Logics for Data and Knowledge Representation

2a. Exercises in FOL

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Outline

- Examples of logical formalisation:
  - FOL: intuitive meaning;
  - Formalizing English Sentences in FOL;
  - FOL Interpretation and Satisfiability.

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FOL: Intuitive Meaning

Examples

- $bought(Frank, dvd)$
  "Frank bought a dvd."
- $\exists x.bought(Frank, x)$
  "Frank bought something."
- $\forall x.(bought(Frank, x) \rightarrow bought(Susan, x))$
  "Susan bought everything that Frank bought."
- $\forall x.bought(Frank, x) \rightarrow \forall x.bought(Susan, x)$
  "If Frank bought everything, so did Susan."
- $\forall x \exists y.bought(x, y)$
  "Everyone bought something."
- $\exists x \forall y.bought(x, y)$
  "Someone bought everything."

FOL: Intuitive Meaning

Example

Which of the following formulas is a formalization of the sentence:
"There is a computer which is not used by any student"

- $\exists x.(Computer(x) \wedge \forall y.(\neg Student(y) \wedge \neg Uses(y,x)))$
- $\exists x.(Computer(x) \rightarrow \forall y.(Student(y) \rightarrow \neg Uses(y,x)))$
- $\exists x.(Computer(x) \wedge \forall y.(Student(y) \rightarrow \neg Uses(y,x)))$

Formalizing English Sentences in FOL

Examples

- All Students are smart.
  $\forall x.(Student(x) \rightarrow Smart(x))$
- There exists a student.
  $\exists x.Student(x)$
- There exists a smart student
  $\exists x.(Student(x) \wedge Smart(x))$
- Every student loves some student
  $\forall x.(Student(x) \rightarrow \exists y.(Student(y) \wedge Loves(x, y)))$
- Every student loves some other student.
  $\forall x.(Student(x) \rightarrow \exists y.(Student(y) \wedge \neg(x = y) \wedge Loves(x, y)))$

Formalizing English Sentences in FOL

Examples

- There is a student who is loved by every other student.
  $\exists x.(Student(x) \wedge \forall y.(Student(y) \wedge \neg(x = y) \rightarrow Loves(y, x)))$
- Bill is a student.
  $Student(Bill)$
- Bill takes either Analysis or Geometry (but not both).
  $Takes(Bill, Analysis) \leftrightarrow \neg Takes(Bill, Geometry)$
- Bill takes Analysis and Geometry.
  $Takes(Bill, Analysis) \wedge Takes(Bill, Geometry)$
- Bill doesn't take Analysis.
  $\neg Takes(Bill, Analysis)$

Formalizing English Sentences in FOL

Examples

- No students love Bill.
  $\neg\exists x.(Student(x) \wedge Loves(x, Bill))$
- Bill has at least one sister.
  $\exists x.SisterOf(x, Bill)$
- Bill has no sister.
  $\neg\exists x.SisterOf(x, Bill)$
- Bill has at most one sister.
  $\forall x \forall y.(SisterOf(x, Bill) \wedge SisterOf(y, Bill) \rightarrow x = y)$
- Bill has (exactly) one sister.
  $\exists x.(SisterOf(x, Bill) \wedge \forall y.(SisterOf(y, Bill) \rightarrow x = y))$
- Bill has at least two sisters.
  $\exists x \exists y.(SisterOf(x, Bill) \wedge SisterOf(y, Bill) \wedge \neg(x = y))$

Formalizing English Sentences in FOL

Examples

- Every student takes at least one course.
  $\forall x.(Student(x) \rightarrow \exists y.(Course(y) \land Takes(x, y)))$
- Only one student failed Geometry.
  $\exists x.(Student(x) \land Failed(x, Geometry) \land \forall y.(Student(y) \land Failed(y, Geometry) \rightarrow x = y))$
- No student failed Geometry but at least one student failed Analysis.
  $\neg\exists x.(Student(x) \land Failed(x, Geometry)) \land \exists x.(Student(x) \land Failed(x, Analysis))$
- Every student who takes Analysis also takes Geometry.
  $\forall x.(Student(x) \land Takes(x, Analysis) \rightarrow Takes(x, Geometry))$

Homework

Formalise sentences 1–4 in FOL and write the intuitive meaning of formulae 5 and 6.

