Mathematical Logic

Reasoning in First Order Logic

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Reasoning tasks in FOL

Model checking

Question: Is ϕ true in the interpretation I with the assignment a?
Answer: Yes if $\mathcal{I} \models \phi[a]$. No otherwise

Query answering

Question: Which values for $x_1, \ldots, x_n$, makes $\phi(x_1, \ldots, x_n)$ true in I?
Answer: $\left\{ \langle a(x_1), \ldots, a(x_n) \rangle \;\middle|\; I \models \phi(x_1, \ldots, x_n)[a] \right\}$

Satisfiability

Question: Does there exists an interpretation and an assignment that satisfies ϕ?
Answer: Yes if there is an I such that $I \models \phi[a]$ for some a, No otherwise

Validity

Question: Is ϕ true in all the interpretation and for all the assignments?
Answer: Yes if $\models \phi$, No Otherwise.

Logical consequence

Question: Is ϕ a logical consequence of a set of formulas Γ?
Answer: Yes if $\Gamma \models \phi$, No otherwise.

Hilbert style Reasoning

Extends the axioms and rules for propositional connectives ($\supset$ and $\neg$) to the case of quantifiers.

To minimize the set of axioms, Hilbert considers $\exists$ as a shortcut for $\neg\forall\neg$ (**Exercise**: Why is this correct?)

Hilbert deduction of ϕ from Γ (same as for Propositional Logic)

A deduction of a formula ϕ from a set of formulae Γ is a sequence of formulas $\phi_1, \ldots, \phi_n$, with $\phi_n = \phi$, such that ϕ_k (for $1 \leq k \leq n$)

- is an axiom, or
- it is in Γ (an assumption), or
- it is derived form previous formulae via inference rules.

ϕ is derivable from Γ, in symbols $\Gamma \vdash \phi$, if there is a deduction of ϕ from Γ
ϕ is provable, in symbols $\vdash \phi$, if there is a deduction of ϕ from $\emptyset$.

Remember that the main objective of Hilbert was minimality: find the smallest set of axioms and inference rules from which it was possible to derive all the tautologies.

Hilbert style Reasoning

Axioms and rules for propositional connectives

A1 $\phi \supset (\psi \supset \phi)$

A2 $(\phi \supset (\psi \supset \theta)) \supset ((\phi \supset \psi) \supset (\phi \supset \theta))$

A3 $(\neg\psi \supset \neg\phi) \supset ((\neg\psi \supset \phi) \supset \psi)$

MP $\dfrac{\phi \quad \phi \supset \psi}{\psi}$

Axioms and rules for quantifiers

A4 $\forall x.\phi(x) \supset \phi(t)$ if t is free for x in $\phi(x)$

A5 $\forall x.(\phi \supset \psi) \supset (\phi \supset \forall x.\psi)$ if x does not occur free in ϕ

Gen $\dfrac{\phi}{\forall x.\phi}$

Soundness & Completeness of Natural Deduction

Theorem

$\Gamma \vdash_{HIL} A$ *if and only if* $\Gamma \models A$.

Using the Hilbert axiomatization we can prove all and only the logical consequences of First Order Logic.
We will not prove it for Hilbert but for Natural Deduction (the proof is simpler).

Soundness of Hilbert axioms and inference rules

Problem with free variables

A rule is sound if when the premises are true then, the conclusions are also true. In first order logics, formulas are true w.r.t. an assignment. Therefore, if a formula ϕ contains a free variable x, we cannot say that $I \models \phi$ without considering an assignment for x.

Definition (Soundness of axioms and inference rules)

- An axioms ϕ is sound if and only if for every interpretation I and for every assignment a to the free variables of ϕ we have that $I \models \phi[a]$.
- A Hilbert style rule $\frac{\phi_1, \ldots, \phi_n}{\phi}$ is sound if for every interpretation I, if $I \models \phi_k[a]$ for every $1 \leq k \leq n$ and for every assignment a, then $I \models \phi[b]$ for every assignment b.

