Mathematical Logic

Natural Deduction and Hilbert style Propositional reasoning. Introduction to decision procedures

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5, 12 March 2013

Deciding logical consequence

Problem

Is there an algorithm to determine whether a formula ϕ is the logical consequence of a set of formulas Γ?

Naïve solution

- Apply directly the definition of logical consequence i.e., for all possible interpretations $\mathcal{I}$ determine if $\mathcal{I} \models \Gamma$, if this is the case then check if $\mathcal{I} \models A$ too.
- This solution can be used when Γ is finite, and there is a finite number of relevant interpretations.

Deciding logical consequence, is not always possible

Propositional Logics

The truth table method enumerates all the possible interpretations of a formula and, for each formula, it computes the relation $\models$.

Other logics

For first order logic and modal logics There no general algorithm to compute the logical consequence. There are some algorithms computing the logical consequence for first order logic sub-languages and for sub-classes of structures (as we will see further on).

Propositional logical consequence

Exercize (Logical consequence via truth table)

Determine, Via truth table, if the following statements about logical consequence holds

- $p \models q$
- $p \supset q \models q \supset p$
- $p, \neg q \supset \neg p \models q$
- $\neg q \supset \neg p \models p \supset q$

Complexity of the logical consequence problem

The truth table method is Exponential

The problem of determining if a formula A containing n primitive propositions, is a logical consequence of the empty set, i.e., the problem of determining if A is valid, ($\models A$), takes an n-exponential number of steps. To check if A is a tautology, we have to consider 2^n interpretations in the truth table, corresponding to 2^n lines.

More efficient algorithms?

Are there more efficient algorithms? I.e. Is it possible to define an algorithm which takes a polinomial number of steps in n, to determine the validity of A? This is an unsolved problem

$P \stackrel{?}{=} NP$

The existence of a polinomial algorithm for checking validity is still an open problem, even it there are a lot of evidences in favor of non-existence

Propositional reasoning: Proofs and deductions (or derivations)

proof

A proof of a formula ϕ is a sequence of formulas $\phi_1, \ldots, \phi_n$, with $\phi_n = \phi$, such that each ϕ_k is either

- an axiom or
- it is derived from previous formulas by reasoning rules

ϕ is provable, in symbols $\vdash \phi$, if there is a proof for ϕ.

Deduction of ϕ from Γ

A deduction of a formula ϕ from a set of formulas Γ is a sequence of formulas $\phi_1, \ldots, \phi_n$, with $\phi_n = \phi$, such that ϕ_k

- is an axiom or
- it is in Γ (an assumption)
- it is derived form previous formulas bhy reasoning rules

ϕ is derivable from Γ, in symbols $\Gamma \vdash \phi$, if there is a proof for ϕ from formulas in Γ.

Historical notes

Natural deduction (ND) was invented by G. Gentzen in 1934. The idea was to have a system of derivation rules that as closely as possible reflects the logical steps in an informal rigorous proof.

Natural Deduction

Introduction and elimination rules

For each connective $\circ$,

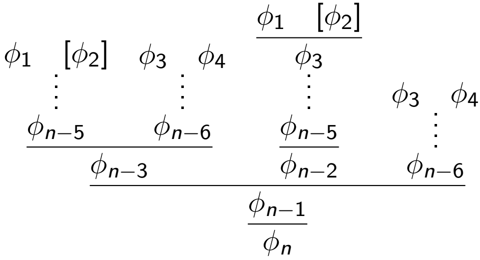
- there is an introduction rule $(\circ I)$ which can be seen as a definition of the truth conditions of a formula with $\circ$ given in terms of the truth values of its component(s);
- there is an elimination rule $(\circ E)$ that allows to exploit such a definition to derive truth of the components of a formula whose main connective is $\circ$.

Assumptions

In the process of building a deduction one can make new assumptions and can discharge already done assumptions.

Natural deduction Derivation

A derivation is a tree where the nodes are the rules and the leafs are the assumptions of the derivation. The root of the tree is the conclusion of the derivation.



