Logics for Data and Knowledge Representation

1. Introduction to First order logic

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Outline

- Why First Order Logic (FOL)?
- Syntax and Semantics of FOL;
- Examples of First Order Theories;
- Reasoning in FOL:
  - general concepts;
  - Hilbert style axiomatization;
  - Natural deduction.

;

Expressivity of propositional logic - I

Question

Try to express in Propositional Logic the following statements:

- Mary is a person
- John is a person
- Mary is mortal
- Mary and John are siblings

Expressivity of propositional logic - I

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A solution

Through atomic propositions:

- `Mary-is-a-person`
- `John-is-a-person`
- `Mary-is-mortal`
- `Mary-and-John-are-siblings`

Problem with previous solution

- `Mary-is-a-person`
- `John-is-a-person`
- `Mary-is-mortal`
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How do we link `Mary` of the first sentence to `Mary` of the third sentence? And how we link `Mary` and `Mary-and-John`?

Expressivity of propositional logic - II

Question

Try to express in Propositional Logic the following statements:

- All persons are mortal;
- There is a person who is a spy.

Expressivity of propositional logic - II

Question

Try to express in Propositional Logic the following statements:

- All persons are mortal;
- There is a person who is a spy.

A solution

We can give all people a name and express this fact through atomic propositions:

- $\texttt{Mary-is-mortal} \wedge \texttt{John-is-mortal} \wedge \texttt{Chris-is-mortal} \wedge \ldots \wedge \texttt{Michael-is-mortal}$
- $\texttt{Mary-is-a-spy} \vee \texttt{John-is-a-spy} \vee \texttt{Chris-is-a-spy} \vee \ldots \vee \texttt{Michael-is-a-spy}$

Problem with previous solution

- `Mary-is-mortal` $\wedge$ `John-is-mortal` $\wedge$ `Chris-is-mortal` $\wedge \ldots \wedge$ `Michael-is-mortal`
- `Mary-is-a-spy` $\vee$`John-is-a-spy` $\vee$`Chris-is-a-spy` $\vee \ldots \vee$ `Michael-is-a-spy`

Problem with previous solution

- Mary-is-mortal $\wedge$ John-is-mortal $\wedge$ Chris-is-mortal $\wedge \ldots \wedge$ Michael-is-mortal
- Mary-is-a-spy $\vee$John-is-a-spy $\vee$Chris-is-a-spy $\vee \ldots \vee$ Michael-is-a-spy

The representation is not compact and generalization patterns are difficult to express.

Problem with previous solution

- Mary-is-mortal $\wedge$ John-is-mortal $\wedge$ Chris-is-mortal $\wedge \ldots \wedge$ Michael-is-mortal
- Mary-is-a-spy $\vee$John-is-a-spy $\vee$Chris-is-a-spy $\vee \ldots \vee$ Michael-is-a-spy

The representation is not compact and generalization patterns are difficult to express.
What is we do not know all the people in our "universe"? How can we express the statement independently from the people in the "universe"?

Expressivity of propositional logic - III

Question

Try to express in Propositional Logic the following statements:

- Every natural number is either even or odd

Expressivity of propositional logic - III

Question

Try to express in Propositional Logic the following statements:

- Every natural number is either even or odd

A solution

We can use two families of propositions $even_i$ and odd_i for every $i \geq 1$, and use the set of formulas

$$\{odd_i \vee even_i | i \geq 1\}$$

Problem with previous solution

$$\{odd_i \vee even_i | i \geq 1\}$$

What happens if we want to state this in one single formula? To do this we would need to write an infinite formula like:

$$(odd_1 \vee even_1) \wedge (odd_2 \vee even_2) \wedge \ldots$$

and this cannot be done in propositional logic.

Expressivity of propositional logic -IV

Question

Express the statements:

- the father of Luca is Italian

Solution (Partial)

- $\texttt{mario-is-father-of-luca} \supset \texttt{mario-is-italian}$
- $\texttt{michele-is-father-of-luca} \supset \texttt{michele-is-italian}$
- ...

Problem with previous solution

- `mario-is-father-of-luca` $\supset$ `mario-is-italian`
- `michele-is-father-of-luca` $\supset$ `michele-is-italian`
- ...

This statement strictly depend from a fixed set of people. What happens if we want to make this statement independently of the set of persons we have in our universe?

Why first order logic?

Because it provides a way of representing information like the following one:

1. Mary is a person;
2. John is a person;
3. Mary is mortal;
4. Mary and John are siblings
5. Every person is mortal;
6. There is a person who is a spy;
7. Every natural number is either even or odd;
8. The father of Luca is Italian

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and also to infer the third one from the first one and the fifth one.

First order logic

Whereas propositional logic assumes world contains facts, first-order logic (like natural language) assumes the world contains:

- Constants: mary, john, 1, 2, 3, red, blue, world war 1, world war 2, 18th Century...
- Predicates: Mortal, Round, Prime, Brother of, Bigger than, Inside, Part of, Has color, Occurred after, Owns, Comes between, ...
- Functions: Father of, Best friend, Third inning of, One more than, End of, ...

Constants and Predicates

- Mary is a person
- John is a person
- Mary is mortal
- Mary and John are siblings

In FOL it is possible to build an atomic propositions by applying a predicate to constants

- $Person(mary)$
- $Person(john)$
- $Mortal(mary)$
- $Siblings(mary, john)$

Quantifiers and variables

- Every person is mortal;
- There is a person who is a spy;
- Every natural number is either even or odd;

In FOL it is possible to build propositions by applying universal (existential) quantifiers to variables. This allows to quantify to arbitrary objects of the universe.

- $\forall x.Person(x) \supset Mortal(x)$;
- $\exists x.Person(x) \supset Spy(x)$;
- $\forall x.(Odd(x) \vee Even(x))$

Functions

- The father of Luca is Italian.

In FOL it is possible to build propositions by applying a function to a constant, and then a predicate to the resulting object.

- $Italian(fatherOf(Mario))$

Syntax of FOL

Logical symbols

- the logical constant $\perp$
- propositional logical connectives $\wedge$, $\vee$, $\supset$, $\neg$, $\equiv$
- the quantifiers $\forall$, $\exists$
- the set of variable symbols $x_1, x_2, \dots$
- the equality symbol $=$. (optional)

Non Logical symbols

- a set $c_1, c_2, \dots$ of constant symbols
- a set $f_1, f_2, \dots$ of functional symbols each of which is associated with its *arity* (i.e., number of arguments)
- a set $P_1, P_2, \dots$ of *relational symbols* each of which is associated with its *arity* (i.e., number of arguments)

Terms and formulas of FOL

Terms

- every constant c_i and every variable x_i is a term;
- if $t_1, \ldots, t_n$ are terms and f_i is a functional symbol of arity equal to n, then $f(t_1, \ldots, t_n)$ is a term

Well formed formulas

- if t_1 and t_2 are terms then $t_1 = t_2$ is a formula
- If $t_1, \ldots, t_n$ are terms and P_i is relational symbol of arity equal to n, then $P_i(t_1, \ldots, t_n)$ is formula
- if A and B are formulas then $\perp$, $A \wedge B$, $A \supset B$, $A \vee B$ $\neg A$ are formulas
- if A is a formula and x a variable, then $\forall x.A$ and $\exists x.A$ are formulas.

Examples of terms and formulas

Example (Terms)

- x_i,
- c_i,
- $f_i(x_j, c_k)$, and
- $f(g(x, y), h(x, y, z), y)$

Example (formulas)

- $f(a, b) = c$,
- $P(c_1)$,
- $\exists x(A(x) \vee B(y))$, and
- $P(x) \supset \exists y.Q(x, y)$.

