Mathematical Logics

7. Model theory

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Recap of what is Σ-structure

Σ-structure

Given a vocabulary $\Sigma = \langle c_1, c_2, \dots, f_1, f_2, \dots, R_1, R_2, \dots \rangle$ a Σ-structure is $\mathcal{M}$ is composed of a non empty set $\Delta_{\mathcal{M}}$ and an interpretation function such that

- $c_i^{\mathcal{M}} \in |\mathcal{M}|$
- $f_i^{\mathcal{M}} \in |\mathcal{M}|^{arity(f_i)} \longrightarrow |\mathcal{M}|$
- $R_i^{\mathcal{M}} \in |\mathcal{M}|^{arity(R_i)}$

Substructures and isomorphic structures

Substructure

A Σ-structure $\mathcal{M}$ is a *substructure* of a Σ-structure $\mathcal{N}$, in symbols $\mathcal{M} \subseteq \mathcal{N}$ if

- $|\mathcal{M}| \subseteq |\mathcal{N}|$
- $c^{\mathcal{M}} = c^{\mathcal{N}}$
- $f^{\mathcal{M}}$ is the restriction of $f^{\mathcal{N}}$ to the set $|\mathcal{M}|$, i.e., for all $a_1, \dots, a_n \in |\mathcal{M}|$, $f^{\mathcal{M}}(a_1, \dots, a_n) = f^{\mathcal{N}}(a_1, \dots, a_n)$.
- $R^{\mathcal{M}} = R^{\mathcal{N}} \cap |\mathcal{M}|^n$

where n is the arity of f and R.

Isomorphic structures

Two Σ-structures $\mathcal{M}$ and $\mathcal{N}$ are isomorphic, in symbols $\mathcal{M} \simeq \mathcal{N}$, if there is a bijection $i : |\mathcal{M}| \to |\mathcal{N}|$ such that

- $i(c^{\mathcal{M}}) = c^{\mathcal{N}}$ for every constant c
- $i(f^{\mathcal{M}}(a_1, \dots, a_n) = f^{\mathcal{N}}(i(a_1), \dots, i(a_n))$.
- $\langle a_1, \dots, a_n \rangle \in R^{\mathcal{M}}$ iff $\langle i(a_1), \dots, i(a_n) \rangle \in R^{\mathcal{N}}$

Elementary equivalent structures

Elementary equivalent structures

Two Σ-structures $\mathcal{M}$ and $\mathcal{M}$ are elementary equivalent, in symbols $\mathcal{M} \equiv \mathcal{N}$, if for all sentences φ

$$\mathcal{M} \models \varphi \iff \mathcal{N} \models \varphi$$

Theorem

if $\mathcal{M} \simeq \mathcal{N}$ then $\mathcal{M} \equiv \mathcal{N}$.

The viceversa of the above theorem does not hold. There are pairs of structure which are elementary equivalent but they are not isomorphic.

Example

$\langle \mathbb{Q}, < \rangle \equiv \langle \mathbb{R}, < \rangle$. (the order on rational numbers is elementary equivalent with the order on real numbers). But these two structures cannot be isomorphic since one has numerable cardinality and the other is not. Which implies that there cannot exist an isomorphism. We therefore conclude that $\langle \mathbb{Q}, < \rangle \not\simeq \langle \mathbb{R}, < \rangle$.

Definability

We can consider the expressiveness of first order logic by observing which are the mathematical objects (actually the relations) that can be defined.
For example we can define the unit circle as the binary relation $\{\langle x, y\rangle \,|x^2 + y^2 = 1\}$ on $\mathbb{R}$. We can also define the symmetry property for a binary relation R as $\forall x \forall y(xRy \leftrightarrow yRx)$ which is satisfied by all symmetric binary relations including the circle relations.

- definability within a fixed Σ-Structure
- definability within a class of Σ-Structure.

