DTFT Properties

- Example - Determine the DTFT $Y(e^{j\omega})$ of

$$y[n] = (n+1)\alpha^n \mu[n], \ |\alpha| < 1$$

- Let $x[n] = \alpha^n \mu[n], |\alpha| < 1$
- We can therefore write

$$y[n] = n\,x[n] + x[n]$$

- From Table 3.3, the DTFT of $x[n]$ is given by

$$X(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}$$

DTFT Properties

- Using the differentiation property of the DTFT given in Table 3.2, we observe that the DTFT of $nx[n]$ is given by

$$j\frac{dX(e^{j\omega})}{d\omega} = j\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right) = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$$

- Next using the linearity property of the DTFT given in Table 3.4 we arrive at

$$Y(e^{j\omega}) = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2} + \frac{1}{1-\alpha e^{-j\omega}} = \frac{1}{(1-\alpha e^{-j\omega})^2}$$

DTFT Properties

- Example - Determine the DTFT $V(e^{j\omega})$ of the sequence $v[n]$ defined by

$$d_0 v[n] + d_1 v[n-1] = p_0 \delta[n] + p_1 \delta[n-1]$$

- From Table 3.3, the DTFT of $\delta[n]$ is 1
- Using the time-shifting property of the DTFT given in Table 3.4 we observe that the DTFT of $\delta[n-1]$ is $e^{-j\omega}$ and the DTFT of $v[n-1]$ is $e^{-j\omega}V(e^{j\omega})$

DTFT Properties

- Using the linearity property of Table 3.4 we then obtain the frequency-domain representation of

$$d_0 v[n] + d_1 v[n-1] = p_0 \delta[n] + p_1 \delta[n-1]$$

as

$$d_0 V(e^{j\omega}) + d_1 e^{-j\omega} V(e^{j\omega}) = p_0 + p_1 e^{-j\omega}$$

- Solving the above equation we get

$$V(e^{j\omega}) = \frac{p_0 + p_1 e^{-j\omega}}{d_0 + d_1 e^{-j\omega}}$$

Energy Density Spectrum

- The total energy of a finite-energy sequence $g[n]$ is given by

$$E_g = \sum_{n=-\infty}^{\infty} \left| g[n] \right|^2$$

- From Parseval's relation given in Table 3.4 we observe that

$$E_g = \sum_{n=-\infty}^{\infty} \left| g[n] \right|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| G(e^{j\omega}) \right|^2 d\omega$$

Energy Density Spectrum

- The quantity

$$S_{gg}(\omega) = \left|G(e^{j\omega})\right|^2$$

is called the **energy density spectrum**

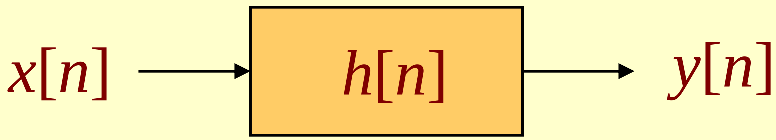
- The area under this curve in the range $-\pi \leq \omega \leq \pi$ divided by 2π is the energy of the sequence

The Frequency Response

- Most discrete-time signals encountered in practice can be represented as a linear combination of a very large, maybe infinite, number of sinusoidal discrete-time signals of different angular frequencies
- Thus, knowing the response of the LTI system to a single sinusoidal signal, we can determine its response to more complicated signals by making use of the superposition property

The Frequency Response

- Consider the LTI discrete-time system with an impulse response $\{h[n]\}$ shown below



- Its input-output relationship in the time-domain is given by the convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$

The Frequency Response

- The quantity $H(e^{j\omega})$ is called the **frequency response** of the LTI discrete-time system
- $H(e^{j\omega})$ provides a frequency-domain description of the system
- $H(e^{j\omega})$ is precisely the DTFT of the impulse response $\{h[n]\}$ of the system

The Frequency Response

- $H(e^{j\omega})$, in general, is a complex function of ω with a period 2π
- It can be expressed in terms of its real and imaginary parts

$$H(e^{j\omega}) = H_{re}(e^{j\omega}) + j\,H_{im}(e^{j\omega})$$

or, in terms of its magnitude and phase,

$$H(e^{j\omega}) = \left|H(e^{j\omega})\right| e^{j\theta(\omega)}$$

where

$$\theta(\omega) = \arg H(e^{j\omega})$$

The Frequency Response

- The function $\left|H(e^{j\omega})\right|$ is called the **magnitude response** and the function $\theta(\omega)$ is called the **phase response** of the LTI discrete-time system
- Design specifications for the LTI discrete-time system, in many applications, are given in terms of the magnitude response or the phase response or both