An example of representation in FOL

Example (Language)

constants	functions (arity)	Predicate (arity)
Aldo	mark (2)	attend (2)
Bruno	best-friend (1)	friend (2)
Carlo		student (1)
MathLogic		course (1)
DataBase		less-than (2)
0, 1, . . . , 10		

Example (Terms)

Intuitive meaning	term
an individual named Aldo	Aldo
the mark 1	1
Bruno's best friend	best-friend(Bruno)
anything	x
Bruno's mark in MathLogic	mark(Bruno,MathLogic)
somebody's mark in DataBase	mark(x,DataBase)
Bruno's best friend mark in MathLogic	mark(best-friend(Aldo),MathLogic)

An example of representation in FOL (cont'd)

Example (Formulas)

Intuitive meaning	Formula
Bob and Roberto are the same person	$Bob = Roberto$
Carlo is a person and MathLogic is a course	$person(Carlo) \wedge course(MathLogic)$
Aldo attends MathLogic	$attend(Aldo, MathLogic)$
Courses are attended only by students	$\forall x(attend(x, y) \supset course(y) \supset student(x))$
every course is attended by somebody	$\forall x(course(x) \supset \exists y\ attend(y, x))$
every student attends a course	$\forall x(student(x) \supset \exists y\ attend(x, y))$
a student who attends all the courses	$\exists x(student(x) \wedge \forall y(course(y) \supset attend(x, y)))$
a course has at least two attenders	$\forall x(course(x) \supset \exists y \exists z$ $(attend(y, x) \wedge attend(z, x) \wedge \neg y = z))$
Aldo's best friend attend the same courses attended by Aldo	$\forall x(attend(Aldo, x) \supset$ $attend(best\text{-}friend(Aldo), x))$
best-friend is symmetric	$\forall x(best\text{-}friend(best\text{-}friend(x)) = x)$
Aldo and his best friend have the same mark in MathLogic	$mark(\text{best-friend}(Aldo), MathLogic) =$ $mark(Aldo, MathLogic)$
A student can attend at most two courses	$\forall x \forall y \forall z \forall w(attend(x, y) \wedge attend(x, z) \wedge$ $attend(x, w) \supset (y = z \vee z = w \vee y = w))$

Common Mistakes

- Use of $\wedge$ with $\forall$

  $\forall x At(FBK, x) \wedge Smart(x)$

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- Use of $\supset$ with $\exists$

  $\exists x At(FBK, x) \supset Smart(x)$ is true if there is an x who is not at FBK

Common Mistakes

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  $\forall x At(FBK, x) \wedge Smart(x)$ means "Everyone is at FBK and everyone is smart"

  "Everyone at FBK is smart" is formalized as
  $\forall x At(FBK, x) \supset Smart(x)$

- Use of $\supset$ with $\exists$

  $\exists x At(FBK, x) \supset Smart(x)$ is true if there is an x who is not at FBK

  "There is an FBK smart person" is formalized as
  $\exists x At(FBK, x) \wedge Smart(x)$

Representing variations of quantifiers in FOL

Example

Represent the statement at most 2 students attend the KR course

$$\forall x_1 \forall x_2 \forall x_3 (attend(x_1, KR) \wedge attend(x_2, KR) \wedge attend(x_2, KR) \supset x_1 = x_3 \vee x_2 = x_3 \vee x_1 = x_3)$$

At most n ...

$$\forall x_1 \ldots x_{n+1} \left(\bigwedge_{i=1}^{n+1} \phi(x_i) \supset \bigvee_{i \neq j=1}^{n+1} x_i = x_j \right)$$

Representing variations quantifiers in FOL

Example

Represent the statement at least 2 students attend the KR course

$$\exists x_1 \exists x_2 (attend(x_1, KR) \wedge attend(x_2, KR) \wedge x_1 \neq x_3)$$

At least n ...

$$\exists x_1 \ldots x_n \left(\bigwedge_{i=1}^{n} \phi(x_i) \wedge \bigwedge_{i \neq j = 1}^{n} x_i \neq x_j \right)$$

Semantics of FOL

FOL interpretation for a language L

A first order interpretation for the language $L = \langle c_1, c_2, \ldots, f_1, f_2, \ldots, P_1, P_2, \ldots \rangle$ is a pair $\langle \Delta, \mathcal{I} \rangle$ where

- Δ is a non empty set called interpretation domain
- $\mathcal{I}$ is is a function, called interpretation function
    - $\mathcal{I}(c_i) \in \Delta$ (elements of the domain)
    - $\mathcal{I}(f_i) : \Delta^n \to \Delta$ (n-ary function on the domain)
    - $\mathcal{I}(P_i) \subseteq \Delta^n$ (n-ary relation on the domain)

where n is the arity of f_i and P_i.

Example of interpretation

Example (Of interpretation)

Symbols

Constants: *alice*, *bob*, *carol*, *robert*
Function: *mother-of* (with arity equal to 1)
Predicate: *friends* (with arity equal to 2)

Domain

$\Delta = \{1, 2, 3, 4, \dots\}$

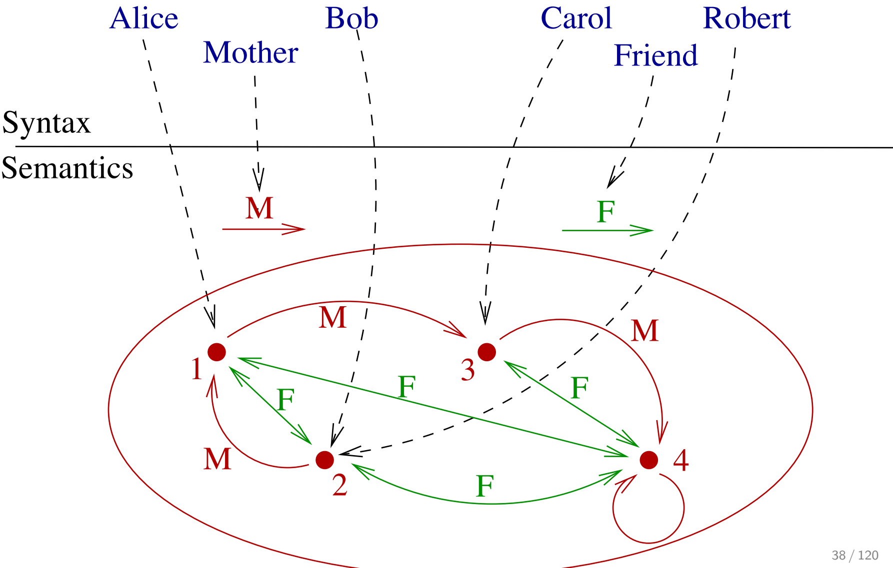
Interpretation

$\mathcal{I}(alice) = 1$, $\mathcal{I}(bob) = 2$, $\mathcal{I}(carol) = 3$,
$\mathcal{I}(robert) = 2$

$$\mathcal{I}(mother\text{-}of) = M \quad \begin{array}{l} M(1) = 3 \\ M(2) = 1 \\ M(3) = 4 \\ M(n) = n + 1 \text{ for } n \geq 4 \end{array}$$

$$\mathcal{I}(friends) = F = \left\{ \begin{array}{lll} \langle 1, 2 \rangle, & \langle 2, 1 \rangle, & \langle 3, 4 \rangle, \\ \langle 4, 3 \rangle, & \langle 4, 2 \rangle, & \langle 2, 4 \rangle, \\ \langle 4, 1 \rangle, & \langle 1, 4 \rangle, \langle 4, 4 \rangle & \end{array} \right\}$$

Example (cont'd)



Interpretation of terms

Definition (Assignment)

An assignment a is a function from the set of variables to Δ.

