1. Classical Categorial Grammar

- **Aim:** To build a language recognition device.
- **Who:** Lesniewski (1929), Ajdukiewicz (1935), Bar-Hillel (1953).
- **How:** Linguistic strings are seen as the result of function applications starting from the categories assigned to lexicon items.
- **Language:** Given a set of basic categories ATOM, the set of categories CAT is the smallest set such that:
  - if $X \in \mathsf{ATOM}$, then $X \in \mathsf{CAT}$;
  - if $X, Y \in \mathsf{ATOM}$, then $X/Y, Y\backslash X \in \mathsf{CAT}$
- **Rules:** The above categories can be composed by means of functional application rules

$$\frac{X/Y \quad Y}{X}\,[\mathrm{MP_r}] \qquad\qquad \frac{Y \quad Y\backslash X}{X}\,[\mathrm{MP_l}]$$

2. Classical Categorial Grammar. Examples

Given $\mathsf{ATOM} = \{np, s, n, pp\}$, we can build the following lexicon:

Lexicon

John, Mary	$\in$	np	the	$\in$	np/n
student	$\in$	n	to	$\in$	pp/np
walks	$\in$	$np\backslash s$	talks	$\in$	$(np\backslash s)/pp$
sees	$\in$	$(np\backslash s)/np$	some student	$\in$	$s/(np\backslash s)$

Analysis

John walks $\in s$? $\rightsquigarrow$ $np, np\backslash s \Rightarrow s$? Yes

$$\frac{np \quad np\backslash s}{s}\,[\mathrm{MP_l}]$$

John sees Mary $\in s$? $\rightsquigarrow$ $np, (np\backslash s)/np, np \Rightarrow s$? Yes

$$\frac{np \qquad \dfrac{(np\backslash s)/np \quad np}{np\backslash s}\,[\mathrm{MP_r}]}{s}\,[\mathrm{MP_l}]$$

3. Categories and Types

We can define the following translation tr from types to categories.

$$
\begin{array}{lcl}
\mathsf{tr}(e) & = & np \\
\mathsf{tr}(t) & = & s \\
\mathsf{tr}(\langle a, b\rangle) & = & \mathsf{tr}(a)/\mathsf{tr}(b) \\
 & = & \mathsf{tr}(b)\backslash\mathsf{tr}(a)
\end{array}
\qquad
\begin{array}{lll}
\mathtt{m}_e & \text{iff} & np : \mathtt{m} \\
\mathsf{S}_t & \text{iff} & s : \mathsf{S} \\
\mathtt{W}_{\langle a,b\rangle} & \text{iff} & \mathsf{tr}(b)/\mathsf{tr}(a) : \mathtt{W} \\
 & & \text{or } \mathsf{tr}(a)\backslash\mathsf{tr}(b) : \mathtt{W}
\end{array}
$$

Modus ponens corresponds to functional application.

$$
\frac{X/Y : t \quad Y : r}{X : t(r)}\,[\mathrm{MP_r}] \qquad\qquad \frac{Y : r \quad Y\backslash X : t}{X : t(r)}\,[\mathrm{MP_l}]
$$

Example

$$
\frac{np : \mathtt{john} \quad np\backslash s : \mathtt{walk}}{s : \mathtt{walk}(\mathtt{john})}\,[\mathrm{MP_l}]
$$

$$
np\backslash s : \lambda x.\mathtt{walk}(x) \quad (\lambda x.\mathtt{walk}(x))(\mathtt{john}) \leadsto_{\lambda-\mathrm{conv.}} \mathtt{walk}(\mathtt{john})
$$

4. Lambek Calculus

Jim Lambek [1958] defines the logic behind Catergorial Grammar, considering categories as formulae and $\backslash, /$ as logic connectives.

Rules: Natural Deduction proof format [Elimination and Introduction rules]

Besides functional applications rules – which correspond to the elimination of $\backslash, /$ – we have their introduction rules. $\Gamma \vdash A$ means that A derives from Γ; Γ, Δ stand for structures, A, B, C for logic formulae.

$$\frac{\Delta \vdash B/A \quad \Gamma \vdash A}{\Delta, \Gamma \vdash B}\ [/\mathrm{E}] \qquad \frac{\Gamma \vdash A \quad \Delta \vdash A\backslash B}{\Gamma, \Delta \vdash B}\ [\backslash \mathrm{E}]$$

$$\frac{\Delta, B \vdash C}{\Delta \vdash C/B}\ [/\mathrm{I}] \qquad \frac{B, \Delta \vdash C}{\Delta \vdash B\backslash C}\ [\backslash \mathrm{I}]$$

John and some student went to the park $\in s$?

'and' conjunct expressions of the same category;

We have: John $\in np$, some student $\in s/(np\backslash s)$;

Hence we need $s/(np\backslash s) \vdash np$ or $np \vdash s/(np\backslash s)$.

$$\dfrac{\dfrac{\dfrac{\text{john} \in np}{np \vdash np}[\text{Lex}] \quad [np\backslash s \vdash np\backslash s]^1}{np \circ np\backslash s \vdash s}[\backslash\text{E}]}{np \vdash s/(np\backslash s)}[/\text{I}]^1$$

$$\dfrac{\dfrac{\dfrac{\text{john} \in np : \texttt{john}}{np \vdash np : \texttt{john}} \quad [np\backslash s \vdash np\backslash s : P]^1}{np \circ np\backslash s \vdash s : P(\texttt{john})}[\backslash\text{E}]}{np \vdash s/(np\backslash s) : \lambda P.P(\texttt{john})}[/\text{I}]^1$$

The introduction rules correspond to λ-abstraction.

