Logic: Propositional Logic Tableaux

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Contents

1. Summary of last lesson

We have studied:

- Interpretation function [exercises]
- Domain [exercises]
- Model [exercises]
- Entailment [exercises]
- Satisfiability, unsatisfiability, validity [exercises]
- Counter-example [exercises]

2. Satisfiability and Validity

An interpretation $\mathcal{I}$ is a **model** of ϕ: $\mathcal{I} \models \phi$
A formula ϕ is

- **satisfiable**, if there is some $\mathcal{I}$ that satisfies ϕ,
- **unsatisfiable**, if ϕ is not satisfiable,
- **falsifiable**, if there is some $\mathcal{I}$ that does not satisfy ϕ,
- **valid** (i.e., a **tautology**), if every $\mathcal{I}$ is a model of ϕ.

3. Tableaux Calculus

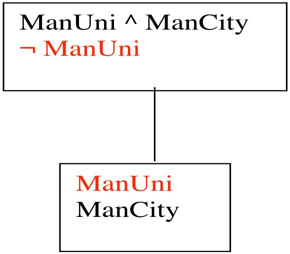
- The Tableaux Calculus is a decision procedure solving the problem of satisfiability.
- If a formula is satisfiable, the procedure will constructively exhibit a model of the formula.
- The basic idea is to incrementally build the model by looking at the formula, by decomposing it in a top/down fashion. The procedure exhaustively looks at all the possibilities, so that it can eventually prove that no model could be found for unsatisfiable formulas.

  Satisfiability If we have to check whether a set of statements *is satifiable* we build the tableaux and check whether there is *at least one branch that does not close*. If it exists, this branch gives us the interpretation that satisfies the statements. Otherwise the set is unsatisfiable.

  We have to understand what does it mean that a branch closes.

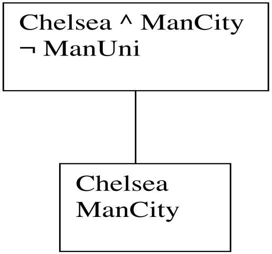
4. Example

KB= ManUni ∧ ManCity, ¬ ManUni



Closed

KB= Chelsea ∧ ManCity, ¬ ManUni



5. The Calculus

$$\frac{\phi \wedge \psi}{\begin{array}{c}\phi \\ \psi\end{array}}$$

If a model satisfies a conjunction, then it also satisfies *each of* the conjuncts

$$\frac{\phi \vee \psi}{\phi \mid \psi}$$

If a model satisfies a disjunction, then it also satisfies *one of* the disjuncts. It is a non-deterministic rule, and it generates two *alternative* branches of the tableaux.

6. Summary: all rules

$\begin{array}{c} A \wedge B \\ A \\ B \end{array}$	$\begin{array}{ccc} & A \vee B & \\ & \diagup \quad \diagdown & \\ A & & B \end{array}$	$\begin{array}{ccc} & A \rightarrow B & \\ & \diagup \quad \diagdown & \\ \neg A & & B \end{array}$
$\begin{array}{ccc} & A \leftrightarrow B & \\ & \diagup \quad \diagdown & \\ A \wedge B & & \neg A \wedge \neg B \end{array}$	$\begin{array}{c} \neg\neg A \\ A \end{array}$	$\begin{array}{ccc} & \neg(A \wedge B) & \\ & \diagup \quad \diagdown & \\ \neg A & & \neg B \end{array}$
$\begin{array}{c} \neg(A \vee B) \\ \neg A \\ \neg B \end{array}$	$\begin{array}{c} \neg(A \rightarrow B) \\ A \\ \neg B \end{array}$	$\begin{array}{ccc} & \neg(A \leftrightarrow B) & \\ & \diagup \quad \diagdown & \\ A \wedge \neg B & & \neg A \wedge B \end{array}$

7. Models

- The *not closed branch* of the tree gives a *model* of KB: the KB is *satisfiable*. Since all formulas have been reduced to literals (i.e., either positive or negative atomic propositions), it is possible to find an assignment of truth and falsity to atomic sentences which make all the sentences in the branch true.
- If *every branch closes*, then it is not possible to find an assignment making the original KB true: the KB is *unsatisfiable*. In fact, the original formulas from which the tree is constructed can not be true simultaneously.

8. Exercises

Determine the status of the following set of logical forms by means of truth tables first and then by tableaux method:

$$\{\neg B \rightarrow B, \neg(A \rightarrow B), \neg A \vee \neg B\}$$

$$\{\neg A \vee B, \neg(B \wedge \neg C), C \rightarrow D, \neg(\neg A \vee D)\}$$

9. Heuristics

Apply non-branching rules before branching rules.

Efficiency: order of rule applications. E.g.

$$KB = p \wedge q, \neg p, (a \wedge b) \wedge c$$

10. Efficiency: comparison with truth tables

- The complexity of truth tables depends on the number of atomic formulas appearing in the KB,
- the complexity of tableaux depends on the syntactic structure of the formulas in KB.

Try:

$KB = ((p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee r) \wedge (\neg p \vee \neg r))$

11. Tableaux as a Decision Procedure

Tableaux is a decision procedure for computing satisfiability, validity, and entailment in propositional logics:

- it is a sound algorithm
- it is a complete algorithm

12. Home work

- Study Kelly: Ch. 2
- Tomorrow 2 hrs Lab!!! Truth Tables vs. Tableaux.
- Next Lecture (3rd of November): again Tableaux, more on refutation, and more complex problems.