$a[x/d]$ denotes the assignment that coincides with a on all the variables but x, which is associated to d.

Definition

Interpretation of terms The interpretation of a term t w.r.t. the assignment a, in symbols $\mathcal{I}(t)[a]$ is recursively defined as follows:

$$\begin{aligned}
\mathcal{I}(x_i)[a] &= a(x_i) \\
\mathcal{I}(c_i)[a] &= \mathcal{I}(c_i) \\
\mathcal{I}(f(t_1, \ldots, t_n))[a] &= \mathcal{I}(f)(\mathcal{I}(t_1)[a], \ldots, \mathcal{I}(t_n)[a])
\end{aligned}$$

FOL Satisfiability of formulas

Definition (Satisfiability of a formula w.r.t. an assignment)

An interpretation $\mathcal{I}$ satisfies a formula ϕ w.r.t. the assignment a according to the following rules:

$$
\begin{array}{rcl}
\mathcal{I} \models t_1 = t_2[a] & \text{iff} & \mathcal{I}(t_1)[a] = \mathcal{I}(t_2)[a] \\
\mathcal{I} \models P(t_1, \dots, t_n)[a] & \text{iff} & \langle \mathcal{I}(t_1)[a], \dots, \mathcal{I}(t_n)[a] \rangle \in \mathcal{I}(P) \\
\mathcal{I} \models \phi \wedge \psi[a] & \text{iff} & \mathcal{I} \models \phi[a] \text{ and } \mathcal{I} \models \psi[a] \\
\mathcal{I} \models \phi \vee \psi[a] & \text{iff} & \mathcal{I} \models \phi[a] \text{ or } \mathcal{I} \models \psi[a] \\
\mathcal{I} \models \phi \supset \psi[a] & \text{iff} & \mathcal{I} \not\models \phi[a] \text{ or } \mathcal{I} \models \psi[a] \\
\mathcal{I} \models \neg\phi[a] & \text{iff} & \mathcal{I} \not\models \phi[a] \\
\mathcal{I} \models \phi \equiv \psi[a] & \text{iff} & \mathcal{I} \models \phi[a] \text{ iff } \mathcal{I} \models \psi[a] \\
\mathcal{I} \models \exists x \phi[a] & \text{iff} & \text{there is a } d \in \Delta \text{ such that} \mathcal{I} \models \phi[a[x/d]] \\
\mathcal{I} \models \forall x \phi[a] & \text{iff} & \text{for all } d \in \Delta, \mathcal{I} \models \phi[a[x/d]]
\end{array}
$$

Example (cont'd)

Exercise

Check the satisfiability of the following statements, considering the interpretation defined few slides ago:

1. $\mathcal{I} \models Alice = Bob[a]$
2. $\mathcal{I} \models Robert = Bob[a]$
3. $\mathcal{I} \models x = Bob[a[x/2]$

Example (cont.)

$$\mathcal{I}(\textit{mother-of}(\textit{alice}))[a] = 3$$

$$\mathcal{I}(\textit{mother-of}(x))[a[x/4]] = 5$$

$\mathcal{I}(\textit{friends}(x, y)) =$

$x :=$	$y :=$
1	2
2	1
4	1
1	4
4	2
2	4
4	3
3	4
4	4

$\mathcal{I}(\textit{friends}(x, x)) =$

$x :=$
4

$\mathcal{I}(\textit{friends}(x, y) \wedge x = y) =$

$x :=$	$y :=$
4	4

$\mathcal{I}(\exists x \textit{friends}(x, y)) =$

$y :=$
2
1
4
3

$\mathcal{I}(\forall x \textit{friends}(x, y)) =$

$y :=$
4

Free variable and free terms

Intuition

A free occurrence of a variable x is an occurrence of x which is not bounded by a (universal or existential) quantifier.

Definition (Free occurrence)

- any occurrence of x in t_k is free in $P(t_1, \ldots, t_k, \ldots, t_n)$
- any free occurrence of x in ϕ or in ψ is also fee in $\phi \wedge \psi$, $\psi \vee \phi$, $\psi \supset \phi$, and $\neg\phi$
- any free occurrence of x in ϕ, is free in $\forall y.\phi$ and $\exists y.\phi$ if y is distinct from x.

Definition (Ground/Closed Formula)

A formula ϕ is ground or closed if it does not contain free occurrences of variables.

Free variable and free terms

A variable x is free in ϕ (denote by $\phi(x)$) if there is at least a free occurrence of x in ϕ.
Free variables represents individuals which must be instantiated to make the formula a meaningful proposition.

- x is free in $friends(alice, x)$.
- x is free in $P(x) \supset \forall x.Q(x)$ (the occurrence of x in red is free the one in green is not free.

Free variable and free terms - example

Definition (Term free for a variable)

A term is free for x in ϕ, if all the occurrences of x in ϕ are not in the scope of a quantifier for a variable occurring in t.

An occurrence of a variable x can be safely instantiated by a term free for x in a formula ϕ,
If you replace x with a terms which is not free for x in ϕ, you can have unexpected effects:
E.g., replacing x with $\mathit{mother\text{-}of}(y)$ in the formula $\exists y.\mathit{friends}(x, y)$ you obtain the formula

$$\exists y.\mathit{friends}(\mathit{mother\text{-}of}(y), y)$$

Satisfiability and Validity

Definition (Model, satisfiability and validity)

An interpretation $\mathcal{I}$ is a model of ϕ under the assignment a, if

$$\mathcal{I} \models \phi[a]$$

A formula ϕ is satisfiable if there is some $\mathcal{I}$ and some assignment a such that $\mathcal{I} \models \phi[a]$.
A formula ϕ is unsatisfiable if it is not satisfiable.
A formula ϕ is valid if every $\mathcal{I}$ and every assignment a $\mathcal{I} \models \phi[a]$

Definition (Logical Consequence)

A formula ϕ is a logical consequence of a set of formulas Γ, in symbols $\Gamma \models \phi$, if for all interpretations $\mathcal{I}$ and for all assignment a

$$\mathcal{I} \models \Gamma[a] \quad \Longrightarrow \quad \mathcal{I} \models \phi[a]$$

where $\mathcal{I} \models \Gamma[a]$ means that $\mathcal{I}$ satisfies all the formulas in Γ under a.

Note: Validity of ϕ can be defined in terms of logical consequence as

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Note: Validity of ϕ can be defined in terms of logical consequence as $\emptyset \models \phi$

Logical Consequence and reasoning

The notion of logical consequence enables us to determine if “Mary is mortal” is a consequence of the facts that “Mary is a person” and “All persons are mortal”.
What we need to do is to determine if

$$Person(mary), \forall x Person(x) \supset Mortal(x) \models Mortal(mary)$$

We'll come back to this in the next lecture.