ND rules for propositional connectives

$\wedge$

$$\frac{\phi \quad \psi}{\phi \wedge \psi}\ \wedge I \qquad \frac{\phi \wedge \psi}{\phi}\ \wedge E_1 \qquad \frac{\phi \wedge \psi}{\psi}\ \wedge E_2$$

$\supset$

$$\frac{\begin{matrix}[\phi] \\ \vdots \\ \psi\end{matrix}}{\phi \supset \psi}\ \supset I \qquad \frac{\phi \quad \phi \supset \psi}{\psi}\ \supset E$$

$\vee$

$$\frac{\phi}{\phi \vee \psi}\ \vee I_1 \qquad \frac{\psi}{\phi \vee \psi}\ \vee I_2 \qquad \frac{\phi \vee \psi \quad \begin{matrix}[\phi] \\ \vdots \\ \theta\end{matrix} \quad \begin{matrix}[\psi] \\ \vdots \\ \theta\end{matrix}}{\theta}\ \vee E$$

ND rules for propositional connectives

The connective $\neg$ for negation

ND does not provide rules for the $\neg$ connective. Instead, the logical constant $\bot$ is introduced,
$\bot$ stands for the unsatisfiable formula, i.e., the formula that is false in all interpretations.
$\neg A$ is defined to be a syntactic sugar for $A \supset \bot$
(**exercise:** Verify that $\neg A \equiv (A \supset \bot)$ is a valid formula).

$\bot$

$$\dfrac{\begin{array}{c}[\neg\phi]\\ \vdots \\ \bot\end{array}}{\phi}\ \bot_c$$

Definition (Deduction)

A deduction Π of A with undischarged assumption $A_1, \ldots, A_n$, is a tree with root A, obtained by applying the ND rules, and every assumption in Π, but $A_1, \ldots, A_n$ is discharged, by the application of one of the ND rules.

Definition ($\Gamma \vdash_{ND} A$)

A formula A is derivable from a set of formulas Γ, if there is a deduction of A with undischarged assumption contained in Γ. In this case we write

$$\Gamma \vdash_{ND} A$$

If no ambiguity arises we omit the subscript ND and use $\Gamma \vdash A$

Examples

For each of the following statements provide a proof in natural deduction.

1. $\vdash_{ND} A \supset (B \supset A)$
2. $\vdash_{ND} \neg(A \wedge \neg A)$
3. $\vdash_{ND} \neg\neg A \leftrightarrow A$
4. $\vdash_{ND} (A \vee A) \equiv (A \vee \bot)$
5. $(A \wedge B) \wedge C \vdash_{ND} A \wedge (B \wedge C)$
6. $\vdash_{ND} A \vee \neg A;$
7. $\vdash_{ND} (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
8. $\vdash_{ND} (A \supset B) \leftrightarrow (\neg A \vee B)$
9. $\vdash_{ND} A \vee (A \supset B)$
10. $\neg(A \supset \neg B) \vdash_{ND} (A \wedge B)$
11. $A \supset (B \supset C), A \vee C, \neg B \supset \neg A \vdash_{ND} C$

Examples

1. $\vdash_{ND} A \supset (B \supset A)$

$$\cfrac{\cfrac{A^1}{B \supset A} \supset I}{A \supset (B \supset A)} \supset I_{(1)}$$

Examples

2. $\vdash_{ND} \neg(A \wedge \neg A)$

$$\dfrac{\dfrac{\dfrac{A \wedge \neg A^1}{A}\ \wedge E \qquad \dfrac{A \wedge \neg A^1}{\neg A}\ \wedge E}{\bot}\ \supset E}{\neg(A \wedge \neg A)}\ \bot c_{(1)}$$

Examples

3. $\vdash_{ND} \neg\neg A \leftrightarrow A$

$$\dfrac{\dfrac{\dfrac{\neg\neg A^2 \quad \neg A^1}{\bot} \supset E}{A} \bot c_{(1)}}{\neg\neg A \supset A} \supset I_{(2)}$$

$$\dfrac{\dfrac{\dfrac{A^2 \quad \neg A^1}{\bot} \supset E}{\neg\neg A} \bot c_{(1)}}{A \supset \neg\neg A} \supset I_{(2)}$$

