Mathematical Logic Exam

10 June 2014

Instructions

- Answer in English and write in ink unless the question paper gives other instructions.
- Write clearly; illegible answers will not be marked.
- Take care to identify each answer clearly with:
  - the number of the exercise.
  - where appropriate, the part of the exercise you are answering.
- Clearly cross out rough working, or unwanted answers before handing in your answers.
- If you take the exam to recover one of the midterms, Please state clearly which part (Propositional Logic or First Order + Modal Logic) you intend to re-do. If you do not state this in an explicit manner, we will assume that you are taking the entire exam, and the midterm marks will not be taken into account anymore.

Propositional Logic

Exercise 1 (PL Theory)**. [6 points]** Show that the propositional α-rule

$$R_\wedge \ \frac{\phi \wedge \psi}{\begin{array}{c}\phi \\ \psi\end{array}}$$

preserves the satisfiability of the tableau (that is, $R_\wedge$ extends a satisfiable branch β to a branch β' that is also satisfiable)

Solution.

- *let $\mathcal{I}$ be an interpretation that satisfies β, i.e., $\mathcal{I} \models \beta$*
- *since $\phi \wedge \psi \in \beta$ then $\mathcal{I} \models \phi \wedge \psi$*
- *which implies that $\mathcal{I} \models \phi$ and $\mathcal{I} \models \psi$*

- *which implies that $\mathcal{I} \models \beta'$ with $\beta' = \beta \cup \{\phi, \psi\}$.*

Exercise 2 (PL modeling)**. [6 points]** Brown, Jones, and Smith are three friends. They say the following:

- **Brown**: *"Jones is drunk and Smith is sober"*.
- **Jones**: *"If Brown is drunk then so is Smith"*.
- **Smith**: *"I'm sober, but at least one of the others is drunk"*.

Let B, J, and S be the statements "*Brown is drunk*", "*Jones is drunk*", and "*Smith is drunk*", respectively, and consider being sober as the negation of being drunk. Do the following:

1. Express the sentence of each friend as a PL formula.
2. Write a truth table for the three sentences.
3. Use the truth table to answer the following questions:
   (a) Are the three sentences satisfiable (together)?
   (b) The sentence of one of the friends follows from that of another. Which from which?
   (c) Assuming that all sentences are true, who is sober and who is drunk?
   (d) Assuming that the sober friends told the truth and the drunk friends told lies, who is sober and who is drunk?

Solution.

1. *The three statements can be expressed as $J \wedge \neg S$, $B \supset S$, and $\neg S \wedge (B \vee J)$.*

2.

	B	J	S	$J \wedge \neg S$	$B \supset S$	$\neg S \wedge (B \vee J)$
(1)	T	T	T	F	T	F
(2)	T	T	F	T	F	T
(3)	T	F	T	F	T	F
(4)	T	F	F	F	F	T
(5)	F	T	T	F	T	F
(6)	F	T	F	T	T	T
(7)	F	F	T	F	T	F
(8)	F	F	F	F	T	F

3. (a) *Yes, assigment (6) makes them all true*
   (b) $J \wedge \neg S \models \neg S \wedge (B \vee J)$
   (c) *Assuming that all sentences are true corresponds to assignment (6). In this case Jones is drunk and the others are sober.*

(d) *We have to search for an assignment such that if B (resp. J and S) is false then the sentence of B (resp. J and S) is true and that if B (resp. J and S) is true, then the sentence of B (resp. J and S) is false. The only assignment satisfying this restriction is assignment (3) in which Jones is sober and Brown and Smith are drunk.*

Exercise 3 (PL Reasoning)**.** [**6 points**] Apply DPLL procedure to check if the following set of clauses is satisfiable, and if it is so, return a partial assignment that makes all the fomulas true.

1. $p \vee u$
2. $\neg u \vee \neg v$
3. $q \vee \neg v$
4. $\neg q \vee s$
5. $\neg s \vee \neg u \vee m$
6. $\neg m \vee u \vee \neg s$

In the solution you have to specify all the applications of unit propagation rule, and all the choices you take when Unit propagation is not applicable.

Solution.