Excercises

Say where these formulas are valid, satisfiable, or unsatisfiable

- $\forall x P(x)$
- $\forall x P(x) \supset \exists y P(y)$
- $\forall x . \forall y . (P(x) \supset P(y))$
- $P(x) \supset \exists y P(y)$
- $P(x) \vee \neg P(y)$
- $P(x) \wedge \neg P(y)$
- $P(x) \supset \forall x . P(x)$
- $\forall x \exists y . Q(x, y) \supset \exists y \forall x Q(x, y)$
- $x = x$
- $\forall x . P(x) \equiv \forall y . P(y)$
- $x = y \supset \forall x . P(x) \equiv \forall y . P(y)$
- $x = y \supset (P(x) \equiv P(y))$
- $P(x) \equiv P(y) \supset x = y$

Properties of quantifiers

Proposition

The following formulas are valid

- $\forall x(\phi(x) \wedge \psi(x)) \equiv \forall x \phi(x) \wedge \forall x \psi(x)$
- $\exists x(\phi(x) \vee \psi(x)) \equiv \exists x \phi(x) \vee \exists x \psi(x)$
- $\forall x \phi(x) \equiv \neg \exists x \neg \phi(x)$
- $\forall x \exists x \phi(x) \equiv \exists x \phi(x)$
- $\exists x \forall x \phi(x) \equiv \forall x \phi(x)$

Proposition

The following formulas are not valid

- $\forall x(\phi(x) \vee \psi(x)) \equiv \forall x \phi(x) \vee \forall x \psi(x)$
- $\exists x(\phi(x) \wedge \psi(x)) \equiv \exists x \phi(x) \wedge \exists x \psi(x)$
- $\forall x \phi(x) \equiv \exists x \phi(x)$
- $\forall x \exists y \phi(x, y) \equiv \exists y \forall x \phi(x, y)$

Expressing properties in FOL

For each property write a formula expressing the property, and for each formula writhe the property it formalises.

- Every Man is Mortal
- Every Dog has a Tail
- There are two dogs
- Not every dog is white
- $\exists x.Dog(x) \wedge \exists y.Dog(y)$
- $\forall x, y(Dog(x) \wedge Dog(y) \supset x = y)$

Expressing properties in FOL

For each property write a formula expressing the property, and for each formula writhe the property it formalises.

- Every Man is Mortal
  $\forall x.Man(x) \supset Mortal(x)$
- Every Dog has a Tail
  $\forall x.Dog(x) \supset \exists y(PartOf(x, y) \wedge Tail(y))$
- There are two dogs
  $\exists x, y(Dog(x) \wedge Dog(y) \wedge x \neq y)$
- Not every dog is white
  $\neg\forall x.Dog(x) \supset White(x)$
- $\exists x.Dog(x) \wedge \exists y.Dog(y)$
  There is a dog
- $\forall x, y(Dog(x) \wedge Dog(y) \supset x = y)$
  There is at most one dog

Open and Closed Formulas

- Note that for closed formulas, satisfiability, validity and logical consequence do not depend on the assignment of variables.
- For closed formulas, we therefore omit the assignment and write $\mathcal{I} \models \phi$.
- More in general $\mathcal{I} \models \phi[a]$ if and only if $\mathcal{I} \models \phi[a']$ when $[a]$ and $[a']$ coincide on the variables free in ϕ (they can differ on all the others)

First order theories

- Mathematics focuses on the study of properties of certain structures. E.g. Natural/Rational/Real/Complex numbers, Algebras, Monoids, Lattices, Partially-ordered sets, Topological spaces, fields, ...
- In knowledge representation, mathematical structures can be used as a reference abstract model for a real world feature. e.g.,
  - natural/rational/real numbers can be used to represent linear time;
  - trees can be used to represent possible future evolutions;
  - graphs can be used to represent maps;
  - ...
- Logics provides a rigorous way to describe certain classes of mathematical structures.

First order theory

Definition (First order theory)

A first order theory is a set of formulas of the FOL language closed under the logical consequence relation. That is, T is a theory iff $T \models A$ implies that $A \in T$

Remark

A FOL theory always contains an infinite set of formulas. Indeed any theory T contains at least all the valid formulas (which are infinite).

Definition (Set of axioms for a theory)

A set of formulas Ω is a set of axioms for a theory T if for all $\phi \in T$, $\Omega \models \phi$.

First order theory (cont'd)

Definition

Finitely axiomatizable theory A theory T is finitely axiomatizable if it has a finite set of axioms.

Definition (Axiomatizable structure)

Given a class of mathematical structures C for a language L, we say that a theory T is a sound and complete axiomatization of C if and only if

$$T \models \phi \quad \Longleftrightarrow \quad \mathcal{I} \models \phi \text{ for all } \mathcal{I} \in C$$

Examples of first order theories

Number theory (or Peano Arithmetic) PA $\mathcal{L}$ contains the constant symbol 0, the 1-nary function symbol s, (for successor) and two 2-nary function symbol $+$ and $*$

1. $0 \neq s(x)$
2. $s(x) = s(y) \supset x = y$
3. $x + 0 = x$
4. $x + s(y) = s(x + y)$
5. $x * 0 = 0$
6. $x * s(y) = (x * y) + x$
7. the Induction axiom schema: $\phi(0) \wedge \forall x.(\phi(x) \supset \phi(s(x))) \supset \forall x.\phi(x)$, for every formula $\phi(x)$ with at least one free variable

K. Gödel 1931 It's false that $\mathcal{I} \models PA$ if and only if $\mathcal{I}$ is isomorphic to the standard models for natural numbers.

Logical Consequence and reasoning

The notion of logical consequence enables us to determine if "Mary is mortal" is a consequence of the facts that "Mary is a person" and "All persons are mortal".
What we need to do is to determine if

$$Person(mary), \forall x Person(x) \supset Mortal(x) \models Mortal(mary)$$

Goal of this part: Understand how we determine this.

Deciding logical consequence

Problem

Is there an algorithm to determine whether a formula ϕ is the logical consequence of a set of formulas Γ?

Naïve solution

- Apply directly the definition of logical consequence. That is:
    - build all the possible interpretations $\mathcal{I}$;
    - determine for which interpretations $\mathcal{I} \models \Gamma$;
    - for those interpretations check if $\mathcal{I} \models A$
- This solution can be used when Γ is finite, and there is a finite number of relevant interpretations.

Deciding logical consequence, is not always possible

Propositional Logics

The truth table method enumerates all the possible interpretations of a formula and, for each formula, it computes the relation $\models$.

Other logics

For first order logic There no general algorithm to compute the logical consequence. This because there may be an infinite number of relevant interpretations. There are some algorithms computing the logical consequence for sub-languages of first order logic (e.g., the set of formulas you can build using only two variables) and for sub-classes of structures (as you will see further on).

The Naïve solution in Propositional logic

Exercise (Logical consequence via truth table)

Determine, Via truth table, if the following statements about logical consequence holds

- $p \models q$
- $p \supset q \models q \supset p$
- $p, \neg q \supset \neg p \models q$
- $\neg q \supset \neg p \models p \supset q$

Complexity of the propositional logical consequence problem

The truth table method is Exponential

The problem of determining if a formula A containing n primitive propositions, is a logical consequence of the empty set, i.e., the problem of determining if A is valid, ($\models A$), takes an n-exponential number of steps. To check if A is a tautology, we have to consider 2^n interpretations in the truth table, corresponding to 2^n lines.

More efficient algorithms?

Are there more efficient algorithms? That is, is it possible to define an algorithm which takes a polinomial number of steps in n, to determine the validity of A? This is an unsolved problem

$P \stackrel{?}{=} NP$

The existence of a polinomial algorithm for checking validity is still an open problem, even it there are a lot of evidences in favor of non-existence

The Inference approach

- Instead of building all possible interpretations of Γ and check whether $\Gamma \models \phi$, try to obtain ϕ from Γ using axioms and reasoning rules.

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- Here Hilbert style and Natural Deduction style inference rules.