Examples

4. $\vdash_{ND} (A \vee A) \equiv (A \vee \bot)$

$$\dfrac{\dfrac{A \vee A^2 \quad \dfrac{A^1}{A \vee \bot}\vee I \quad \dfrac{A^1}{A \vee \bot}\vee I}{A \vee \bot}\vee E_{(1)}}{(A \vee A) \supset (A \vee \bot)} \supset I_{(2)}$$

$$\dfrac{\dfrac{A \vee \bot^2 \quad \dfrac{A^1}{A \vee A}\vee I \quad \dfrac{\bot^1}{A \vee A}\bot_c}{A \vee A}\vee E_{(1)}}{(A \vee \bot) \supset (A \vee A)} \supset I_{(2)}$$

Examples

5. $(A \wedge B) \wedge C \vdash_{ND} A \wedge (B \wedge C)$

$$\dfrac{\dfrac{\dfrac{(A \wedge B) \wedge C}{A \wedge B}\wedge E}{A}\wedge E \qquad \dfrac{\dfrac{\dfrac{(A \wedge B) \wedge C}{A \wedge B}\wedge E}{B}\wedge E \qquad \dfrac{(A \wedge B) \wedge C}{C}\wedge E}{B \wedge C}\wedge I}{A \wedge (B \wedge C)}\wedge I$$

Examples

6. $\vdash_{ND} A \vee \neg A$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A^1}{A \vee \neg A}\,\vee I \qquad \neg(A \vee \neg A)^2}{\bot}\supset E}{\neg A}\,\bot_{c(1)}}{A \vee \neg A}\,\vee I \qquad \neg(A \vee \neg A)^2}{\bot}\supset E}{A \vee \neg A}\,\bot_{c(2)}
$$

Examples

7. $\vdash_{ND} (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \supset (B \supset C)^3 \quad A^1}{B \supset C} \supset E \quad \dfrac{A \supset B^2 \quad A^1}{B} \supset E}{C} \supset E}{A \supset C} \supset I_{(1)}}{(A \supset B) \supset (A \supset C)} \supset I_{(2)}}{(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))} \supset I_{(3)}$$

Examples

8.a $\vdash_{ND} (A \supset B) \supset (\neg A \vee B)$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \supset B^3 \quad A^1}{B} \supset E}{\neg A \vee B} \vee I \qquad \neg(\neg A \vee B)^2}{\bot} \supset E}{\neg A} \bot c_{(1)}}{\neg A \vee B} \vee I \qquad \neg(\neg A \vee B)^2}{\bot} \supset E}{\neg A \vee B} \bot c_{(2)}}{(A \supset B) \supset (\neg A \vee B)} \supset I_{(3)}
$$

Examples

8.b $\vdash_{ND} (\neg A \vee B) \supset (A \supset B)$

$$\dfrac{\dfrac{\neg A \vee B^3 \qquad \dfrac{\dfrac{\dfrac{\neg A^2 \quad A^1}{\bot} \supset E}{B} \bot c}{A \supset B} \supset I_{(1)} \qquad \dfrac{B^2}{A \supset B} \supset I}{A \supset B} \vee E_{(2)}}{(\neg A \vee B) \supset (A \supset B)} \supset I_{(3)}$$

9. $\vdash_{ND} A \lor (A \supset B)$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{A^1}{A \lor (A \supset B)}\ \lor I
          \qquad
          \neg(A \lor (A \supset B))^2
        }{\bot}\ \supset E
      }{B}\ \bot_c
    }{A \supset B}\ \supset I_{(1)}
  }{A \lor (A \supset B)}\ \lor I
  \qquad
  \neg(A \lor (A \supset B))^2
}{
  \dfrac{\bot}{A \lor (A \supset B)}\ \bot_{c(2)}
}\ \supset E
$$

10. $\neg(A \supset \neg B) \vdash_{ND} (A \wedge B)$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{A^1 \quad \neg A^2}{\bot}\supset E
}{\neg B}\bot c
}{A \supset \neg B}\supset I_{(1)}
\quad \neg(A \supset \neg B)
}{\bot}\supset E
}{A}\bot c_{(2)}
\qquad
\dfrac{
\dfrac{
\dfrac{\neg B^3}{A \supset \neg B}\supset I
\quad \neg(A \supset \neg B)
}{\bot}
}{B}\bot c_{(3)}
}{A \wedge B}\wedge I
$$