1. *Let ϕ the CNF of the conjunction of 1–6. ϕ does not contain unit clause, which implies that unit propagation is not applicable.*
2. *therefore, we select a literal (say $\neg u$) and set $\mathcal{I}(u) = false$*
3. *Compute $\phi|_{\neg u}$:*
$$\phi|_{\neg u} = \{\{p\},\ \{q, \neg v\},\ \{\neg q, s\},\ \{\neg m, \neg s\}\}$$
4. *$\phi|_{\neg u}$ contains the unit clause $\{p\}$, we therefore extend the partial interpretation with $\mathcal{I}(p) =$ True. We then apply unit propagation with $\{p\}$ as unit clause, obtaining*
$$\phi|_{\neg u,p} = \{\{q, \neg v\},\ \{s\},\ \{\neg m, \neg s\}\}$$
5. *$\phi|_{\neg u,p}$ contains the unit clause $\{s\}$, we therefore extend the partial interpretation with $\mathcal{I}(s) =$ True. We then apply unit propagation with $\{s\}$ as unit clause, obtaining*
$$\phi|_{\neg u,p,s} = \{\{q, \neg v\},\ \{\neg m\}\}$$
6. *$\phi|_{\neg u,p,s}$ contains the unit clause $\{\neg m\}$, we therefore extend the partial interpretation with $\mathcal{I}(m) =$ False. We then apply unit propagation with $\{\neg m\}$ as unit clause, obtaining*
$$\phi|_{\neg u,p,s,\neg m} = \{\{q, \neg v\},\ \}$$

7. *$\phi|_{\neg u,p,s,\neg m}$ does not contain unit clause, which implies that unit propagation is not applicable. We, therefore, select a literal (say q) and set $\mathcal{I}(q) = True$. We then compute $\phi|_{\neg u,p,s,\neg m,q} = \{\}$. Which implies that the initial formula is satisfiable, by the partial assignment:*

$$\begin{array}{lll} I(u) = \mathit{False} & I(p) = \mathit{True} & I(s) = \mathit{True} \\ I(m) = \mathit{False} & I(q) = \mathit{True} & \end{array}$$

Exercise 4 (FOL Theory)**. [6 points]** Let $\mathcal{L}$ be a first order language on a signatore containing

- the constant symbols a and b,
- the binary function symbol f, and
- the binary predicate symbol P.

Answer to the following questions:

1. What is the Herbrand Universe for $\mathcal{L}$ (2 point)
2. Does $\mathcal{L}$ have a finite model? If yes define it, if not explain why. (2 point)
3. Let $\mathcal{T}$ be a theory containing the following axioms
   (a) $\forall y.\neg P(x,x)$ (P is irreflexive)
   (b) $\forall xyz.(P(x,y) \wedge P(y,z) \supset P(x,z))$ (P is transitive)
   (c) $\forall xy.(P(x, f(x,y)) \wedge P(y, f(x,y))$

   Is $\mathcal{T}$ satisfiable?. If yes can you provide a model for $\mathcal{T}$ (2 points)

Solution.

1. *The Herbrand Universe for $\mathcal{L}$ is the set of ground terms that can be built starting from the constants by applying the function symbols. In this case it is the following infinite set of terms.*

$$\begin{array}{l} \{a, b, f(a,a), f(a,b), f(b,a), f(b,b), \\ f(a, f(a,a)), f(a, f(a,b)), f(a, f(b,a)), f(a, f(b,b)), \\ f(b, f(a,a)), f(b, f(a,b)), f(b, f(b,a)), f(b, f(b,b)) \dots\} \end{array}$$

2. *$\mathcal{L}$ has a finite model. For instance $\mathcal{I} = \left\langle \Delta^{\mathcal{I}} = \{0\}, f^{\mathcal{I}}(0,0) = 0, P^{\mathcal{I}} = \emptyset \right\rangle$ is a model of $\mathcal{L}$, and it is finite since $|\Delta^{\mathcal{I}}| = 1$ i.e., the cardinality of the domain of $\mathcal{I}$ is a finite number. namely 1.*

3. *$\mathcal{T}$ is satisfiable. Consider the herbrand interpretation $\mathcal{H}$ defined on the domain which is the herbrand universe, where P is interpreted in the following binary relation:*

$$\langle t, t'\rangle \in P^{\mathcal{H}} \quad \textit{if and only if} \quad t \textit{ is a substring of } t'$$