Theorem (Soundness)

A1–A5 *are sound axioms, and* **MP**, **Gen** *are sound inference rules.*

Proof

A4: We have to prove that $\mathcal{I} \models \forall x.\phi(x) \supset \phi(t)$, for all $\mathcal{I}$ and a

$$
\begin{array}{rcll}
\mathcal{I} \models \forall x.\phi(x) \supset \phi(t)[a] & \text{iff} & \mathcal{I} \models \forall x.\phi(x)[a] \text{ implies } \mathcal{I} \models \phi(t)[a] & \\
\mathcal{I} \models \forall x.\phi(x)[a] & \Longrightarrow & \mathcal{I} \models \phi(x)[a[x/d]] \text{ for all } d \in \Delta & \\
& \Longrightarrow & \mathcal{I} \models \phi(x)[a[x/\mathcal{I}(t)]] & (1) \\
& \Longrightarrow & \mathcal{I} \models \phi(t)[a] & (2)
\end{array}
$$

Assignment 1 Show that, to prove (1) $\Longrightarrow$ (2), you need the hypothesis that t is free for x in ϕ

Soundness - cont.

Gen:

$$\begin{aligned} \models \phi \quad & \text{iff} \quad \mathcal{I} \models \phi[a] \text{ for all } \mathcal{I} \text{ and } a \\ & \text{iff} \quad \mathcal{I} \models \phi[a[x/d]] \text{ for any } d \in \Delta \\ & \text{iff} \quad \mathcal{I} \models \forall x \phi[a] \text{ for any } \mathcal{I} \text{ and } a \\ & \text{iff} \quad \models \forall x \phi \end{aligned}$$

Assignment 2 Prove that **A1–A3**, **A5** and **MP** are sound.

Natural deduction for classical FOL

Propositional classical natural deduction is extended with the rules for introduction and elimination of quantifiers ($\forall$ and $\exists$)

$$\frac{\phi(x)}{\forall x.\phi(x)}\ \forall I \qquad\qquad \frac{\forall x.\phi(x)}{\phi(t)}\ \forall E$$

$$\frac{\phi(t)}{\exists x.\phi(x)}\ \exists I \qquad\qquad \frac{\exists x.\phi(x) \quad \begin{matrix}[\phi(x)] \\ \vdots \\ \theta\end{matrix}}{\theta}\ \exists E$$

Restrictions

- $\forall I$: p does not occur free in any assumption from which ϕ depends on. In other words, x must be "new".
- $\exists E$: x does not occur in θ and in any assumption θ depends on (with the exception of $\phi(x)$.

Why restrictions?

- Consider the following ND proof which violates the $\forall I$ restriction.

$$\dfrac{\dfrac{P(a) \supset Q(x) \quad P(a)}{Q(x)} \supset E}{\forall x.Q(x)} \forall I$$

Is it the case that $\{P(a) \supset Q(x), P(a)\} \models \forall x.Q(x)$?

- consider the following ND proof which violates the $\exists E$ restriction.

$$\dfrac{\exists x.P(x) \qquad \dfrac{P(x) \supset Q(x) \quad [P(x)]}{Q(x)} \supset E}{Q(x)} \exists E$$

Is it the case that $\{P(x) \supset Q(x), \exists x.P(x)\} \models Q(x)$?

Why restrictions?

- Consider the following ND proof which violates the $\forall I$ restriction.

$$\dfrac{\dfrac{P(a) \supset Q(x) \quad P(a)}{Q(x)} \supset E}{\forall x.Q(x)} \forall I$$

  Is it the case that $\{P(a) \supset Q(x), P(a)\} \models \forall x.Q(x)$?

- consider the following ND proof which violates the $\exists E$ restriction.

$$\dfrac{\exists x.P(x) \qquad \dfrac{P(x) \supset Q(x) \quad [P(x)]}{Q(x)} \supset E}{Q(x)} \exists E$$

  Is it the case that $\{P(x) \supset Q(x), \exists x.P(x)\} \models Q(x)$?

Natural deduction for classical FOL

...and for the equality symbol (=).

$$\frac{}{t = t} = I \qquad \frac{\phi(t) \quad x = t}{\phi(x)} = E$$

Natural deduction for classical FOL

Assignment 3 Show the deduction for the following first order valid formulas.