□

6. Lambek calculus. Advantages

- **Hypothetical reasoning:** Having added $[\backslash I], [/I]$ gives the system the right expressivity to reason about hypothesis and abstract over them.
- **Curry Howard Correspondence:** Curry-Howard correspondence holds between proofs and terms. This means that parsed structures are assigned an interpretation into a model via the connection 'categories-terms'.
- **Logic:** We have moved from a grammar to a logic. Hence its behavior can be studied. The system is sound, complete and decidable.

7. Lambek calculus. Limits

- **No explicit structural reasoning**: There is no way to speak about the structures and have control on them. If we consider the system commutative and/or associative overgeneration problems arise. If we do not the system will undergenerate.

1. The book that Dodgson wrote $\in np$?

$$\dfrac{\text{the} \vdash np/n \qquad \dfrac{\text{book} \vdash n \qquad \dfrac{\text{that} \vdash (n\backslash n)/(s/np) \qquad \dfrac{\begin{array}{c}[x_1 \vdash np]^1 \\ \vdots \\ \text{D (wrote } x_1) \vdash s\end{array}}{\text{D wrote} \vdash s/np}\,[/I]^1}{\text{that (D wrote)} \vdash s/np}\,[/E]}{\text{book (that (D wrote))} \vdash n}\,[\backslash E]}{\text{the (book (that (D wrote)))} \vdash np}\,[/E]$$

2. that Dogson dedicated to Liddell $\in n\backslash n$

$$\dfrac{\text{that} \vdash (n\backslash n)/(s/np) \quad \dfrac{\dfrac{[x \vdash np]^1}{\vdots} \atop (\text{D (dedicated } x)(\text{to L})) \vdash s}{\text{D (dedicated (to L))} \vdash s/np}[/\text{I}]}{\text{that (D (dedicated (to L)))} \vdash n\backslash n}[\backslash \text{E}]$$

3. The Mad Hatter loves himself vs. * The Mad Hatter thinks Alice loves himself.

$$\dfrac{\dfrac{\text{think} \vdash (np\backslash s)/s \quad \overset{\vdots}{\text{Alice (loves } x) \vdash s}}{\text{thinks (Alice (loves } x)) \vdash np\backslash s}[/\text{E}]}{\text{thinks (Alice loves)} \vdash (np\backslash s)/np}[/\text{I}]$$

$$\text{himself} \vdash ((np\backslash s)/np)\backslash(np\backslash s) : \lambda P_{\text{tv}} z_{np}.P(z)(z)$$

$$(\underbrace{\text{The Mad Hatter}}_{np})((\underbrace{\text{loves}}_{tv})\text{himself}) = (\underbrace{\text{The Mad Hatter}}_{np})((\underbrace{\text{thinks Alice loves}}_{tv})\text{himself})$$

8. Multimodal Lambek Calculus

Frames $F = \langle W, R^2, R^3 \rangle$

W: 'signs', resources, expressions

R^3: 'Merge', grammatical composition

R^2: 'feature checking', structural control

Models $\mathcal{M} = \langle F, V \rangle$

Valuation $V : \mathsf{TYPE} \mapsto \mathcal{P}(W)$: types as sets of expressions

Interpretation of the constants

$$\begin{array}{rcl} V(\diamond_s A) & = & \{x \mid \exists y(R^2_s xy \;\&\; y \in V(A)\} \\ V(\boxdot_s^{\downarrow} A) & = & \{x \mid \forall y(R^2_s yx \;\Rightarrow y \in V(A)\} \end{array}$$

$$\begin{array}{rcl} V(C/B) & = & \{x \mid \forall y \forall z[(R^3 zxy \;\&\; y \in V(B)) \Rightarrow z \in V(C)]\} \\ V(A \backslash C) & = & \{y \mid \forall x \forall z[(R^3 zxy \;\&\; x \in V(A)) \Rightarrow z \in V(C)]\} \end{array}$$

9. Proof System

- **Logic Rules**: Besides the logic rules of $(\backslash, /)$ we have the introduction and elimination rules for the unary operators $(\Diamond_s, \Box_s^{\downarrow})$

$$\frac{\Delta \vdash \Diamond_s A \quad \Gamma[\langle A \rangle^s] \vdash B}{\Gamma[\Delta] \vdash B}\,[\Diamond_s E] \qquad \frac{\Gamma \vdash A}{\langle \Gamma \rangle^s \vdash \Diamond_s A}\,[\Diamond_s I]$$

$$\frac{\Gamma \vdash \Box_s^{\downarrow} A}{\langle \Gamma \rangle^s \vdash A}\,[\Box_s^{\downarrow} E] \qquad \frac{\langle \Gamma \rangle^s \vdash A}{\Gamma \vdash \Box_s^{\downarrow} A}\,[\Box_s^{\downarrow} I]$$

- **Structural Rules**:

Distribution

$$\frac{\Gamma[\langle \Delta_1 \circ \Delta_2 \rangle^u] \vdash A}{\Gamma[\langle \Delta_1 \rangle^u \circ \langle \Delta_2 \rangle^u] \vdash A}\,[\mathrm{Pol}_u]$$

Computation

$$\frac{\Gamma[\langle \langle \Delta \rangle^u \rangle^s] \vdash A}{\Gamma[\langle \Delta \rangle^v] \vdash A}\,[\mathrm{Pol}_{u,s}]$$

where $s, u, v \in \{+, -\}$ and $v = sg(u, s)$.