Hilbert axiomatization for propositional logic

Axioms

A1 $\phi \supset (\psi \supset \phi)$

A2 $(\phi \supset (\psi \supset \theta)) \supset ((\phi \supset \psi) \supset (\phi \supset \theta))$

A3 $(\neg\psi \supset \neg\phi) \supset ((\neg\psi \supset \phi) \supset \psi)$

Inference rule(s)

MP $\dfrac{\phi \quad \phi \supset \psi}{\psi}$

Why there are no axioms for $\wedge$ and $\vee$ and $\equiv$?

The connectives $\wedge$ and $\vee$ are rewritten into equivalent formulas containing only $\supset$ and $\neg$.

$$
\begin{array}{rcl}
A \wedge B & \equiv & \neg(A \supset \neg B) \\
A \vee B & \equiv & \neg A \supset B \\
A \equiv B & \equiv & \neg((A \supset B) \supset \neg(B \supset A))
\end{array}
$$

Hilbert axiomatization for FOL

Add to the axioms and rules for propositional logic the following:

Axioms and rules for quantifiers

A4 $\forall x.\phi(x) \supset \phi(t)$ if t is free for x in $\phi(x)$

A5 $\forall x.(\phi \supset \psi) \supset (\phi \supset \forall x.\psi)$ if x does not occur free in ϕ

Gen $\dfrac{\phi}{\forall x.\phi}$

Why there are no axioms for $\exists$? Left as an excercise.

Proofs and deductions (or derivations)

proof

A proof of a formula ϕ is a sequence of formulas $\phi_1, \ldots, \phi_n$, with $\phi_n = \phi$, such that each ϕ_k is either

- an axiom or
- it is derived from previous formulas by MP or **Gen**

ϕ is provable, in symbols $\vdash \phi$, if there is a proof for ϕ.

Deduction of ϕ from Γ

A deduction of a formula ϕ from a set of formulas Γ is a sequence of formulas $\phi_1, \ldots, \phi_n$, with $\phi_n = \phi$, such that ϕ_k

- is an axiom or
- it is in Γ (an assumption)
- it is derived form previous formulas by MP or **Gen**

ϕ is derivable from Γ in symbols $\Gamma \vdash \phi$ if there is a proof for ϕ.

The deduction theorem

Theorem

$\Gamma, A \vdash B$ *if and only if* $\Gamma \vdash A \supset B$

Proof.

If A and B are equal, then we know that $\vdash A \supset B$ (see previous example), and by monotonicity $\Gamma \vdash A \supset B$.

Suppose that A and B are distinct formulas. Let $\pi = (A_1, \ldots, A_n = B)$ be a deduction of $\Gamma, A \vdash B$, we proceed by induction on the length of π.

Base case $n = 1$ If $\pi = (B)$, then either $B \in \Gamma$ or B is an axiom If $B \in \Gamma$, then

Axiom **A1**	$B \supset (A \supset B)$
$B \in \Gamma$ or B is an axiom	B
by **MP**	$A \supset B$

is a deduction of $A \supset B$ from Γ or from the empty set, and therefore $\Gamma \vdash A \supset B$.

□

The deduction theorem

Proof.

Step case If $A_n = B$ is either an axiom or an element of Γ, then we can reason as the previous case.
If B is derived by **MP** form A_i and $A_j = A_i \supset B$. Then, A_i and $A_j = A_i \supset B$, are provable in less then n steps and, by induction hypothesis, $\Gamma \vdash A \supset A_i$ and $\Gamma \vdash A \supset (A_1 \supset B)$. Starting from the deductions of these two formulas from Γ, we can build a deduction of $A \supset B$ form Γ as follows:

By induction	$\vdots$ deduction of $A \supset (A_i \supset B)$ form Γ
	$A \supset (A_i \supset B)$
By induction	$\vdots$ deduction of $A \supset A_i$ form Γ
	$A \supset A_i$
A2	$(A \supset (A_i \supset B)) \supset ((A \supset A_i) \supset (A \supset B))$
MP	$(A \supset A_i) \supset (A \supset B)$
MP	$A \supset B$

$\square$

Deduction and proof - example

Example (Proof of $A \supset A$)

1.	$A1$	$A \supset ((A \supset A) \supset A)$
2.	$A2$	$(A \supset ((A \supset A) \supset A)) \supset ((A \supset (A \supset A)) \supset (A \supset A))$
3.	$MP(1,2)$	$(A \supset (A \supset A)) \supset (A \supset A)$
4.	$A1$	$(A \supset (A \supset A))$
5.	$MP(4,3)$	$A \supset A$

Deduction and proof - other examples

Example (proof of $\neg A \supset (A \supset B)$)

We prove that $A, \neg A \vdash B$ and by deduction theorem we have that $\neg A \vdash A \supset B$ and that $\vdash \neg A \supset (A \supset B)$

We label with Hypothesis the formula on the left of the $\vdash$ sign.

1.	*hypothesis*	A
2.	$A1$	$A \supset (\neg B \supset A)$
3.	$MP(1,2)$	$\neg B \supset A$
4.	*hypothesis*	$\neg A$
5.	$A1$	$\neg A \supset (\neg B \supset \neg A)$
6.	$MP(4,5)$	$\neg B \supset \neg A$
7.	$A3$	$(\neg B \supset \neg A) \supset ((\neg B \supset A) \supset B)$
8.	$MP(6,7)$	$(\neg B \supset A) \supset B$
9.	$MP(3,8)$	B

Hilbert axiomatization

Minimality

The main objective of Hilbert was to find the smallest set of axioms and inference rules from which it was possible to derive all the tautologies.

Unnatural

Proofs and deductions in Hilbert axiomatization are awkward and unnatural. Other proof styles, such as Natural Deductions, are more intuitive. As a matter of facts, nobody is practically using Hilbert calculus for deduction.

Why it is so important

Providing an Hilbert style axiomatization of a logic describes with simple axioms the entire properties of the logic. Hilbert axiomatization is the "identity card" of the logic.

Soundness & Completeness

How can we be sure that we derive exactly what we can logically infer?

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Theorem

Soundness We do not prove "wrong" logical consequences.
If $\Gamma \vdash A$ *then* $\Gamma \models A$*.*

Soundness & Completeness

How can we be sure that we derive exactly what we can logically infer?

Theorem

Soundness We do not prove "wrong" logical consequences.
If $\Gamma \vdash A$ *then* $\Gamma \models A$.

Theorem

Completeness We can prove all logical consequences.
If $\Gamma \models A$ *then* $\Gamma \vdash A$.

Soundness & Completeness of the Hilbert axiomatization

Theorem

$\Gamma \vdash A$ *if and only if* $\Gamma \models A$.

Using the Hilbert style axiomatization we can prove all and only the logical consequences of FOL.

Decidability of FOL

Definition

A logical system is decidable if there is an effective method for determining whether arbitrary formulas are logically valid.

- Propositional logic is decidable, because the truth-table method can be used to determine whether an arbitrary propositional formula is logically valid.
- First-order logic is not decidable in general; in particular, the set of logical validities in any signature that includes equality and at least one other predicate with two or more arguments is not decidable.

More efficient reasoning systems

Hilbert style is not easy implementable

Checking if $\Gamma \models \phi$ by searching for a Hilbert-style deduction of ϕ from Γ is not an easy task for computers. Indeed, in trying to generate a deduction of ϕ from Γ, there are to many possible actions a computer could take:

- adding an instance of one of the three axioms (infinite number of possibilities)
- applying **MP** to already deduced formulas,
- adding a formula in Γ

More efficient methods

- **Resolution** to check if a formula is *not satisfiable*
- **SAT** DP, DPLL to *search for an interpretation that satisfies a formula*
- **Tableaux** *search for a model of a formula* guided by its structure

Natural Deduction

Historical notes

Natural deduction (ND) was invented by G. Gentzen in 1934. The idea was to have a system of derivation rules that as closely as possible reflects the logical steps in an informal rigorous proof.