1. $\exists x.\forall y.R(x,y) \supset \forall y.\exists x.R(x,y)$
2. $\exists x.(P(x) \supset \forall x.P(x))$
3. $\exists x.(P(x) \vee Q(x)) \supset (\exists x.P(x) \vee \exists x.Q(x))$
4. $\exists x.(P(x) \wedge Q(x)) \supset \exists x.P(x) \wedge \exists x.Q(x))$
5. $(\exists x.P(x) \wedge \forall x.Q(x)) \supset \exists x.(P(x) \wedge Q(x))$
6. $\forall x.(P(x) \supset Q) \supset (\exists x.P(x) \supset Q)$, where x is not free in Q.
7. $\forall x.\exists y.x = y$
8. $\forall xyzw.((x = z \wedge y = w) \supset (R(x,y) \supset R(z,w)))$, where $\forall xyzw \ldots$ stands for $\forall x.(\forall y.(\forall z.(\forall w \ldots)))$.

Natural deduction for classical FOL

Assignment 3 Show the deduction for the following first order valid formulas.

1. $(A \supset \forall x.B(x)) \equiv \forall x(A \supset B(x))$ where x does not occur free in A
2. $\exists x(A(x) \vee B(x)) \equiv (\exists x A(x) \vee \exists x B(x))$
3. $\neg \exists x A(x) \equiv \forall x \neg A(x)$
4. $\forall x(A(x) \vee B) \equiv \forall x A(x) \vee B$ where x does not occur free in B
5. $\exists x(A(x) \supset B) \equiv (\forall x A(x) \supset B)$ where x does not occur free in B
6. $\exists x(A \supset B(x)) \equiv (A \supset \exists x B(x))$ where x does not occur free in A
7. $\forall x(A(x) \supset B) \equiv (\exists x A(x) \supset B)$ where x does not occur free in B

Soundness & Completeness of Natural Deduction

Theorem

$\Gamma \vdash_{ND} A$ if and only if $\Gamma \models A$.

Using the Natural Deduction rules we can prove all and only the logical consequences of First order Logic.

We first prove soundness ($\Gamma \vdash_{ND} A$ implies $\Gamma \models A$), and then completeness ($\Gamma \models A$ implies $\Gamma \vdash_{ND} A$).

Soundness

Soundness If $\Gamma \vdash_{ND} \phi$ then $\Gamma \models \phi$

Proof By induction on the length $L(\phi)$ of derivation (proof tree).

Base case If $\Gamma \vdash_{ND} \phi$ with a derivation of length 1 then we have to cases:

- ϕ is one of the assumptions in Γ, or
- ϕ is the formula $t = t$ for some term t

In both cases it is trivial to show that $\Gamma \models \phi$.

Soundness - cont'd

Inductive step Let us assume that we have proven the theorem for all derivations of length $\leq n$ and let us prove the theorem for a derivation of length $n+1$.
Strategy: Consider a general derivation $\Gamma \vdash_{ND} \phi$. It will be of the form:

$$\dfrac{\begin{array}{cccc} \Gamma_1 & \Gamma_2 & & \Gamma_n \\ \Pi_1 & \Pi_2 & & \Pi_n \\ \phi_1 & \phi_2 & \ldots & \phi_n \end{array}}{\phi}\ \text{ND rule}$$

with $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_n$.
Let us assume that $\mathcal{I} \models \Gamma[a]$. Then $\mathcal{I} \models \Gamma_i[a]$ for $1 \leq i \leq n$, and from the inductive hypothesis

$$\Gamma_1 \models \phi_1[a] \quad \Gamma_2 \models \phi_2[a] \quad \ldots \quad \Gamma_n \models \phi_n[a]$$

we can infer that

$$\mathcal{I} \models \phi_1[a] \quad \mathcal{I} \models \phi_2[a] \quad \ldots \quad \mathcal{I} \models \phi_n[a] \tag{3}$$

Thus, what we have to prove is that from (3) we we can infer

$$\mathcal{I} \models \phi[a]$$

for all the ND rules used in the last step of the derivation.