Where t is a substring of t' means that when t' is of the form $f(\ldots t \ldots)$ It's easy to check that the three axioms of $\mathcal{T}$ are all satisfied by $\mathcal{B}$

Exercise 5 (FOL tableaux)**. [6 points]** Show by means of tableaux that the following formula is valid:

$$\forall xyz(P(x,y) \land P(x,z) \supset P(y,z)) \supset \forall x(\exists w P(w,x) \supset P(x,x))$$

Solution.

$$\neg(\forall xyz(P(x,y) \land P(x,z) \supset P(y,z)) \supset \forall x(\exists w P(w,x) \supset P(x,x)))$$

$$|$$

$$\forall xyz(P(x,y) \land P(x,z) \supset P(y,z))$$
$$\neg\forall x(\exists w P(w,x) \supset P(x,x))$$

$$|$$

$$\neg(\exists w P(w,a) \supset P(a,a))$$

$$|$$

$$\exists w P(w,a)$$
$$\neg P(a,a)$$

$$|$$

$$P(b,a)$$

$$|$$

$$P(b,a) \land P(b,a) \supset P(a,a)$$

$$\begin{array}{ccc}
\multicolumn{2}{c}{\neg(P(b,a) \land P(b,a))} & P(a,a) \\
\neg P(b,a) & \neg P(b,a) & \times \\
| & | & \\
\times & \times &
\end{array}$$

Exercise 6 (Modal logics Modal axioms)**. [6 points]** Consider the axiom schema $\Box\phi \supset \phi$. Say which is the property P such that (1) holds.

$$\mathcal{F} \models \Box\phi \supset \phi \text{ if and only if } \mathcal{F} \text{ has the property } P \tag{1}$$

Prove (1).

Solution. *We have to prove*

***Soundness*:** *If $\mathcal{F}$ is a frame that satisfies the property P, then $\Box\phi \supset \phi$ is a valid formula in $\mathcal{F}$.*

***Completeness*:** *If $\Box\phi \supset \phi$ is a valid formula in a frame $\mathcal{F}$, then $\mathcal{F}$ is a frame that satisfies the property P. For the completeness we prove the (equivalent) contropositive statement, i.e., that if $\mathcal{F}$ does not satisfy the property P then $\Box\phi \supset \phi$ is not valid in $\mathcal{F}$. We do this by building a countermodel $\mathcal{M} = \langle F, V\rangle$ for $\Box\phi \supset \phi$, by providing an assignment V to propositional variables on $\mathcal{F}$, and by selecting a world of w in $\mathcal{F}$ so that $\mathcal{M}, w \not\models \Box\phi \supset \phi$.*

(T): $\Box\phi \supset \phi$ *P is equal to* Reflexivity*, i.e., $\forall w \in W,\ wRw$.*

***Soundness*:** *Let $\mathcal{M}$ be a model on a reflexive frame $\mathcal{F} = \langle W, R\rangle$ and w any world in W. We prove that $\mathcal{M}, w \models \Box\phi \supset \phi$.*

1. *Since R is reflexive then wRw*
2. *Suppose that $\mathcal{M}, w \models \Box\phi$ (Hypothesis)*
3. *From the satisfiability condition of $\Box$, $\mathcal{M}, w \models \Box\phi$, and wRw imply that $\mathcal{M}, w \models \phi$ (Thesis)*
4. *Since from (Hypothesis) we have derived (Thesis), we can conclude that $\mathcal{M}, w \models \Box\phi \supset \phi$.*

***Completeness*:** *Suppose that a frame $\mathcal{F} = \langle W, R\rangle$ is not reflexive.*

1. *If R is not reflexive then there is a $w \in W$ which does not access to itself. I.e., for some $w \in W$ it does not hold that wRw.*
2. *Let $\mathcal{M}$ be any model on $\mathcal{F}$, and let ϕ be the propositional formula p. Let V the set p true in all the worlds of W but w where p is set to be false.*
3. *From the fact that w does not access to itself, we have that in all the worlds w accessible from w, p is true, i.e, $\forall w',\ wRw',\ \mathcal{M}, w' \models p$.*
4. *Form the satisfiability condition of $\Box$ we have that $\mathcal{M}, w \models \Box p$.*
5. *since $\mathcal{M}, w \not\models p$, we have that $\mathcal{M}, w \not\models \Box p \supset p$.*