Examples

11. $A \supset (B \supset C), A \vee C, \neg B \supset \neg A \vdash_{ND} C$

$$
\dfrac{A \vee C \qquad \dfrac{\dfrac{A \supset (B \supset C) \quad A^2}{B \supset C}\supset E \qquad \dfrac{\dfrac{\dfrac{\dfrac{\neg B \supset \neg A \quad \neg B^1}{\neg A}\supset E \qquad A^2}{\bot}\supset E}{B}\bot_{c(1)}}{C}\supset E \qquad C^2}{C}\vee E_{(2)}
$$

Proof Strategies

1: $\vdash_{ND} \psi \supset \phi$

- assume ψ and try to deduce ϕ (simplest solution)
- as an alternative, assume $\neg\phi$ and ψ and try to deduce $\bot$

2: $\vdash_{ND} \phi_1 \supset (\phi_2 \supset \phi_3)$

- apply recursively the strategy in 1

3: $\vdash_{ND} \psi \wedge \phi$

- try to deduce ψ and try to deduce ϕ (separately) and then apply $\wedge I$

4: $\vdash_{ND} \psi \vee \phi$

- try to deduce ψ or (alternatively) ϕ and then apply $\vee I$ ... usually it doesn't work.
- assume $\neg\psi$, try to derive ϕ and proceed by contradiction:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg\psi^1 \\ \vdots \\ \phi}{\psi \vee \phi}\vee I \quad \neg(\psi \vee \phi)^2}{\bot}\supset E}{\psi}\bot c_{(1)}}{\psi \vee \phi}\vee I \quad \neg(\psi \vee \phi)^2}{\bot}\supset E}{\psi \vee \phi}\bot c_{(2)}
$$

alternatively, assume $\neg\phi$, try to derive ψ and proceed by contradiction in the same way

Proof Strategies

5: $\vdash_{ND} (\phi_1 \vee \phi_2) \supset \phi_3$

1. assume ϕ_1 and deduce ϕ_3
2. assume ϕ_2 and deduce ϕ_3
3. assume $\phi_1 \vee \phi_1$ and apply $\vee E$

$$\dfrac{\phi_1 \vee \phi_2 \qquad \begin{matrix} \phi_1^1 \\ \vdots \\ \phi_3 \end{matrix} \qquad \begin{matrix} \phi_2^1 \\ \vdots \\ \phi_3 \end{matrix}}{\phi_3} \; \vee E_{(1)}$$

Soundness & Completeness of Natural Deduction

Theorem

$\Gamma \vdash_{ND} A$ *if and only if* $\Gamma \models A$.

Using the Natural Deduction rules we can prove all and only the logical consequences of Propositional Logic.
We will not prove it for Natural Deduction but for the Hilbert Axiomatization.

Hilbert axioms for classical propositional logic

Axioms

A1 $\phi \supset (\psi \supset \phi)$

A2 $(\phi \supset (\psi \supset \theta)) \supset ((\phi \supset \psi) \supset (\phi \supset \theta))$

A3 $(\neg\psi \supset \neg\phi) \supset ((\neg\psi \supset \phi) \supset \psi)$

Inference rule(s)

MP $\dfrac{\phi \quad \phi \supset \psi}{\psi}$

Why there are no axioms for $\wedge$ and $\vee$ and $\equiv$?

The connectives $\wedge$ and $\vee$ are rewritten into equivalent formulas containing only $\supset$ and $\neg$.

$$
\begin{aligned}
A \wedge B &\equiv \neg(A \supset \neg B) \\
A \vee B &\equiv \neg A \supset B \\
A \equiv B &\equiv \neg((A \supset B) \supset \neg(B \supset A))
\end{aligned}
$$

Proofs and deductions (or derivations)

proof

A proof of a formula ϕ is a sequence of formulas $\phi_1, \ldots, \phi_n$, with $\phi_n = \phi$, such that each ϕ_k is either

- an axiom or
- it is derived from previous formulas by MP

ϕ is provable, in symbols $\vdash \phi$, if there is a proof for ϕ.