Soundness of $\vee E$

Assume that the last rule used is $\vee E$. Then the derivation tree is of the form

$$\cfrac{\begin{array}{c}\Gamma_1\\ \Pi_1\\ A \vee B\end{array} \qquad \begin{array}{c}\Gamma_2, [A]\\ \Pi_2\\ C\end{array} \qquad \begin{array}{c}\Gamma_3, [B]\\ \Pi_3\\ C\end{array}}{C}\ \vee E$$

From the inductive hypothesis we know that $\mathcal{I} \models A \vee B$. and from the definition of $\models$ we have that

$$\mathcal{I} \models A[a], \text{ or} \tag{4}$$
$$\mathcal{I} \models B[a] \tag{5}$$

- Assume that $\mathcal{I} \models A[a]$. Since $\mathcal{I} \models \Gamma_2[a]$, then $\mathcal{I} \models C[a]$ because Π_2 is a proof tree shorter than $n+1$ and we can apply the inductive hypothesis.
- Assume that $\mathcal{I} \models B[a]$. Since $\mathcal{I} \models \Gamma_3[a]$, then $\mathcal{I} \models C[a]$ because Π_3 is a proof tree shorter than $n+1$ and we can apply the inductive hypothesis.

Thus in both cases $\mathcal{I} \models C[a]$ and we can therefore conclude that $\mathcal{I} \models C[a]$.

Assignment: prove completeness for all the propositional ND rules.

Soundness of $\forall I$

Assume that the last rule used is $\forall I$. Then the derivation tree is of the form

$$\cfrac{\begin{array}{c}\Gamma \\ \Pi \\ A(x)\end{array}}{\forall x.A(x)}\ \forall I$$

with x not free in Γ. Let $\mathcal{I}, a$ be such that $\mathcal{I} \models \Gamma[a]$.

From the inductive hypothesis we know that $\mathcal{I} \models \phi(x)[a]$.

Since x does not appear free in Γ, then $\mathcal{I} \models \Gamma[a[x/d]]$ holds for all $d \in \Delta$.

Therefore from the inductive hypothesis $\mathcal{I} \models \phi(x)[a[x/d]]$ holds for all $d \in \Delta$.

Then for the definition of $\models$, we have that $\mathcal{I} \models \forall x.\phi(x)[a]$.

Assignment 3 Prove that the rules $\forall E$, $\exists I$ and $\exists E$ are sound.

Completeness

Completeness If $\Gamma \models \phi$ then $\Gamma \vdash \phi$.

The Completeness Theorem is a consequence of the following Lemma

Lemma (Extended Completeness)

Is Γ is consistent then it has a model.

Proof (Completeness)

By contradiction:

- *If $\Gamma \nvdash \phi$, then $\Gamma \cup \{\neg\phi\}$ is consistent*
- *By the extended completeness lemma $\Gamma \cup \{\neg\phi\}$ is satisfiable (i.e., it has a model)*
- *there is an interpretation $\mathcal{I}$ and an assignment a such that $\mathcal{I} \models \Gamma[a]$ and $\mathcal{I} \not\models \phi[a]$*
- *contradiction with the hypothesis that $\Gamma \models \phi$.*

Note: once the extended completeness Lemma is proven, the proof of completeness is the same as for the propositional case.
Completeness for First Order Logic was first proved by Kurt Gödel in 1929 with an extremely complex proof which didn't make use of the Extended Completeness Lemma.

Extended Completeness

How do we build a (canonical) model for a consistent set of formulae Γ?

Construction method (Leon Henkin, 1949) The strategy extends the one for the propositional case. We start from a consistent set Γ and build a model $\mathcal{I}$ for it, following the steps:

1. Extend Γ with a set of witnesses for existential quantified formulas (**new step**)
2. Saturate Γ with either ϕ or $\neg\phi$ for every ϕ obtaining maximally consistent set Σ.
3. Starting from Σ construct an interpretation $\mathcal{I}$
4. Show that $\mathcal{I}$ is a model for Σ,
5. Since $\Gamma \subseteq \Sigma$, $\mathcal{I}$ is also a model for Γ.

Witnesses - intuition

Problem: $\mathcal{I} \models \exists x \phi(x)[a]$ does not implies $\mathcal{I} \models \phi(t)[a]$ for some ground term t.