Natural Deduction

Introduction and elimination rules

For each connective $\circ$,

- there is an introduction rule $(\circ I)$ which can be seen as a definition of the truth conditions of a formula with $\circ$ given in terms of the truth values of its component(s);
- there is an elimination rule $(\circ E)$ that allows to exploit such a definition to derive truth of the components of a formula whose main connective is $\circ$.

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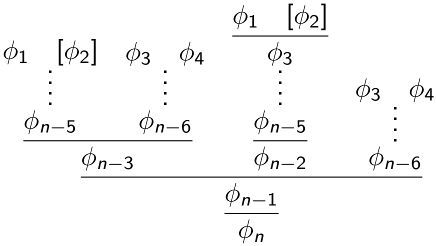
Assumptions

In the process of building a deduction one can make new assumptions and can discharge already done assumptions.

Natural Deduction

Natural deduction Derivation

A derivation is a tree where the nodes are the rules and the leafs are the assumptions of the derivation. The root of the tree is the conclusion of the derivation.



ND rules for propositional connectives

$\wedge$

$$\frac{\phi \quad \psi}{\phi \wedge \psi}\ \wedge I \qquad \frac{\phi \wedge \psi}{\phi}\ \wedge E_1 \qquad \frac{\phi \wedge \psi}{\psi}\ \wedge E_2$$

ND rules for propositional connectives

$\wedge$

$$\frac{\phi \quad \psi}{\phi \wedge \psi}\ \wedge I \qquad \frac{\phi \wedge \psi}{\phi}\ \wedge E_1 \qquad \frac{\phi \wedge \psi}{\psi}\ \wedge E_2$$

$\supset$

$$\frac{\begin{array}{c}[\phi] \\ \vdots \\ \psi\end{array}}{\phi \supset \psi}\ \supset I \qquad \frac{\phi \quad \phi \supset \psi}{\psi}\ \supset E$$

ND rules for propositional connectives

$\wedge$

$$\frac{\phi \quad \psi}{\phi \wedge \psi}\ \wedge I \qquad \frac{\phi \wedge \psi}{\phi}\ \wedge E_1 \qquad \frac{\phi \wedge \psi}{\psi}\ \wedge E_2$$

$\supset$

$$\frac{\begin{array}{c}[\phi] \\ \vdots \\ \psi\end{array}}{\phi \supset \psi}\ \supset I \qquad \frac{\phi \quad \phi \supset \psi}{\psi}\ \supset E$$

$\vee$

$$\frac{\phi}{\phi \vee \psi}\ \vee I_1 \qquad \frac{\psi}{\phi \vee \psi}\ \vee I_2 \qquad \frac{\phi \vee \psi \quad \begin{array}{c}[\phi] \\ \vdots \\ \theta\end{array} \quad \begin{array}{c}[\psi] \\ \vdots \\ \theta\end{array}}{\theta}\ \vee E$$

ND rules for propositional connectives

The connective $\neg$ for negation

ND does not provide rules for the $\neg$ connective. Instead, the logical constant $\perp$ is introduced,

$\perp$ stands for the unsatisfiable formula, i.e., the formula that is false in all interpretations.

$\neg A$ is defined to be a syntactic sugar for $A \supset \perp$

(**exercise:** Verify that $\neg A \equiv (A \supset \perp)$ is a valid formula).

ND rules for propositional connectives

The connective $\neg$ for negation

ND does not provide rules for the $\neg$ connective. Instead, the logical constant $\bot$ is introduced,
$\bot$ stands for the unsatisfiable formula, i.e., the formula that is false in all interpretations.
$\neg A$ is defined to be a syntactic sugar for $A \supset \bot$
(**exercise:** Verify that $\neg A \equiv (A \supset \bot)$ is a valid formula).

$\bot$

$$\dfrac{\begin{matrix}[\neg\phi] \\ \vdots \\ \bot\end{matrix}}{\phi}\ \bot_c$$

Extending ND to FOL: quantifiers

∀

$$\frac{\phi(x)}{\forall x.\phi(x)}\ \forall I \qquad \frac{\forall x.\phi(x)}{\phi(t)}\ \forall E$$

∃

$$\frac{\phi(t)}{\exists x.\phi(x)}\ \exists I \qquad \frac{\exists x.\phi(x) \qquad \begin{array}{c}[\phi(x)]\\ \vdots \\ \theta\end{array}}{\theta}\ \exists E$$

Restrictions $\forall I$: x does not occur free in any assumption from which ϕ depends on.
$\exists E$: x does not occur free in θ and in any assumption θ depends on with the exception of $phi(x)$.

Extending ND to FOL: equality

=

$$\frac{}{t = t} = I \qquad \frac{\phi(t) \quad x = t}{\phi(x)} = E$$

Natural Deduction Rules

$$\frac{\phi \quad \psi}{\phi \wedge \psi}\ \wedge I \qquad \frac{\phi \wedge \psi}{\phi} \quad \frac{\phi \wedge \psi}{\psi}\ \wedge E \qquad \frac{\phi}{\phi \vee \psi} \quad \frac{\psi}{\phi \vee \psi}\ \vee I \qquad \frac{\phi \vee \psi \quad \begin{array}{c}[\phi]\\ \vdots \\ \theta\end{array} \quad \begin{array}{c}[\psi]\\ \vdots \\ \theta\end{array}}{\theta}\ \vee E$$

$$\frac{\begin{array}{c}[\phi]\\ \vdots \\ \psi\end{array}}{\phi \supset \psi}\ \supset I \qquad \frac{\phi \quad \phi \supset \psi}{\psi}\ \supset E \qquad \frac{\begin{array}{c}[\neg \phi]\\ \vdots \\ \bot\end{array}}{\phi}\ \bot_c$$

$$\frac{\phi(x)}{\forall x . \phi(x)}\ \forall I \qquad \frac{\forall x . \phi(x)}{\phi(t)}\ \forall E \qquad \frac{\phi(t)}{\exists x . \phi(x)}\ \exists I \qquad \frac{\exists x . \phi(x) \quad \begin{array}{c}[\phi(x)]\\ \vdots \\ \theta\end{array}}{\theta}\ \exists E$$

$$\frac{}{t = t}\ = I \qquad \frac{\phi(t) \quad x = t}{\phi(x)}\ = E$$

Natural Deduction

Definition (Deduction)

A deduction Π of A with undischarged assumption $A_1, \ldots, A_n$, is a tree with root A, obtained by applying the ND rules, and every assumption in Π, but $A_1, \ldots, A_n$ is discharged, by the application of one of the ND rules.

Definition ($\Gamma \vdash_{ND} A$)

A formula A is derivable from a set of formulas Γ, if there is a deduction of A with undischarged assumption contained in Γ. In this case we write

$$\Gamma \vdash_{ND} A$$

If no ambiguity arises we omit the subscript ND and use $\Gamma \vdash A$

Soundness & Completeness of Natural Deduction

Theorem

$\Gamma \vdash_{ND} A$ *if and only if* $\Gamma \models A$.

Using the Natural Deduction rules we can prove all and only the logical consequences of FOL.

Examples

For each of the following statements provide a proof in natural deduction.