1. Every Man is Mortal
   $\forall x.Man(x) \supset Mortal(x)$
2. Every Dog has a Tail
   $\forall x.Dog(x) \supset \exists y(PartOf(x,y) \wedge Tail(y))$
3. There are two dogs
   $\exists x,y(Dog(x) \wedge Dog(y) \wedge x \neq y)$
4. Not every dog is white
   $\neg\forall x.Dog(x) \supset White(x)$
5. $\exists x.Dog(x) \wedge \exists y.Dog(y)$
   There is a dog
6. $\forall x,y(Dog(x) \wedge Dog(y) \supset x = y)$
   There is at most one dog

Homework

Define an appropriate language and formalize the following sentences in FOL:

- someone likes Mary.
- nobody likes Mary.
- nobody loves Bob but Bob loves Mary.
- if David loves someone, then he loves Mary.
- if someone loves David, then he (someone) loves also Mary.
- everybody loves David or Mary.

Homework

Define an appropriate language and formalize the following sentences in FOL:

- there is at least one person who loves Mary.
- there is at most one person who loves Mary.
- there is exactly one person who loves Mary.
- there are exactly two persons who love Mary.
- if Bob loves everyone that Mary loves, and Bob loves David, then Mary doesn't love David.
- Only Mary loves Bob.

Formalizing English Sentences in FOL

Example

Define an appropriate language and formalize the following sentences in FOL:

- "A is above C, D is on E and above F."
- "A is green while C is not."
- "Everything is on something."
- "Everything that has nothing on it, is free."
- "Everything that is green is free."
- "There is something that is red and is not free."
- "Everything that is not green and is above B, is red."

Formalizing English Sentences in FOL

Non Logical symbols

Constants: A, B, C, D, E, F;
Predicates: $On^2, Above^2, Free^1, Red^1, Green^1$.

Example

- "A is above C, D is above F and on E."
  $\phi_1 : Above(A, C) \wedge Above(E, F) \wedge On(D, E)$
- "A is green while C is not."
  $\phi_2 : Green(A) \wedge \neg Green(C)$
- "Everything is on something."
  $\phi_3 : \forall x \exists y . On(x, y)$
- "Everything that has nothing on it, is free."
  $\phi_4 : \forall x . (\neg \exists y . On(y, x) \rightarrow Free(x))$

Formalizing English Sentences in FOL

Non Logical symbols

Constants: A, B, C, D, E, F;
Predicates: $On^2, Above^2, Free^1, Red^1, Green^1$.

Example

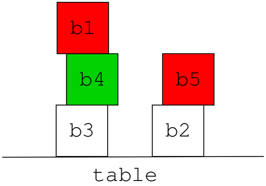
- "Everything that is green is free."
  $\phi_5 : \forall x.(Green(x) \rightarrow Free(x))$
- "There is something that is red and is not free."
  $\phi_6 : \exists x.(Red(x) \wedge \neg Free(x))$
- "Everything that is not green and is above B, is red."
  $\phi_7 : \forall x.(\neg Green(x) \wedge Above(x, B) \rightarrow Red(x))$

An interpretation $\mathcal{I}_1$ in the Blocks World

Non Logical symbols

Constants: A, B, C, D, E, F;

Predicates: $On^2, Above^2, Free^1, Red^1, Green^1$.



Interpretation $\mathcal{I}_1$

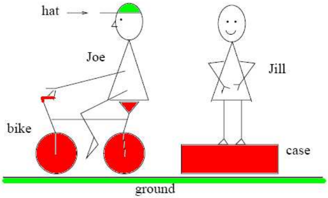
- $\mathcal{I}_1(A) = b_1$, $\mathcal{I}_1(B) = b_2$, $\mathcal{I}_1(C) = b_3$, $\mathcal{I}_1(D) = b_4$, $\mathcal{I}_1(E) = b_5$, $\mathcal{I}_1(F) = table$
- $\mathcal{I}_1(On) = \{\langle b_1, b_4\rangle, \langle b_4, b_3\rangle, \langle b_3, table\rangle, \langle b_5, b_2\rangle, \langle b_2, table\rangle\}$
- $\mathcal{I}_1(Above) = \{\langle b_1, b_4\rangle, \langle b_1, b_3\rangle, \langle b_1, table\rangle, \langle b_4, b_3\rangle, \langle b_4, table\rangle, \langle b_3, table\rangle, \langle b_5, b_2\rangle, \langle b_5, table\rangle, \langle b_2, table\rangle\}$
- $\mathcal{I}_1(Free) = \{\langle b_1\rangle, \langle b_5\rangle\}$, $\mathcal{I}_1(Green) = \{\langle b_4\rangle\}$, $\mathcal{I}_1(Red) = \{\langle b_1\rangle, \langle b_5\rangle\}$

A different interpretation $\mathcal{I}_2$

Non Logical symbols

Constants: A, B, C, D, E, F;

Predicates: $On^2, Above^2, Free^1, Red^1, Green^1$.