Example

Consider a FOL language containing the constant symbol c, no functional symbols, and the predicate P.
Let $\mathcal{I}$ be such that

- $\Delta = \{1, 2, 3, 4\}$
- $\mathcal{I}(c) = 1$
- $\mathcal{I}(P) = \{3\}$.

Clearly $\mathcal{I} \models \exists x.P(x)[a]$, but there is no term in the language that denotes the element 3, and therefore we cannot have $\mathcal{I} \models P(t)[a]$ for any term t.

Witnesses - intuition

Problem: $\mathcal{I} \models \exists x \phi(x)[a]$ does not implies $\mathcal{I} \models \phi(t)[a]$ for some ground term t.

Why is this a problem?

Because if we want to construct an interpretation $\mathcal{I}$ starting from a maximally consistent set Σ, and $\exists x \phi(x)$ is one of the formulae in Σ, then we need to be sure that the domain of our model contains an element that satisfies the existential formula.

Witnesses - intuition

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Witnesses - intuition

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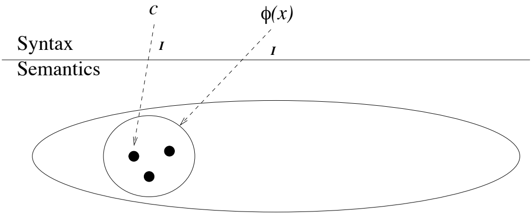
Why is this a problem?
Because if we want to construct an interpretation $\mathcal{I}$ starting from a maximally consistent set Σ, and $\exists x \phi(x)$ is one of the formulae in Σ, then we need to be sure that the domain of our model contains an element that satisfies the existential formula.

Idea: Add a witness c for each existential statement
If $\mathcal{I} \models \exists x.\phi(x) \supset \phi(c)$ then:



FOL Completeness - witnesses

Definition (Witness)

A set of constant symbols C is a set of witnesses for a set of formulas Γ in a language $\mathcal{L}$, if for every formula $\phi(x)$ of $\mathcal{L}$, with at most one free variable x,

$$\Gamma \vdash \exists x.\phi(x) \supset \phi(c)$$

Lemma (witnesses extension)

A consistent set of formulae Γ can be extended to a consistent set of formula Γ' which has a set C of witnesses.

Proof Let C be an infinite set of new constants

$$c_0, c_1, c_2, \ldots, c_n, \ldots$$

and $\mathcal{L}' = \mathcal{L} \cup C$ the first order language obtained extending $\mathcal{L}$ with the set C of constants.
Let

$$\phi_0(x_0), \phi_1(x_1), \phi_2(x_2), \ldots \phi_n(x_n), \ldots$$

be an enumeration of all the formulas of $\mathcal{L}'$ with exactly one free variable.
Let $\Gamma = \Gamma_0 \subset \Gamma_1 \subset \Gamma_2 \ldots$ with

$$\Gamma_{n+1} = \Gamma_n \cup \{\exists x_n.\phi_n(x_n) \supset \phi_n(c_n)\}$$

where c_n is a constant of C not occurring in Γ_n and $\phi_n(x_n)$.

Let $\Gamma' = \bigcup_{n \geq 0} \Gamma_n$.

FOL Completeness - Step 1: add witnesses

The fact that Γ' has a set of witnesses can be easily shown by construction.
To prove that Γ' is consistent we first prove that each Γ_{n+1} is consistent by induction on n.

Base case ($n = 0$) Γ_0 is consistent by hypothesis.

Step Case Assume that Γ_n is consistent and show that Γ_{n+1} is consistent.
We prove it by contradiction. That is, we suppose that Γ_n is consistent and Γ_{n+1} is not consistent.
If Γ_{n+1} is not consistent then there is a deduction

$$\begin{array}{c} \Gamma'_n \quad \exists x(\phi_n(x) \supset \phi(c)) \\ \Pi \\ \bot \end{array}$$

with Γ'_n a finite subset of Γ_n. Since c does not appear in Γ_n, we can consider it as a free variable and we can build the following deduction of $\bot$ from Γ_n

$$\dfrac{\exists y.(\exists x_n.\phi_n(x_n) \supset \phi_n(y)) \qquad \begin{array}{c} \Gamma'_n \quad [\exists x(\phi_n(x) \supset \phi(c))] \\ \vdots \\ \bot \end{array}}{\bot} \; \exists E$$

Thus Γ_n is not consistent. But this is a contradiction, and therefore Γ_{n+1} is consistent.