Definability within a structure

Definability of a relation w.r.t. a structure

An n-ary relation R defined over the domain $|\mathcal{M}|$ of a Σ-structure $\mathcal{M}$ is definable in $\mathcal{M}$ if there is a formula φ that contains n free variables (in symbols $\phi(x_1, \ldots, x_n)$) such that for every n-tuple of elements $a_1, \ldots, a_n \in |\mathcal{M}|$

$$\langle a_1, \ldots, a_n \rangle \in R \quad \textit{iff} \quad \mathcal{M} \models \varphi(x_1, \ldots, x_n)[a_1, \ldots a_n]$$

Definability within a structure (cont'd)

Example (Definition of 0 in different structures)

- In the structure of ordered natural numbers $\langle \mathbb{N}, < \rangle$, the singleton set (= unary relation containing only one element) $\{0\}$ is defined by the following formula

$$\forall y (y \neq x \rightarrow x < y)$$

- In the structure of ordered real numbers $\langle \mathbb{R}, < \rangle$, $\{0\}$ has no special property that distinguish it from the other real numbers, and therefore it cannot be defined.
- In the structure of real numbers with sum $\langle \mathbb{R}, + \rangle$, $\{0\}$ can be defined in two alternatives way:

$$\forall y (x + y = y) \qquad x + x = x$$

- In the structure of real numbers with product $\langle \mathbb{R}, \cdot \rangle$, $\{0\}$ can be defined by the following formula:

$$\forall y (x + y = y)$$

  Notice that unlike the previous case $\{0\}$ cannot be defined by $x \cdot x = x$ since also $\{1\}$ satisfies this property $(1 \cdot 1 = 1)$

Definability within a structure (cont'd)

Example (Definition of reachability relation in a graph)

Consider a graph structure $G = \langle V, E \rangle$, we would like to define the reachability relation between two nodes. I.e., the relation

$$Reach = \{\langle x, y \rangle \in V^2 | \text{there is a path from } x \text{ to } y \text{ in } G\}$$

We can scompose *Reach* in the following relations
"y is reachable from x in 1 step" or
"y is reachable from x in 2 steps" or ....
And define each single relation for all $n \geq 0$ as follows:

$$reach_1(x, y) \equiv E(x, y) \tag{1}$$

$$reach_{n+1}(x, y) \equiv \exists z (reach_n(x, z) \wedge E(z, y)) \tag{2}$$

If V is finite, then the relation Reach can be defined by the formula

$$reach_0(x, y) \vee reach_1(x, y) \vee \cdots \vee reach_{|V||}(x, y)$$

if V is infinite, then reachability is not definable in first order logic.

Definability within a class of structures

Class of structures defined by a (set of) formula(s)

Given a formula φ of the alphabet Σ we define $mod(\phi)$ as the class of Σ-structures that satisfies φ. i.e.,

$$mod(\varphi) = \{\mathcal{M} \mid \mathcal{M} \text{ is a } \Sigma\text{-structures and } \mathcal{M} \models \varphi\}$$

Given a set of formulas T, $mod(T)$ is the class of Σ structures that satisfies each formula in T.

Example

$$mod(\forall xy\ x = y) = \{\mathcal{M} \mid |\mathcal{M}| = 1\}$$

The question we would like to answer is: What classes of Σ-structures can we describe using first order sentences? For instance can we describe the class of all connected graphs?

Definability within a class of structures (cont'd)

Example (Classes definable with a single formula)

- The class of undirected graphs

$$\varphi_{UG} = \forall x\, \neg E(x,x) \wedge \forall xy\, (E(x,y) \equiv E(y,x))$$

- the class of partial orders:

$$\begin{aligned} \varphi_{PO} = & \forall x R(x,x) \wedge \\ & \forall xy (R(x,y) \wedge R(y,x) \rightarrow x = y) \wedge \\ & \forall xyz (R(x,y) \wedge R(y,z) \rightarrow R(x,z)) \end{aligned}$$

- the class of total orders:

$$\varphi_{TO} = \varphi_{PO} \wedge \forall xy (R(x,y) \vee R(y,x))$$

Definability within a class of structures (cont'd)

Example (Classes definable with a single formula)

- the class of groups:

$$\begin{aligned}\varphi_G = & \forall x(x + 0 = x \wedge 0 + x = x) \wedge \\ & \forall x \exists y(x + y = 0 \wedge y + x = 0) \wedge \\ & \forall xyz((x + y) + z = x + (y + z))\end{aligned}$$

- the class of abelian groups:

$$\varphi_{AG} = \varphi_G \wedge \forall xy(x + y = y + x)$$

- the class of structures that contains at most n elements

$$\varphi_n = \forall x_0 \ldots x_n \bigvee_{0 \leq i < j \leq n} x_i = x_j$$

Remark

Classes of Structures characterizable by an infinite set of formulas

Theorem

The class of infinite structures is characterizable by the following infinite set of formulas:

there are at least 2 elements	$\varphi_2 = \exists x_1 x_2\ x_1 \neq x_2$
there are at least 3 elements	$\varphi_3 = \exists x_1 x_2 x_3 (x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_2 \neq x_3)$
there are at least n elements	$\varphi_n = \exists x_1 x_2 x_3 \dots x_n \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j$

Finite satisfiability and compactness

Definition (Finite satisfiability)

A set Φ of formulas is finitely satisfiable if every finite subset of Φ is satisfiable.