1. $\vdash_{ND} A \supset (B \supset A)$
2. $\vdash_{ND} \neg(A \wedge \neg A)$
3. $\vdash_{ND} \neg\neg A \leftrightarrow A$
4. $\vdash_{ND} (A \vee A) \equiv (A \vee \bot)$
5. $(A \wedge B) \wedge C \vdash_{ND} A \wedge (B \wedge C)$
6. $\vdash_{ND} A \vee \neg A$;
7. $\vdash_{ND} (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
8. $\vdash_{ND} (A \supset B) \leftrightarrow (\neg A \vee B)$
9. $\vdash_{ND} A \vee (A \supset B)$
10. $\neg(A \supset \neg B) \vdash_{ND} (A \wedge B)$
11. $A \supset (B \supset C), A \vee C, \neg B \supset \neg A \vdash_{ND} C$

Examples

1. $\vdash_{ND} A \supset (B \supset A)$

$$\dfrac{\dfrac{A^1}{B \supset A}\supset I}{A \supset (B \supset A)}\supset I_{(1)}$$

Examples

2. $\vdash_{ND} \neg(A \wedge \neg A)$

$$\dfrac{\dfrac{\dfrac{A \wedge \neg A^1}{A}\wedge E \quad \dfrac{A \wedge \neg A^1}{\neg A}\wedge E}{\bot}\supset E}{\neg(A \wedge \neg A)}\bot c_{(1)}$$

Examples

3. $\vdash_{ND} \neg\neg A \leftrightarrow A$

$$\cfrac{\cfrac{\cfrac{\neg\neg A^2 \quad \neg A^1}{\bot}\supset E}{A}\bot c_{(1)}}{\neg\neg A \supset A}\supset I_{(2)}$$

$$\cfrac{\cfrac{\cfrac{A^2 \quad \neg A^1}{\bot}\supset E}{\neg\neg A}\bot c_{(1)}}{A \supset \neg\neg A}\supset I_{(2)}$$

Examples

4. $\vdash_{ND} (A \vee A) \equiv (A \vee \bot)$

$$\dfrac{\dfrac{A \vee A^2 \quad \dfrac{A^1}{A \vee \bot}\,\vee I \quad \dfrac{A^1}{A \vee \bot}\,\vee I}{A \vee \bot}\,\vee E_{(1)}}{(A \vee A) \supset (A \vee \bot)}\,\supset I_{(2)}$$

$$\dfrac{\dfrac{A \vee \bot^2 \quad \dfrac{A^1}{A \vee A}\,\vee I \quad \dfrac{\bot^1}{A \vee A}\,\bot c}{A \vee A}\,\vee E_{(1)}}{(A \vee \bot) \supset (A \vee A)}\,\supset I_{(2)}$$

Examples

5. $(A \wedge B) \wedge C \vdash_{ND} A \wedge (B \wedge C)$

$$\dfrac{\dfrac{\dfrac{(A \wedge B) \wedge C}{A \wedge B}\wedge E}{A}\wedge E \qquad \dfrac{\dfrac{\dfrac{(A \wedge B) \wedge C}{A \wedge B}\wedge E}{B}\wedge E \qquad \dfrac{(A \wedge B) \wedge C}{C}\wedge E}{B \wedge C}\wedge I}{A \wedge (B \wedge C)}\wedge I$$

Examples

6. $\vdash_{ND} A \vee \neg A$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A^1}{A \vee \neg A}\vee I \qquad \neg(A \vee \neg A)^2}{\bot}\supset E}{\neg A}\bot_{c(1)}}{A \vee \neg A}\vee I \qquad \neg(A \vee \neg A)^2}{\bot}\supset E}{A \vee \neg A}\bot_{c(2)}$$

Examples

7. $\vdash_{ND} (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \supset (B \supset C)^3 \quad A^1}{B \supset C} \supset E \quad \dfrac{A \supset B^2 \quad A^1}{B} \supset E}{C} \supset E}{A \supset C} \supset I_{(1)}}{(A \supset B) \supset (A \supset C)} \supset I_{(2)}}{(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))} \supset I_{(3)}
$$

Examples

8.a $\vdash_{ND} (A \supset B) \supset (\neg A \vee B)$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \supset B^3 \quad A^1}{B}\supset E}{\neg A \vee B}\vee I \qquad \neg(\neg A \vee B)^2}{\bot}\supset E}{\neg A}\bot c_{(1)}}{\neg A \vee B}\vee I \qquad \neg(\neg A \vee B)^2}{\bot}\supset E}{\neg A \vee B}\bot c_{(2)}}{(A \supset B) \supset (\neg A \vee B)}\supset I_{(3)}
$$

Examples

8.b $\vdash_{ND} (\neg A \vee B) \supset (A \supset B)$

$$
\dfrac{\dfrac{\neg A \vee B^3 \qquad \dfrac{\dfrac{\dfrac{\neg A^2 \quad A^1}{\bot}\supset E}{B}\bot c}{A \supset B}\supset I_{(1)} \qquad \dfrac{B^2}{A \supset B}\supset I}{A \supset B}\vee E_{(2)}}{(\neg A \vee B) \supset (A \supset B)}\supset I_{(3)}
$$

Examples

9. $\vdash_{ND} A \vee (A \supset B)$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A^1}{A \vee (A \supset B)}\,\vee I \qquad \neg(A \vee (A \supset B))^2}{\bot}\,\supset E}{B}\,\bot c}{A \supset B}\,\supset I_{(1)}}{A \vee (A \supset B)}\,\vee I \qquad \neg(A \vee (A \supset B))^2}{\bot}\,\supset E}{A \vee (A \supset B)}\,\bot c_{(2)}
$$

Examples

10. $\neg(A \supset \neg B) \vdash_{ND} (A \wedge B)$

$$
\begin{prooftree}
\AxiomC{A^1}
\AxiomC{$\neg A^2$}
\RightLabel{$\supset E$}
\BinaryInfC{$\bot$}
\RightLabel{$\bot c$}
\UnaryInfC{$\neg B$}
\RightLabel{$\supset I_{(1)}$}
\UnaryInfC{$A \supset \neg B$}
\AxiomC{$\neg(A \supset \neg B)$}
\RightLabel{$\supset E$}
\BinaryInfC{$\bot$}
\RightLabel{$\bot c_{(2)}$}
\UnaryInfC{A}
\AxiomC{$\neg B^3$}
\RightLabel{$\supset I$}
\UnaryInfC{$A \supset \neg B$}
\AxiomC{$\neg(A \supset \neg B)$}
\BinaryInfC{$\bot$}
\RightLabel{$\bot c_{(3)}$}
\UnaryInfC{B}
\RightLabel{$\wedge I$}
\BinaryInfC{$A \wedge B$}
\end{prooftree}
$$

Examples

11. $A \supset (B \supset C), A \vee C, \neg B \supset \neg A \vdash_{ND} C$

$$\dfrac{A \vee C \qquad \dfrac{\dfrac{A \supset (B \supset C) \quad A^2}{B \supset C}\supset E \qquad \dfrac{\dfrac{\dfrac{\neg B \supset \neg A \quad \neg B^1}{\neg A}\supset E \quad A^2}{\bot}\supset E}{B}\bot c_{(1)}}{C}\supset E \qquad C^2}{C}\vee E_{(2)}$$

Proof Strategies

1: $\vdash_{ND} \psi \supset \phi$

- assume ψ and try to deduce ϕ (simplest solution)
- as an alternative, assume $\neg\phi$ and ψ and try to deduce $\bot$

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- apply recursively the strategy in 1

Proof Strategies

1: $\vdash_{ND} \psi \supset \phi$

- assume ψ and try to deduce ϕ (simplest solution)
- as an alternative, assume $\neg\phi$ and ψ and try to deduce $\perp$

