definable with a single formula)

- the class of groups:

$$\begin{aligned}\varphi_G = & \forall x(x+0=x \wedge 0+x=x) \wedge \\ & \forall x \exists y(x+y=0 \wedge y+x=0) \wedge \\ & \forall xyz((x+y)+z=x+(y+z))\end{aligned}$$

- the class of abelian groups:

$$\varphi_{AG} = \varphi_G \wedge \forall xy(x+y=y+x)$$

- the class of structures that contains at most n elements

$$\varphi_n = \forall x_0 \dots x_n \bigvee_{0 \leq i < j \leq n} x_i = x_j$$

Remark

Notice that every class of structures that can be defined with a finite set of formulas (as e.g., groups, rings, vector spaces, boolean algebras topological spaces, . . .) can also be defined by a single sentence by taking the finite conjunction of the set of formulas.

Classes of Structures characterizable by an infinite set of formulas

Theorem

The class of infinite structures is characterizable by the following infinite set of formulas:

there are at least 2 elements $\varphi_2 = \exists x_1 x_2\ x_1 \neq x_2$

there are at least 3 elements $\varphi_3 = \exists x_1 x_2 x_3 (x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_2 \neq x_3)$

there are at least n elements $\varphi_n = \exists x_1 x_2 x_3 \dots x_n \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j$

Finite satisfiability and compactness

Definition (Finite satisfiability)

A set Φ of formulas is finitely satisfiable if every finite subset of Φ is satisfiable.

Theorem (Compactness)

A set of formulas Φ is satisfiable iff it is finitely satisfiable

Proof.

An indirect proof of the compactness theorem can be obtained by exploiting the completeness theorem for FOL as follows:
If Φ is not satisfiable, then, by the completeness theorem of FOL, there $\Phi \vdash \bot$. Which means that there is a deduction Π of $\bot$ from Φ. Since Π is a finite structure, it "uses" only a finite subset Φ_f of Φ of hypothesis. This implies that $\Phi_f \vdash \bot$ and therefore, by soundness that Φ_f is not satisfiable; which contradicts the fact that all finite subsets of Φ are satisfiable □

Classes of Structures characterizable by an infinite set of formulas

Theorem

The class $\mathbf{C}_{inf}$ of infinite structures is not characterizable by a finite set of formulas.

Proof.

- Suppose, by contradiction, that there is a sentence ϕ with $mod(\phi) = \mathbf{C}_{inf}$.
- Then $\Phi = \{\neg\phi\} \cup \{\varphi_2, \varphi_2, \dots\}$ (as defined in the previous slides) is not satisfiable,
- by compactness theorem Φ is not finitely satisfiable, and therefore there is an n such that $\Phi_f = \{\neg\phi\} \cup \{\varphi_2, \varphi_2, \dots, \varphi_n\}$ is not satisfiable.
- let $\mathcal{I}$ be a structure with $\Delta^{\mathcal{I}} = n + 1$. Since $\mathcal{I}$ is not infinite then $\mathcal{I} \models \neg\phi$, and since it contains more than k elements for every $k \leq n + 1$ we have that $\mathcal{I} \models \varphi_k$ for $2 \leq k \leq n + 1$.
- Therefore we have that $\mathcal{I} \models \Phi$, i.e., Φ is satisfiable, which contradicts the fact that Φ was derived to be unsatisfiable.

□

First order theory

Theory

A first order theory T over a signature,
$\Sigma = \langle c_1, c_2, \ldots, f_1, f_2, \ldots, R_1, R_2, \ldots \rangle$, or more simply a Σ-theory is a set of sentences over Σ[a] closed logical consequence. I.e

$$T \models \phi \quad \Rightarrow \quad \phi \in T$$

[a] Remember: a sentence is a closed formula. A closed formula is a formula with no free variables

Consistency

A Σ-theory is consistency if T has a model, i.e., if there is a Σ-structure $\mathcal{I}$ such that $\mathcal{I} \models T$.

Theory of a class of Σ-structures

Th(M)

Let $\mathbf{M}$ a class of Σ-structure. The Σ-theory of $\mathbf{M}$ is the set of formulas:

$$th(\mathbf{M}) = \{\alpha \in \text{sent}(\Sigma) | \mathcal{I} \models \alpha, \text{ for all } \mathcal{I} \in \mathbf{M}\}$$

Furthermore $th(\mathbf{M})$ has the following two important properties:

- $th(\mathbf{M})$ is consistent $th(\mathbf{M}) \not\models \bot$
- $th(\mathbf{M})$ is closed under logical consequence

And therefore is a consistent Σ-theory

Remark

Thus, $th(\mathbf{M})$ consists exarcly of all Σ-sentences that hold in all structures in $\mathcal{I}$.

Every theory is a theory for a class of structures

Every Σ-theory T is the Σ-theory of a class $\mathbf{M}$ of Σ structure. in particular $\mathcal{I}$ can be defined as follows:

$$\mathbf{M} = \{\mathcal{I} | \mathcal{I} \text{ is } \Sigma\text{-structure, and } \mathcal{I} \models T\}$$

Axiomatization of a class of Σ-structures

Axiomatization

An (finite) axiomatization of a class of Σ-structures $\mathbf{M}$ is a (finite) set of formulas A such that

$$th(\mathbf{M}) = \{\phi | A \models \phi\}$$

An axiomatization of a (class of) structure(s) $\mathcal{I}$ contains a set of formulas (= axioms) which describes the salient properties of the symbols in Σ (constant, functions and relations) when they are interpreted in the structure $\mathcal{I}$. Every other property of the symbols of Σ in the structure $\mathcal{I}$ are logical consequences of the axioms.

Exercises on axiomatizations

Exercize

Let $\Sigma = \langle root, child(\cdot,\cdot)\rangle$ axiomatize the class of structures isomorphic to a tree of depth less or equal to n

Solution ($Tree_{\leq n}$ be the set of axioms)

- $\forall x.\neg child(x, root)$
- $\forall xyz.(child(y,x) \wedge child(z,x) \supset z = y)$
- $\forall xyz.anchestor(x,y) \equiv$
  $child(x,y) \vee \exists x_1.(child(x,x_1) \wedge child(x_1,y)) \ \vee \dots \vee$
  $\exists x_1,\dots,x_{n-1}(child(x,x_1) \wedge child(x_1,x_2) \wedge \dots \wedge child(x_{n-1},y))$
- $\forall x.\neg anchestor(x,x)$
- $\forall xy.(anchestor(x,y) \supset \neg anchestor(y,x)$
- $\forall x.(x \neq root \supset anchestor(root,x))$

Exercize

Proove that every structure $\mathcal{I}$ that satisfies $Tree_{\leq n}$ is a tree of depth less or equal to n. I.e., a structure constituted of a set A and a binary relation T on A such that there is a vertex $v_0 \in A$ with the property that there exists a unique path of length less then or equal to n in T from v_0 to every other vertex in A, but no path from v_0 to v_0.