Deduction of ϕ from Γ

A deduction of a formula ϕ from a set of formulas Γ is a sequence of formulas $\phi_1, \ldots, \phi_n$, with $\phi_n = \phi$, such that ϕ_k

- is an axiom or
- it is in Γ (an assumption)
- it is derived form previous formulas bhyy MP

ϕ is derivable from Γ in symbols $\Gamma \vdash \phi$ if there is a proof for ϕ.

Deduction and proof - example

Example (Proof of $A \supset A$)

1.	$A1$	$A \supset ((A \supset A) \supset A)$
2.	$A2$	$(A \supset ((A \supset A) \supset A)) \supset ((A \supset (A \supset A)) \supset (A \supset A))$
3.	$MP(1,2)$	$(A \supset (A \supset A)) \supset (A \supset A)$
4.	$A1$	$(A \supset (A \supset A))$
5.	$MP(4,3)$	$A \supset A$

Deduction and proof - other examples

Example (proof of $\neg A \supset (A \supset B)$)

We prove that $A, \neg A \vdash B$ and by deduction theorem we have that $\neg A \vdash A \supset B$ and that $\vdash \neg A \supset (A \supset B)$

We label with Hypothesis the formula on the left of the $\vdash$ sign.

1.	*hypothesis*	A
2.	$A1$	$A \supset (\neg B \supset A)$
3.	$MP(1,2)$	$\neg B \supset A$
4.	*hypothesis*	$\neg A$
5.	$A1$	$\neg A \supset (\neg B \supset \neg A)$
6.	$MP(4,5)$	$\neg B \supset \neg A$
7.	$A3$	$(\neg B \supset \neg A) \supset ((\neg B \supset A) \supset B)$
8.	$MP(6,7)$	$(\neg B \supset A) \supset B$
9.	$MP(3,8)$	B

Hilbert axiomatization

Minimality

The main objective of Hilbert was to find the smallest set of axioms and inference rules from which it was possible to derive all the tautologies.

Unnatural

Proofs and deductions in Hilbert axiomatization are awkward and unnatural. Other proof styles, such as Natural Deductions, are more intuitive. As a matter of facts, nobody is practically using Hilbert calculus for deduction.

Why it is so important

Providing an Hilbert style axiomatization of a logic describes with simple axioms the entire properties of the logic. Hilbert axiomatization is the "identity card" of the logic.

The deduction theorem

Theorem

$\Gamma, A \vdash B$ *if and only if* $\Gamma \vdash A \supset B$

Proof.

If A and B are equal, then we know that $\vdash A \supset B$ (see previous example), and by monotonicity $\Gamma \vdash A \supset B$.
Suppose that A and B are distinct formulas. Let $\pi = (A_1, \ldots, A_n = B)$ be a deduction of $\Gamma, A \vdash B$, we proceed by induction on the length of π.

Base case $n = 1$ If $\pi = (B)$, then either $B \in \Gamma$ or B is an axiom If $B \in \Gamma$, then

Axiom **A1**	$B \supset (A \supset B)$
$B \in \Gamma$ or B is an axiom	B
by **MP**	$A \supset B$

is a deduction of $A \supset B$ from Γ or from the empty set, and therefore $\Gamma \vdash A \supset B$.

□

The deduction theorem

Proof.

Step case If $A_n = B$ is either an axiom or an element of Γ, then we can reason as the previous case.

If B is derived by **MP** form A_i and $A_j = A_i \supset B$. Then, A_i and $A_j = A_i \supset B$, are provable in less then n steps and, by induction hypothesis, $\Gamma \vdash A \supset A_i$ and $\Gamma \vdash A \supset (A_1 \supset B)$. Starting from the deductions of these two formulas from Γ, we can build a deduction of $A \supset B$ form Γ as follows:

By induction	$\vdots$ deduction of $A \supset (A_i \supset B)$ form Γ
	$A \supset (A_i \supset B)$
By induction	$\vdots$ deduction of $A \supset A_i$ form Γ
	$A \supset A_i$
A2	$(A \supset (A_i \supset B)) \supset ((A \supset A_i) \supset (A \supset B))$
MP	$(A \supset A_i) \supset (A \supset B)$
MP	$A \supset B$

□

Soundness of Hilbert axiomatization

Theorem

Soundness of Hilbert axiomatization If $\Gamma \vdash A$ *then* $\Gamma \models A$.