Theorem (Compactness)

A set of formulas Φ is satisfiable iff it is finitely satisfiable

Proof.

An indirect proof of the compactness theorem can be obtained by exploiting the completeness theorem for FOL as follows:
If Φ is not satisfiable, then, by the completeness theorem of FOL, there $\Phi \vdash \perp$. Which means that there is a deduction Π of $\perp$ from Φ. Since Π is a finite structure, it "uses" only a finite subset Φ_f of Φ of hypothesis. This implies that $\Phi_f \vdash \perp$ and therefore, by soundness that Φ_f is not satisfiable; which contradicts the fact that all finite subsets of Φ are satisfiable $\square$

Classes of Structures characterizable by an infinite set of formulas

Theorem

The class $\mathbf{C}_{inf}$ of infinite structures is not characterizable by a finite set of formulas.

Proof.

- Suppose, by contradiction, that there is a sentence ϕ with $mod(\phi) = \mathbf{C}_{inf}$.
- Then $\Phi = \{\neg\phi\} \cup \{\varphi_2, \varphi_2, \dots\}$ (as defined in the previous slides) is not satisfiable,
- by compactness theorem Φ is not finitely satisfiable, and therefore there is an n such that $\Phi_f = \{\neg\phi\} \cup \{\varphi_2, \varphi_2, \dots, \varphi_n\}$ is not satisfiable.
- let $\mathcal{M}$ be a structure with $|\mathcal{M}| = n + 1$. Since $\mathcal{M}$ is not infinite then $\mathcal{M} \models \neg\phi$, and since it contains more than k elements for every $k \leq n + 1$ we have that $\mathcal{M} \models \varphi_k$ for $2 \leq k \leq n + 1$.
- Therefore we have that $\mathcal{M} \models \Phi$, i.e., Φ is satisfiable, which contradicts the fact that Φ was derived to be unsatisfiable.

□

First order theory

Theory

A first order theory T over a signature, $\Sigma = \langle c_1, c_2, \dots, f_1, f_2, \dots, R_1, R_2, \dots \rangle$, or more simply a Σ-theory is a set of sentences over Σ[a] closed logical consequence. I.e

$$T \models \phi \quad \Rightarrow \quad \phi \in T$$

[a]Remember: a sentence is a closed formula. A closed formula is a formula with no free variables

Consistency

A Σ-theory is consistency if T has a model, i.e., if there is a Σ-structure $\mathcal{M}$ such that $\mathcal{M} \models T$.

Theory of a class of Σ-structures

Th(M)

Let $\mathbf{M}$ a class of Σ-structure. The Σ-theory of $\mathbf{M}$ is the set of formulas:

$$th(\mathbf{M}) = \{\alpha \in \mathsf{sent}(\Sigma) | \mathcal{M} \models \alpha, \text{ for all } \mathcal{M} \in \mathbf{M}\}$$

Furthermore $th(\mathbf{M})$ has the following two important properties:

- $th(\mathbf{M})$ is consistent $th(\mathbf{M}) \not\models \bot$
- $th(\mathbf{M})$ is closed under logical consequence

And therefore is a consistent Σ-theory

Remark

Thus, $th(\mathbf{M})$ consists exarcly of all Σ-sentences that hold in all structures in $\mathcal{M}$.

Every theory is a theory for a class of structures

Every Σ-theory T is the Σ-theory of a class $\mathbf{M}$ of Σ structure. in particular $\mathcal{M}$ can be defined as follows:

$$\mathbf{M} = \{\mathcal{M} | \mathcal{M} \text{ is } \Sigma\text{-structure, and } \mathcal{M} \models T\}$$

Axiomatization of a class of Σ-structures

Axiomatization

An (finite) axiomatization of a class of Σ-structures $\mathbf{M}$ is a (finite) set of formulas A such that

$$th(\mathbf{M}) = \{\phi | A \models \phi\}$$

An axiomatization of a (class of) structure(s) $\mathcal{M}$ contains a set of formulas (= axioms) which describes the salient properties of the symbols in Σ (constant, functions and relations) when they are interpreted in the structure $\mathcal{M}$. Every other property of the symbols of Σ in the structure $\mathcal{M}$ are logical consequences of the axioms.