The Frequency Response

- In some cases, the magnitude function is specified in **decibels** as

$$\mathcal{G}(\omega) = 20\log_{10}\left|H(e^{j\omega})\right| \quad dB$$

where $\mathcal{G}(\omega)$ is called the **gain function**

- The negative of the gain function

$$\mathcal{A}(\omega) = -\mathcal{G}(\omega)$$

is called the **attenuation** or **loss function**

The Concept of Filtering

- One application of an LTI discrete-time system is to **pass** certain frequency components in an input sequence without any distortion (if possible) and to **block** other frequency components
- Such systems are called digital filters and one of the main subjects of discussion in this course

The Concept of Filtering

- The key to the filtering process is

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- It expresses an arbitrary input as a linear weighted sum of an infinite number of exponential sequences, or equivalently, as a linear weighted sum of sinusoidal sequences

The Concept of Filtering

- Thus, by appropriately choosing the values of the magnitude function $\left|H(e^{j\omega})\right|$ of the LTI digital filter at frequencies corresponding to the frequencies of the sinusoidal components of the input, some of these components can be selectively heavily attenuated or filtered with respect to the others

The Concept of Filtering

- To understand the mechanism behind the design of frequency-selective filters, consider a real-coefficient LTI discrete-time system characterized by a magnitude function

$$\left|H(e^{j\omega})\right| \cong \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

The Concept of Filtering

- We apply an input

$$x[n] = A\cos\omega_1 n + B\cos\omega_2 n, \qquad 0 < \omega_1 < \omega_c < \omega_2 < \pi$$

  to this system

- Because of linearity, the output of this system is of the form

$$y[n] = A\left|H(e^{j\omega_1})\right|\cos\left(\omega_1 n + \theta(\omega_1)\right)$$

$$+ B\left|H(e^{j\omega_2})\right|\cos\left(\omega_2 n + \theta(\omega_2)\right)$$

The Concept of Filtering

- As

$$\left|H(e^{j\omega_1})\right| \cong 1, \quad \left|H(e^{j\omega_2})\right| \cong 0$$

the output reduces to

$$y[n] \cong A\left|H(e^{j\omega_1})\right|\cos\left(\omega_1 n + \theta(\omega_1)\right)$$

- Thus, the system acts like a **lowpass filter**
- In the following example, we consider the design of a very simple digital filter

The Concept of Filtering

- Example - The input consists of a sum of two sinusoidal sequences of angular frequencies 0.1 rad/sample and 0.4 rad/sample
- We need to design a highpass filter that will pass the high-frequency component of the input but block the low-frequency component
- For simplicity, assume the filter to be an FIR filter of length 3 with an impulse response:

$$h[0] = h[2] = \alpha, \qquad h[1] = \beta$$

The Concept of Filtering

- The convolution sum description of this filter is then given by

$$y[n] = h[0]\,x[n] + h[1]\,x[n-1] + h[2]\,x[n-2]$$
$$= \alpha\, x[n] + \beta\, x[n-1] + \alpha\, x[n-2]$$

- $y[n]$ and $x[n]$ are, respectively, the output and the input sequences
- **Design Objective**: Choose suitable values of α and β so that the output is a sinusoidal sequence with a frequency 0.4 rad/sample

The Concept of Filtering

- Now, the frequency response of the FIR filter is given by

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega}$$

$$= \alpha(1 + e^{-j2\omega}) + \beta e^{-j\omega}$$

$$= 2\alpha\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right)e^{-j\omega} + \beta e^{-j\omega}$$

$$= (2\alpha \cos\omega + \beta)e^{-j\omega}$$

The Concept of Filtering

- The magnitude and phase functions are

$$\left|H(e^{j\omega})\right| = 2\alpha\cos\omega + \beta$$

$$\theta(\omega) = -\omega$$

- In order to block the low-frequency component, the magnitude function at $\omega = 0.1$ should be equal to zero
- Likewise, to pass the high-frequency component, the magnitude function at $\omega = 0.4$ should be equal to one

