# Logic: Propositional Logic
# Truth Tables

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# Contents

# 1. Summary of last lesson

Last time we said:

- Syntax of PL: atomic vs. complex formulas [exercises]
- Semantics of PL: truth tables [exercises]
- Formalization of an argument [exercises]
- Interpretation function [again next time]
- Domain [again next time]
- Model [again next time]
- Entailment [again next time]
- Satisfiability [again next time]

# 2. Satisfiability and Validity

An interpretation $\mathcal{I}$ is a **model** of $\phi$: $\mathcal{I} \models \phi$
A formula $\phi$ is

- **satisfiable**,

  if there is some $\mathcal{I}$ that satisfies $\phi$,

- **unsatisfiable**,

  if $\phi$ is not satisfiable,

- **falsifiable**,

  if there is some $\mathcal{I}$ that does not satisfy $\phi$,

- **valid** (i.e., a **tautology**),

  if every $\mathcal{I}$ is a model of $\phi$.

Two formulas are **logically equivalent**

$(\phi \equiv \psi)$, if for all $\mathcal{I}$:

$$\mathcal{I} \models \phi \text{ iff } \mathcal{I} \models \psi$$

# 3. Exercise

Satisfiable, tautology?

$$
\begin{gathered}
(((a \wedge b) \leftrightarrow a) \rightarrow b) \\
((\neg \phi \rightarrow \neg \psi) \rightarrow (\psi \rightarrow \phi)) \\
(a \vee b \vee \neg c) \wedge (\neg a \vee \neg b \vee d) \wedge (\neg a \vee b \vee \neg d)
\end{gathered}
$$

Equivalent?

$$
\begin{aligned}
\neg(\phi \vee \psi) &\equiv \neg \phi \wedge \neg \psi \\
(\phi \rightarrow \psi) &\equiv (\neg \psi \rightarrow \neg \phi) \\
(\phi \vee (\psi \wedge \chi)) &\equiv ((\phi \vee \psi) \wedge (\psi \wedge \chi))
\end{aligned}
$$

Try to use **truth tables** to support your conclusions.

# 4. Consequences

**Proposition:**

- $\phi$ is a tautology iff $\neg\phi$ is unsatisfiable
- $\phi$ is unsatisfiable iff $\neg\phi$ is a tautology.

**Proposition:** $\phi \equiv \psi$ iff $\phi \leftrightarrow \psi$ is a tautology.

**Theorem:** If $\phi$ and $\psi$ are equivalent, and $\chi'$ results from replacing $\phi$ in $\chi$ by $\psi$, then $\chi$ and $\chi'$ are equivalent.

# 5. Exercise: Counter-example

Recall last time exercise:

Check whether the following argument is valid:

> If Paul lives in Dublin, he lives in Ireland. Paul lives in Ireland. Therefore Paul lives in Dublin.

(i) Give the keys of your formalization using PL; (ii) represent the argument formally; and (iii) Apply the truth table method to prove or disprove the validity of the argument.

Now, iv) Build a counterexample if the argumentation is not valid.

counterexample a model in which the reasoning does not hold. In other words, an interpretation such that the premises are true and the conclusion is false.

# 6. Exercise

Given the graph:

- Italy is connected to Austria
- Austria is connected to Hungary

By means of truth tables, find a coloring function $C : V \rightarrow \{\text{black}, \text{white}\}$ such that no two adjacent countries have the same color.

## 6.1. Graph Coloring Problem

First of all, we have to formalize our general knowledge about graphs and color assignments. Let $1, 2 \ldots n$ be our vertices, and $B$ (blue), $R$ (red), $G$ (green) $\ldots$ be our colors, and let $B_i, C_{i,j}$ stand for "$i$ is of color $B$" and "$i$ is connected to $j$".

- **Reflexivity of edges**: $C_{1,2} \leftrightarrow C_{2,1}, \ldots$ (so for the other connected vertices)
- **Coloring of vertices**: $B_1 \vee G_1 \vee R_1 \ldots$ (so for the colors and for the other vertices)
- **Uniqueness of colors per veritex**: $B_1 \leftrightarrow (\neg G_1 \wedge \neg R_1) \ldots$ (so for the other vertices)

Secondly, we have to add the explicitly given constraints.