Proof.

Let $\pi = (A_1, \ldots, A_n = A)$ be a proof of A form Γ. We prove by induction on n that $\Gamma \models A$

Base case $n = 1$ If π is (A_1), then either $A_1 \in \Gamma$ or A_1 is an instance of **(**A1), **(**A2), or **(**A3). In the first case, by reflexivity we have $A \models A$, and by monotonicity $A \in \Gamma$ implies $\Gamma \models A$. If A_1 is an instance of an axiom, then it is enough to prove that $\models$ **A1**, $\models$ **A2** and n $\models$ **A3** (by exercize)

Step case Suppose that A_n is derived by the application of **MP** to A_i and A_j with $i, j < n$. Then A_j is of the form $A_i \supset A_n$. By induction we have $\Gamma \models A_i$ and $\Gamma \models A_i \supset A_n$. which implies (prove it by exercise) that $\Gamma \models A_n$.

□

Completeness of Hilbert axiomatization

Theorem

If $\Gamma \models A$ *then* $\Gamma \vdash A$.

Completeness proof - 1/5

Definition

- a set of formulas Γ is inconsistent if $\Gamma \vdash \phi$ for every ϕ
- Γ is consistent it is not inconsistent;
- Γ is maximally consistent if it is consistent and any other consistent set $\Sigma \supseteq \Gamma$ is equal to Γ.

Proposition

1. *if* Γ *is consistent and* $\Sigma = \{\phi | \Gamma \vdash \phi\}$ *then* Σ *is consistent.*
2. *if* Γ *is maximally consistent, than* $\Gamma \vdash \phi$ *implies that* $\phi \in \Gamma$
3. Γ *is inconsistent if* $\Gamma \vdash \phi$ *and* $\Gamma \vdash \neg\phi$

Completeness proof - 2/5

Theorem (Lindenbaum's Theorem)

Any consistent set of formulas Σ can be extended to a maximally consistent set of formulas Γ.

Proof.

- Let $\phi_1, \phi_2, \ldots$ an enumeration of all the formulas of the language
- Let $\Sigma = \Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \ldots$, with

$$\Sigma_{n+1} = \begin{cases} \Sigma_n \cup \{\phi_n\} & \text{If } \Sigma_n \cup \{\phi_n\} \text{ is consistent} \\ \Sigma_n & \text{otherwise} \end{cases}$$

  Let $\Gamma = \bigcup_{n \geq 1} \Sigma_n$
- Γ is consistent!
- Γ is maximally consistent!

□

Completeness proof - 3/5

Lemma

If Γ *is maximally consistent then for every formula* ϕ *and* ψ*;*

1. $\phi \in \Gamma$ *if and only if* $\neg\phi \notin \Gamma$*;*
2. $\phi \supset \psi \in \Gamma$ *if and only if* $\phi \in \Gamma$ *implies that* $\psi \in \Gamma$

Proof.

1. ($\Rightarrow$) If $\phi \in \Gamma$, then $\neg\phi \notin \Gamma$ since Γ is consistent
1. ($\Leftarrow$) if $\neg\phi \notin \Gamma$, $\Gamma \cup \phi$ is consistent. Indeed suppose that $\Gamma \cup \phi$ is inconsistent, then $\Gamma \cup \phi \vdash \neg\phi$. By the deduction theorem $\Gamma \vdash \phi \supset \neg\phi$, and since $(\phi \supset \neg\phi) \supset \phi$ is provable, then $\Gamma \models \neg\phi$ (by **MP**). By maximality of Γ, $\Gamma \vdash \neg\phi$ implies that $\neg\phi \in \Gamma$, This contradicts the hypothesis that $\neg\phi \in \Gamma$. The fact that $\Gamma \cup \{\phi\}$ is consisten and the maximality of Γ implies that $\phi \in \Gamma$.
2. ($\Rightarrow$) If $\phi \supset \psi \in \Gamma$ and $\phi \in \Gamma$, then $\Gamma \vdash \psi$, which implies that $\psi \in \Gamma$.
2. ($\Leftarrow$) If $\phi \supset \psi \notin \Gamma$. Then by property 1, $\neg(\phi \supset \psi) \in \Gamma$. Since $\neg(\phi \supset \psi) \supset \phi$ and $\neg(\phi \supset \psi) \supset \neg\psi$, can be proved by the Hilbert axiomatic system, then $\phi \in \Gamma$ and $\neg\psi \in \Gamma$, which implies $\psi \notin \Gamma$. This implies that it is not true that if $\phi \in \Gamma$ then $\psi \in \Gamma$.