The Concept of Filtering

- Thus, the two conditions that must be satisfied are

$$\left|H(e^{j0.1})\right| = 2\alpha\cos(0.1) + \beta = 0$$

$$\left|H(e^{j0.4})\right| = 2\alpha\cos(0.4) + \beta = 1$$

- Solving the above two equations we get

$$\alpha = -6.76195$$

$$\beta = 13.456335$$

The Concept of Filtering

- Thus the output-input relation of the FIR filter is given by

$$y[n] = -6.76195\left(x[n] + x[n-2]\right) + 13.456335\, x[n-1]$$

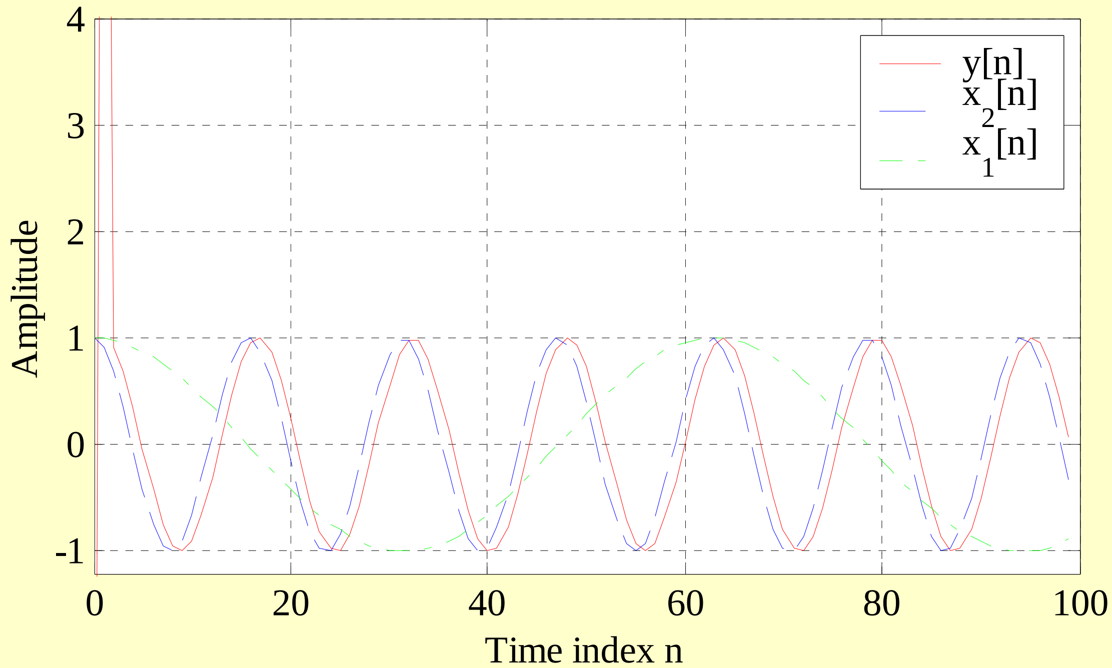
where the input is

$$x[n] = \{\cos(0.1n) + \cos(0.4n)\}\mu[n]$$

- Program 3_3.m can be used to verify the filtering action of the above system

The Concept of Filtering

- Figure below shows the plots generated by running this program



The Concept of Filtering

- The first seven samples of the output are shown below

n	$\cos(0.1n)$	$\cos(0.4n)$	$x[n]$	$y[n]$
0	1.0	1.0	2.0	−13.52390
1	0.9950041	0.9210609	1.9160652	13.956333
2	0.9800665	0.6967067	1.6767733	0.9210616
3	0.9553364	0.3623577	1.3176942	0.6967064
4	0.9210609	−0.0291995	0.8918614	0.3623572
5	0.8775825	−0.4161468	0.4614357	−0.0292002
6	0.8253356	−0.7373937	0.0879419	−0.4161467

The Concept of Filtering

- From this table, it can be seen that, neglecting the least significant digit,

$$y[n] = \cos(0.4(n-1)) \qquad \text{for } n \geq 2$$

- Computation of the present value of the output requires the knowledge of the present and two previous input samples
- Hence, the first two output samples, $y[0]$ and $y[1]$, are the result of assumed zero input sample values at $n = -1$ and $n = -2$

The Concept of Filtering

- Therefore, first two output samples constitute the transient part of the output
- Since the impulse response is of length 3, the steady-state is reached at $n = N = 2$
- Note also that the output is delayed version of the high-frequency component cos(0.4n) of the input, and the delay is one sample period