For instance, the coloring function has to be such that no two connected vertices have the same color:

- **Explicit Constraint**: $C_{1,2} \rightarrow (B_1 \rightarrow \neg B_2) \wedge (R_1 \rightarrow \neg R_2) \wedge (G_1 \rightarrow \neg G_2) \ldots$ (so for the other colors and the other vertices)

## 6.2. Solution

Vertex axioms: (the dual are redundant) [Uniqueness constraint]
$(B_I \leftrightarrow \neg W_I)$
$(B_H \leftrightarrow \neg W_H)$
$(B_A \leftrightarrow \neg W_A)$

Edge axioms: (the dual are redundant) [Explicit Constraint]
$(B_I \leftrightarrow \neg B_A) \wedge (W_I \leftrightarrow \neg W_A)$
$(B_H \leftrightarrow \neg B_A) \wedge (W_H \leftrightarrow \neg W_A)$

By means of an inference procedure we can find the interpretation satisfying the theory. (The problem is satisfiable.)

# 7. Classical Satisfiability Problem: Graph Coloring

Let $KB$ (the set of sentences) be a theory consisting of the general knowledge about colors and graphs, and the constraint that the coloring function has to be such that no two connected vertices have the same color. We could be faced with the following kind of problems:

(A) Given a graph and a coloring function, check whether they satisfy $KB$.

(B) Given a graph, assign colors which satisfy $KB$.

(C) Find a model (a graph and a color assignment) satisfying the KB.

# 8. Summary: Reasoning Problems

## 8.1. Entailment

You are asked to check the validity of a given reasoning, i.e. whether the given set of axioms (premises, $KB$) entails the given sentence (conclusion, $\alpha$) $KB \models \alpha$.

If it does not you could be asked to provide a *counterexample*.

Strategy To solve the problem you have to:

- represent the sentences in a formal language, `PL`.
- use an inference procedure (e.g. Truth Tables or Tableaux) to answer the question.
- prove that the conclusion is true in all cases (for all the interpretations) in which the premises are (all) true.

## 8.2. Satisfiability

(A) You are given a set of sentences and an interpretation (i.e. a domain and an interpretation function), and are asked to *check* whether the sentences are all true in this interpretation.

(B) You are given a set of sentences and a domain, and are asked to *find an interpretation* for the domain satisfying the given sentences.

(C) You are given a set of sentences and are asked to *provide a model* (ie. a domain and an interpretation function satisfying them)

Strategy To solve the problem, you have to:

- represent the axioms in a formal language, `PL`.
- use an inference procedure (e.g. Truth Tables or Tableaux) to answer the question.
- prove that the (found or given) interpretation is such that the set of sentences are all true simultaneously. (To prove validity you have to prove that this holds for all the interpretations.)

# 9. Other kinds of exercises

- Find an interpretation and a formula such that the formula is true in that interpretation (or: the interpretation satisfies the formula).
- Find an interpretation and a formula such that the formula is not true in that interpretation (or: the interpretation does not satisfy the formula).
- Find a formula which can't be true in any interpretation (or: no interpretation can satisfy the formula).

# 10. Properties of Entailment

- $\Theta \cup \{\phi\} \models \psi$ iff $\Theta \models \phi \to \psi$
  $\boxed{\text{(Deduction Theorem)}}$
- $\Theta \cup \{\phi\} \models \neg\psi$ iff $\Theta \cup \{\psi\} \models \neg\phi$
  (Contraposition Theorem)
- $\Theta \cup \{\phi\}$ is unsatisfiable iff $\Theta \models \neg\phi$
  (Contradiction Theorem)

**Question**: Can we decide whether $\Theta \models \psi$ without considering all interpretations?

# 11. Decision Procedures in Logic: soundness

A **decision procedure** solves a problem with YES or NO answers. Our problem is "entailment".

$$\boxed{KB \vdash_i \alpha}$$

- Sentence $\alpha$ can be derived from the set of sentences $KB$ by procedure $i$.
- *Soundness*: procedure $i$ is sound if
  whenever procedure $i$ proves that a sentence $\alpha$ can be derived from a set of sentences $KB$ ($KB \vdash_i \alpha$), then it is also true that $KB$ entails $\alpha$ ($KB \models \alpha$).

  It guarantees that "IF $KB \vdash_i \alpha$ THEN $KB \models \alpha$".