□

Theorem (Extended Completeness)

If set of formulas Σ is consistent then it is satisfiable.

Proof.

We have to prove that there is an interpretation that satisfies all the formulas of Σ.

- By Lindenbaum's Theorem, there is maximally consistent set of formulas $\Gamma \supseteq \Sigma$
- Let $\mathcal{I}$ be the interpretation such that

$$\mathcal{I}(p) = \text{True if and only if } p \in \Gamma$$

- By induction $\mathcal{I}(\phi) = \text{True}$ if and only if $\phi \in \Gamma$
- Since $\Sigma \subseteq \Gamma$, then $\mathcal{I} \models \Gamma$.

□

Completeness proof - 5/5

Theorem (Completeness)

If $\Gamma \models \phi$ *then* $\Gamma \vdash \phi$

Proof.

By contradiction:

- If $\Gamma \nvdash \phi$, then $\Gamma \cup \{\neg\phi\}$ is consistent
- By extended completeness theorem $\Gamma \cup \{\neg\phi\}$ is satisfiable
- there is an interpretation $\mathcal{I} \models \Gamma$ and $\mathcal{I} \not\models \phi$
- contradiction with the hypothesis that $\Gamma \models \phi$.

□

Observation about the completeness proof

- The underlying methodology for the proof of the completeness theorem, is to prove that a consistent set of formulas Γ has a model,
- The model for Γ is build by saturating Γ with formulas
- during the saturation, we have to be careful not to make Γ inconsistent, i.e., every time we add a formula we have to check if a pair of contraddicting formulas are derivable via the set of inference rules, if it is not, we can safely add the formula.
- When Γ is saturated, (but still consistent) it defines a single model for Γ (up to isomorphism) and we have to provide a way to extract such a model form Γ

More efficient reasoning systems

Hilbert style is not easy implementable

Checking if $\Gamma \models \phi$ by searching for a Hilbert-style deduction of ϕ from Γ is not an easy task for computers. Indeed, in trying to generate a deduction of ϕ from Γ, there are to many possible actions a computer could take:

- adding an instance of one of the three axioms (infinite number of possibilities)
- applying **MP** to already deduced formulas,
- adding a formula in Γ

More efficient methods

Resolution to check if a formula is *not satisfiable*

SAT DP, DPLL to *search for an interpretation that satisfies a formula*

Tableaux *search for a model of a formula* guided by its structure

Decision procedures

Four tipes of questions

- Model Checking: $\mathcal{I} \overset{?}{\models} \phi$
- Satisfiability: Is there an $\mathcal{I}$ such that $\mathcal{I} \models \phi$?
- Validity: $\overset{?}{\models} \phi$ (for any model $\mathcal{I}$, is is the case that $\mathcal{I} \models \phi$?)
- Logical consequence: $\Gamma \overset{?}{\models} \phi$ (for any model $\mathcal{I}$ that satisfies Γ, is is the case that $\mathcal{I} \models \phi$?)