Interpretation $\mathcal{I}_2$

- $\mathcal{I}_2(A) = hat$, $\mathcal{I}_2(B) = Joe$, $\mathcal{I}_2(C) = bike$, $\mathcal{I}_2(D) = Jill$, $\mathcal{I}_2(E) = case$, $\mathcal{I}_2(F) = ground$
- $\mathcal{I}_2(On) = \{\langle hat, Joe\rangle, \langle Joe, bike\rangle, \langle bike, ground\rangle, \langle Jill, case\rangle, \langle case, ground\rangle\}$
- $\mathcal{I}_2(Above) = \{\langle hat, Joe\rangle, \langle hat, bike\rangle, \langle hat, ground\rangle, \langle Joe, bike\rangle, \langle Joe, ground\rangle, \langle bike, ground\rangle, \langle Jill, case\rangle, \langle Jill, ground\rangle, \langle case, ground\rangle\}$
- $\mathcal{I}_2(Free) = \{\langle hat\rangle, \langle Jill\rangle\}$, $\mathcal{I}_2(Green) = \{\langle hat\rangle, \langle ground\rangle\}$, $\mathcal{I}_2(Red) = \{\langle bike\rangle, \langle case\rangle\}$

FOL Satisfiability

Example

For each of the following formulas, decide whether they are satisfied by $\mathcal{I}_1$ and/or $\mathcal{I}_2$:

- $\phi_1 : Above(A, C) \wedge Above(E, F) \wedge On(D, E)$
- $\phi_2 : Green(A) \wedge \neg Green(C)$
- $\phi_3 : \forall x \exists y . On(x, y)$
- $\phi_4 : \forall x . (\neg \exists y . On(y, x) \rightarrow Free(x))$
- $\phi_5 : \forall x . (Green(x) \rightarrow Free(x))$
- $\phi_6 : \exists x . (Red(x) \wedge \neg Free(x))$
- $\phi_7 : \forall x . (\neg Green(x) \wedge Above(x, B) \rightarrow Red(x))$

Sol.

- $\mathcal{I}_1 \models \neg\phi_1 \wedge \neg\phi_2 \wedge \neg\phi_3 \wedge \phi_4 \wedge \neg\phi_5 \wedge \neg\phi_6 \wedge \phi_7$
- $\mathcal{I}_2 \models \phi_1 \wedge \phi_2 \wedge \neg\phi_3 \wedge \phi_4 \wedge \neg\phi_5 \wedge \phi_6 \wedge \phi_7$

FOL Satisfiability

Example

Consider the following sentences:

- (1) All actors and journalists invited to the party are late.
- (2) There is at least a person who is on time.
- (3) There is at least an invited person who is neither a journalist nor an actor.

Formalize the sentences and prove that (3) is not a logical consequence of (1) and (2)

FOL Satisfiability

Example

Consider the following sentences:

- All actors and journalists invited to the party are late.
  (1) $\forall x.((a(x) \lor j(x)) \land i(x) \rightarrow l(x))$
- There is at least a person who is on time.
  (2) $\exists x.\neg l(x)$
- There is at least an invited person who is neither a journalist nor an actor.
  (3) $\exists x.(i(x) \land \neg a(x) \land \neg j(x))$

It's sufficient to find an interpretation $\mathcal{I}$ for which the logical consequence does not hold:

	l(x)	a(x)	j(x)	i(x)
Bob	F	T	F	F
Tom	T	T	F	T
Mary	T	F	T	T

Homework

Let $\Delta = \{1, 3, 5, 15\}$ and $\mathcal{I}$ be an interpretation on Δ interpreting the predicate symbols E^1 as 'being even', M^2 as 'being a multiple of' and L^2 as 'being less then', and s.t. $\mathcal{I}(a) = 1, \mathcal{I}(b) = 3, \mathcal{I}(c) = 5, \mathcal{I}(d) = 15$.
Determine whether $\mathcal{I}$ satisfies the following formulas:

$\exists y.E(y) \qquad \forall x.\neg E(x) \qquad \forall x.M(x, a) \qquad \forall x.M(x, b) \qquad \exists x.M(x, d)$

$\exists x.L(x, a) \qquad \forall x.(E(x) \to M(x, a)) \qquad \forall x \exists y.L(x, y) \qquad \forall x \exists y.M(x, y)$

$\forall x.(M(x, b) \to L(x, c)) \qquad \forall x \forall y.(L(x, y) \to \neg L(y, x))$

$\forall x.(M(x, c) \vee L(x, c))$