  - "no wrong inferences are drawn"
  - A sound procedure may fail to find the solution in some cases, when there is actually one; i.e. we could have:

$$KB \nvdash_i \alpha \quad \text{and} \quad KB \models \alpha$$

# 12. Decision Procedures in Logic: completeness

A **decision procedure** solves a problem with YES or NO answers. Our problem is "entailment".

$$\boxed{KB \vdash_i \alpha}$$

- Sentence $\alpha$ can be derived from the set of sentences $KB$ by procedure $i$.
- *Completeness*: procedure $i$ is complete if whenever a set of sentences $KB$ entails a sentence $\alpha$ ($KB \models \alpha$), then procedure $i$ proves that $\alpha$ can be derived from $KB$ ($KB \vdash_i \alpha$).

  It guarantees that "IF $KB \models \alpha$ THEN $KB \vdash_i \alpha$".

  - "all the correct inferences are drawn"
  - A complete procedure may claim to have found a solution in some cases, when there is actually no solution, i.e. we could have:

$$KB \not\models \alpha \quad \text{and} \quad KB \vdash_i \alpha$$

# 13. Sound and Incomplete Algorithms

Sound and incomplete algorithms are very popular:

- they are considered *good* approximations of problem solving procedures.
- they may have a lower complexity than a sound and complete algorithm.
- they are often used due to the inability of programmers to find sound and complete algorithms or the impossibility of finding sound and complete algorithms (i.e. when the problem is undecidable)

# 14. Good Decision procedures

- If an incomplete reasoning mechanism is provided, we can conclude either that the semantics of the representation language does not really capture the meaning of the "world" and of "*what should follow*", or that the algorithms can not infer all the things we would expect.

- Having **sound and complete** reasoning procedures is important!

  they guarantee that "$KB \vdash_i \alpha$ IFF $KB \models \alpha$".

# 15. An extreme example

Let's consider two decision procedures:

- $F$, which always returns the result NO independently from its input
- $T$, which always returns the result YES independently from its input

Let's consider the problem of computing entailment between formulas;

Which of the two procedures is sound and which is complete?

- $F$ is a sound algorithm for computing entailment.
- $T$ is a complete algorithm for computing entailment.

# 16. Propositional Decision Procedures

- Truth tables provide a sound and complete decision procedure for testing satisfiability, validity, and entailment in propositional logic.
  - The proof is based on the observation that truth tables enumerate all possible models.
- Satisfiability, validity, and entailment in propositional logic are thus *decidable* problems.
- For problems involving a large number of atomic propositions the amount of calculation required by using truth tables may be prohibitive (always $2^n$, where $n$ is the number of atomic proposition involved in the formulas).

## 16.1. Recall

- A problem is either decidable (there is always an answer) or undecidable.
- A decision procedure is a program that always terminates and gives a yes or no answer.
- A decision procedure may be sound and/or complete with respect to a given problem.
- If we have a decision procedure that is sound and complete w.r.t. a given problem, then the problem is decidable.

# 17. Reduction to satisfiability

- A formula $\phi$ is satisfiable iff there is some interpretation $\mathcal{I}$ (i.e., a truth value assignment) that satisfies $\phi$ (i.e., $\phi$ is true under $\mathcal{I}$: $\mathcal{I} \models \phi$).
- Validity, equivalence, and entailment can be reduced to satisfiability:
  - $\phi$ is a valid (i.e., a tautology) iff
    $\neg\phi$ is not satisfiable.
  - $\phi$ entails $\psi$ ($\phi \models \psi$) iff $\phi \rightarrow \psi$ is valid (*deduction theorem*).
    - $\phi \models \psi$ iff
      $\phi \wedge \neg\psi$ is not satisfiable.
  - $\phi$ is equivalent to $\psi$ ($\phi \equiv \psi$) iff $\phi \leftrightarrow \psi$ is valid.
    - $\phi \equiv \psi$ iff $\phi \models \psi$ and $\psi \models \phi$
- Hence, a *sound and complete* procedure deciding satisfiability is all we need, and the *tableaux method* is a decision procedure which checks the existence of a model.

# 18. Home work

- Bring at the Lab (October 27th) the solutions for the exercises.
- Today key topics:
  - Interpretation function [exercises]
  - Domain [exercises]
  - Model [exercises]
  - Entailment [exercises]
  - Satisfiability, unsatisfiability, validity [exercises]
  - Counter-example [exercises]
- Next time: Tableaux method. (26.10!!)