Model Checking

Model checking decision procedure

A model checking decision procedure, MCDP is an algorithm that checks if a formula ϕ is satisfied by an interpretation $\mathcal{I}$. Namely

$$\mathrm{MCDP}(\phi, \mathcal{I}) = \textit{true} \quad \text{if and only if} \quad \mathcal{I} \models \phi$$
$$\mathrm{MCDP}(\phi, \mathcal{I}) = \textit{false} \quad \text{if and only if} \quad \mathcal{I} \not\models \phi$$

A simple recursive MCDP

MCDP($\mathcal{I} \models \phi$) applyes one of the following cases:

MCDP($\mathcal{I} \models p$)
  if $I(p) = true$
    then return YES
  else return NO

MCDP($\mathcal{I} \models A \wedge B$)
  if MCDP($I \models A$)
    then return MCDP($I \models B$)
  else return NO

MCDP($\mathcal{I} \models A \vee B$)
  if MCDP($I \models A$)
    then return YES
  else return MCDP($I \models B$)

MCDP($\mathcal{I} \models A \supset B$)
  if MCDP($I \models A$)
    then return MCDP($I \models B$)
  else return YES

MCDP($\mathcal{I} \models A \equiv B$)
  if MCDP($I \models A$)
    then return MCDP($I \models B$)
  else return not(MCDP($I \models B$)

Satisfiability decision procedure

A satisfiability decision procedure SDP is an algorithm that takes in input a formula ϕ and checks if ϕ is (un)satisfiable. Namely

$$\begin{array}{rcl} \mathrm{SDP}(\phi) = \textit{true} & \text{if and only if} & \mathcal{I} \models \phi \text{ for some } \mathcal{I} \\ \mathrm{SDP}(\phi) = \textit{false} & \text{if and only if} & \mathcal{I} \not\models \phi \text{ for all } \mathcal{I} \end{array}$$

When $\mathrm{SDP}(\phi) = \textit{true}$, SDP sometimes returns the interpretation $\mathcal{I}$, i.e., a model of ϕ. Notice that this might not be the only one.

Validity

Validity decision procedure

A decision procedure for Validity, is an algorithm that checks whether a formula is valid. SDP can be used as a satisfiability decision procedure by exploiting the equivalence

$$\phi \text{ is satisfiabile if and only if } \neg\phi \text{ is not Valid}$$

$$\mathrm{SDP}(\neg\phi) = \textit{true} \quad \text{if and only if} \quad \not\models \phi$$
$$\mathrm{SDP}(\neg\phi) = \textit{false} \quad \text{if and only if} \quad \models \phi$$

When $\mathrm{SDP}(\neg\phi)$ returns an interpretation $\mathcal{I}$, this interpretation is a counter-model for ϕ.

Logical consequence

Logical consequence decision procedure

A decision procedure for logical consequence is an algorithm that cheks whether a formula ϕ is a logical consequence of a finite set of formulas $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$. SDP can be used as a satisfiability decision procedure by exploiting the property

$$\Gamma \models \phi \text{ if and only if } \Gamma \cup \{\neg\phi\} \text{ is unsatisfiable}$$

$$\begin{aligned} \mathrm{SDP}(\gamma_1 \wedge \cdots \wedge \gamma_n \wedge \neg\phi) = \textit{true} \quad & \text{if and only if} \quad \Gamma \not\models \phi \\ \mathrm{SDP}(\gamma_1 \wedge \cdots \wedge \gamma_n \wedge \neg\phi) = \textit{false} \quad & \text{if and only if} \quad \Gamma \models \phi \end{aligned}$$

When $\mathrm{SDP}(\gamma_1 \wedge \cdots \wedge \gamma_n \wedge \neg\phi)$ returns an interpretation $\mathcal{I}$, this interpretation is a model for Γ and a counter-model for ϕ.

Proof of the previous property

Theorem

$\Gamma \models \phi$ *if and only if* $\Gamma \cup \{\neg\phi\}$ *is unsatisfiable*

Proof.

- $\Rightarrow$ Suppose that $\Gamma \models \phi$, this means that every interpretation $\mathcal{I}$ that satisfies Γ, it does satisfy ϕ, and therefore $\mathcal{I} \not\models \neg\phi$. This implies that there is no interpretations that satisfies together Γ and $\neg\phi$.
- $\Leftarrow$ Suppose that $\mathcal{I} \models \Gamma$, let us prove that $\mathcal{I} \models \phi$, Since $\Gamma \cup \{\neg phi\}$ is not satisfiable, then $\mathcal{I} \not\models \neg\phi$ and therefore $\mathcal{I} \models \phi$.

□