2: $\vdash_{ND} \phi_1 \supset (\phi_2 \supset \phi_3)$

- apply recursively the strategy in 1

3: $\vdash_{ND} \psi \wedge \phi$

- try to deduce ψ and try to deduce ϕ (separately) and then apply $\wedge I$

Proof Strategies

4: $\vdash_{ND} \psi \vee \phi$

- try to deduce ψ or (alternatively) ϕ and then apply $\vee I$ ... usually it doesn't work.
- assume $\neg\psi$, try to derive ϕ and proceed by contradiction:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\neg\psi^1 \\ \vdots \\ \phi\end{matrix}}{\psi \vee \phi}\,\vee I \qquad \neg(\psi \vee \phi)^2}{\bot}\,\supset E}{\psi}\,\bot_{c(1)}}{\psi \vee \phi}\,\vee I \qquad \neg(\psi \vee \phi)^2}{\bot}\,\supset E}{\psi \vee \phi}\,\bot_{c(2)}$$

alternatively, assume $\neg\phi$, try to derive ψ and proceed by contradiction in the same way

Proof Strategies

5: $\vdash_{ND} (\phi_1 \vee \phi_2) \supset \phi_3$

1. assume ϕ_1 and deduce ϕ_3
2. assume ϕ_2 and deduce ϕ_3
3. assume $\phi_1 \vee \phi_1$ and apply $\vee E$

$$\dfrac{\phi_1 \vee \phi_2 \qquad \begin{matrix}\phi_1^1\\ \vdots \\ \phi_3\end{matrix} \qquad \begin{matrix}\phi_2^1\\ \vdots \\ \phi_3\end{matrix}}{\phi_3}\ \vee E_{(1)}$$

Examples

Prove the validity of the following statements by using natural deduction:

1. $(A \vee B) \vdash_{ND} \neg(\neg A \wedge \neg B)$
2. $((A \supset B) \supset A) \vdash_{ND} A$
3. $(A \supset B) \vdash_{ND} (B \supset C) \supset (A \supset C)$
4. $(A \wedge B) \supset C \vdash_{ND} A \supset (B \supset C)$
5. $\vdash_{ND} (A \supset B) \supset (\neg B \supset \neg A)$

Examples

1. $(A \vee B) \vdash_{ND} \neg(\neg A \wedge \neg B)$

$$\dfrac{A \vee B \qquad \dfrac{\dfrac{A^3 \qquad \dfrac{\neg A \wedge \neg B^1}{\neg A}\wedge E}{\bot}\supset E}{\neg(\neg A \wedge \neg B)}\bot c_{(1)} \qquad \dfrac{\dfrac{B^3 \qquad \dfrac{\neg A \wedge \neg B^2}{\neg B}\wedge E}{\bot}\supset E}{\neg(\neg A \wedge \neg B)}\bot c_{(2)}}{\neg(\neg A \wedge \neg B)}\vee E_{(3)}$$

Examples

2. $((A \supset B) \supset A) \vdash_{ND} A$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A^1 \quad \neg A^3}{\bot}\supset E}{B}\,\bot c}{A \supset B}\supset I_{(1)} \qquad \dfrac{\dfrac{\dfrac{(A \supset B) \supset A \quad A \supset B^2}{A}\supset E \qquad \neg A^3}{\bot}\supset E}{\neg(A \supset B)}\,\bot c_{(2)}}{\bot}\supset E}{A}\,\bot c_{(3)}$$

Examples

3. $(A \supset B) \vdash_{ND} (B \supset C) \supset (A \supset C)$

$$\dfrac{\dfrac{\dfrac{\dfrac{A \supset B \quad A^1}{B}\supset E \qquad B \supset C^2}{C}\supset E}{A \supset C}\supset I_{(1)}}{(B \supset C) \supset (A \supset C)}\supset I_{(2)}$$

Examples

4. $(A \wedge B) \supset C \vdash_{ND} A \supset (B \supset C)$

$$\dfrac{\dfrac{\dfrac{\dfrac{A^2 \quad B^1}{A \wedge B}\wedge I \qquad (A \wedge B) \supset C}{C} \supset E}{B \supset C} \supset I_{(1)}}{A \supset (B \supset C)} \supset I_{(2)}$$

Examples

5. $\vdash_{ND} (A \supset B) \supset (\neg B \supset \neg A)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg B^2 \quad \dfrac{A \supset B^3 \quad A^1}{B} \supset E}{\bot} \supset E}{\neg A} \bot c_{(1)}}{\neg B \supset \neg A} \supset I_{(2)}}{(A \supset B) \supset (\neg B \supset \neg A)} \supset I_{(3)}$$

Exercises

For each of the following formula provide either a proof in natural deduction or a counter-model.

- $(\neg B \supset \neg A) \supset ((\neg B \supset A) \supset A)$
- $A \supset (B \supset C) \equiv (A \wedge B \supset C)$
- $((A \supset B \vee C) \wedge \neg B \wedge \neg C) \supset \neg A$
- $\neg(A \supset B) \supset (B \supset A)$
- $((A \supset C) \vee (B \supset D)) \supset ((A \supset D) \vee (B \supset C))$
- $((A \supset B) \supset B) \supset ((B \supset A) \supset A)$

Exercises

For each of the following propositional classical logical consequences provide a natural deduction proof

- $(A \wedge B) \wedge C \vdash_{ND} A \wedge (B \wedge C)$
- $(A \supset B) \vdash_{ND} (\neg B \supset \neg A)$
- $(A \vee B) \vdash_{ND} \neg(\neg A \wedge \neg B)$
- $((A \supset B) \supset A) \vdash_{ND} A$
- $(A \supset B) \vdash_{ND} ((B \supset C) \supset A \supset C)$
- $((A \wedge B) \supset C) \vdash_{ND} (A \supset (B \supset C))$

Natural deduction for classical FOL

Show the deduction for the following first order valid formulas.

1. $\exists x.\forall y.R(x,y) \supset \forall y.\exists x.R(x,y)$
2. $\exists x.(P(x) \supset \forall x.P(x))$
3. $\exists x.(P(x) \vee Q(x)) \supset (\exists x.P(x) \vee \exists x.Q(x))$
4. $\exists x.(P(x) \wedge Q(x)) \supset \exists x.P(x) \wedge \exists x.Q(x))$
5. $(\exists x.P(x) \wedge \forall x.Q(x)) \supset \exists x.(P(x) \wedge Q(x))$
6. $\forall x.(P(x) \supset Q) \supset (\exists x.P(x) \supset Q)$, where x is not free in Q.
7. $\forall x.\exists y.x = y$
8. $\forall xyzw.((x = z \wedge y = w) \supset (R(x,y) \supset R(z,w)))$, where $\forall xyzw \ldots$ stands for $\forall x.(\forall y.(\forall z.(\forall w \ldots)))$.

Natural deduction for classical FOL

Show the deduction for the following first order valid formulas.

1. $(A \supset \forall x.B(x)) \equiv \forall x(A \supset B(x))$ where x does not occur free in A
2. $\exists x(A(x) \vee B(x)) \equiv (\exists x A(x) \vee \exists x B(x))$
3. $\neg\exists x A(x) \equiv \forall x \neg A(x)$
4. $\forall x(A(x) \vee B) \equiv \forall x A(x) \vee B$ where x does not occur free in B
5. $\exists x(A(x) \supset B) \equiv (\forall x A(x) \supset B)$ where x does not occur free in B
6. $\exists x(A \supset B(x)) \equiv (A \supset \exists x B(x))$ where x does not occur free in A
7. $\forall x(A(x) \supset B) \equiv (\exists x A(x) \supset B)$ where x does not occur free in B