FOL Completeness - Step 1: add witnesses

We can now prove that Γ' is consistent.
If not, there is a finite subset $\overline{\Gamma}$ of Γ' such that $\overline{\Gamma} \vdash \bot$.
By definition of Γ' there is a j such that $\overline{\Gamma} \subseteq \Gamma_j$ (with Γ_j one of the sets in the sequence defined before). Therefore $\Gamma_j \vdash \bot$ and Γ_j is inconsistent.
But this is impossible because we have just shown that every Γ_{i+1} in the sequence $\Gamma = \Gamma_0 \subset \Gamma_1 \subset \Gamma_2 \ldots$ is consistent.
Therefore the assumption that Γ' is inconsistent cannot be, and Γ' is consistent.

Note: The proof relies on the fact that $\exists y.(\exists x, \phi_n(x) \supset \phi_n(y))$ is a valid formula.

Assignment 4 Provide a proof in natural deduction of $\exists y.(\exists x.\phi_n(x) \supset \phi_n(y))$

FOL Completeness - Step 2: saturate

We can extend the set Γ' with witnesses to a maximally consistent set Σ exactly as for propositional logic. We do not need to prove theorems here as the proofs hold also for FOL.

FOL Completeness - Step 3: build the model

Definition ($\sim$)

For all $c, d \in C$, $c \sim d$ iff $c = d \in \Sigma$.

Since Σ is maximally consistent, it is easy to prove that $\sim$ is a congruence (equivalence relation), that is, for all c, d, e

- $c \sim c$,
- $c \sim d$ implies $d \sim c$,
- $c \sim d$ and $d \sim e$ implies $c \sim e$

We indicate with $[c]$ the equivalence class of c. Formally, $[c] = \{d \in C | d \sim c\}$

Now we are ready to construct a model for Σ

FOL Completeness - Step 3: build the model

We build a model whose domain is the set of all the equivalence classes $[c]$

Definition (Canonical model)

A canonical model $\mathcal{I}$ is a pair $\langle \Delta, \mathcal{I} \rangle$ defined as follows:

1. $\Delta = \{[c] | c \in C\}$
2. $\mathcal{I}(c) = [c]$ for all $c \in C$
3. $\mathcal{I}(d) = [c]$ if for some $c \in C$, $c = d \in \Sigma$, for all $d \in \mathcal{L}$
4. $\mathcal{I}(f^n) = \{\langle [c_1], \ldots, [c_n], [c] \rangle \, | f(c_1, \ldots, c_n) = c \in \Sigma\}$
5. $\mathcal{I}(P^n) = \{\langle [c_1], \ldots, [c_n] \rangle \, | P(c_1, \ldots, c_n) \in \Sigma\}$

We should prove that $\mathcal{I}$ is a model. For instance that $\mathcal{I}(d)$ for all $d \in \mathcal{L}$ is defined. But we skip this step. We also restrict to Σ closed.

FOL Completeness - Step 4: $\mathcal{I}$ is a model for Σ

We have to prove that $\mathcal{I}$ is a model for Σ.

Theorem

$$\mathcal{I} \models \phi \textit{ if and only if } \phi \in \Sigma$$

The proof is done by induction on the complexity of formula. We skip the proof in this course. It can be found on Chapter 2 of Chang-Keisler, "Model tehory", North Holland.

FOL Completeness - Step 5: $\mathcal{I}$ is also a model for Γ

Theorem (Completeness)

A set of closed formulas Γ is consistent if it has a model.

Proof.

- Extend Γ to a maximal consistent set Σ with witnesses in C.
- Let $\mathcal{I}$ be the canonical model for Σ.
- Since $\Gamma \subset \Sigma$, then $\mathcal{I} \models \Gamma$.
- Let $\mathcal{I}'$ be the restriction of $\mathcal{I}$ that does not consider the new constant C. Since Γ does not contain any formula about c, then $\mathcal{I}'$ is a model for Γ.

□