Logic: Propositional Logic
Truth Tables

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Contents

1. Knowledge bases

Inference engine	$\longleftarrow$	domain-independent algorithms
Knowledge base	$\longleftarrow$	domain-specific content

- Knowledge base = set of *sentences* in a *formal* language = logical *theory*
- *Declarative* approach to building an intelligent agent:
  TELL it what it needs to know
- Then it can ASK itself what to do—answers should follow from the KB
- Agents can be viewed at the *knowledge level*
  i.e., what they know, regardless of how implemented
- Or at the *implementation level*
  i.e., data structures in KB and algorithms that manipulate them

2. Logic in general

- *Logics* provide formal languages for representing and reasoning with information
- *Syntax* defines the sentences in the language
- *Semantics* define the "meaning" of sentences; i.e., define *truth* of a sentence in a *world*
- E.g., given a language for arithmetic, we can interpret the following formal expressions in the 'world' of natural numbers

  $x + 2 \geq y$ is a sentence; $x2 + y >$ is not a sentence

  $x + 2 \geq y$ is true iff the number $x + 2$ is no less than the number y

  $x + 2 \geq y$ is true in a world if $x = 7, \ y = 1$

  $x + 2 \geq y$ is false in a world if $x = 0, \ y = 6$

  $x + 2 \geq x + 1$ is true in every world independently of the integers assigned to x and y

3. Entailment – Logical Implication

$$\boxed{KB \models \alpha}$$

- Knowledge base KB *entails* sentence α
  if and only if
  α is true in all worlds where KB is true

- E.g., the KB containing "Manchester United won" and "Manchester City won" entails
  "Either Manchester United won or Manchester City won"

4. Propositional Logics: Basic Ideas

Statements:

The elementary building blocks of propositional logic are *atomic statements* that cannot be decomposed any further: *propositions*. E.g.,

- “The block is red”
- “The proof of the pudding is in the eating”
- “It is raining”

and logical connectives “and”, “or”, “not”, by which we can build **propositional formulas**.

5. Propositional Logics: Reasoning

We are interested in the questions:

- when is a statement **logically entailed** by a set of statements, in symbols: $\Theta \models \phi$
- can we define **deduction** in such a way that deduction and entailment coincide?

6. Syntax of Propositional Logic

Countable alphabet Σ of **atomic propositions**: $a, b, c, \ldots$.

Propositional formulas:

$$
\begin{array}{rcll}
\phi, \psi & \longrightarrow & a & \textit{atomic propositions} \\
& \mid & \bot & \textit{false} \\
& \mid & \top & \textit{true} \\
& \mid & \neg\phi & \textit{negation} \\
& \mid & \phi \wedge \psi & \textit{conjunction} \\
& \mid & \phi \vee \psi & \textit{disjunction} \\
& \mid & \phi \rightarrow \psi & \textit{implication} \\
& \mid & \phi \leftrightarrow \psi & \textit{equivalence}
\end{array}
$$

7. From English to Propositional Logic

Eg. If you don't sleep then you will be tired.

Keys: $p =$ you sleep, $q=$ you will be tired. Formula: $\neg p \to q$.

Exercise I:

1. If it rains while the sun shines, a rainbow will appear
2. Charles comes if Elsa does and the other way around
3. Johan comes just when Peter stays at home
4. We are going, unless it is raining
5. Charles and Elsa are brother and sister or nephew and niece
6. If I have lost if I cannot make a move, then I have lost.

Use: http://www.earlham.edu/~peters/courses/log/transtip.htm

8. Semantics: Intuition

- Atomic propositions can be *true* `T` or *false* `F`.
- The truth value of formulas is determined by the truth values of the atoms (*truth value assignment* or *interpretation*).

Example: $(a \vee b) \wedge c$

- If a and b are false and c is true, then the formula is not true.
- Then *logical entailment* could be defined as follows:

  ϕ is entailed by Θ, if ϕ is true in all "the worlds", in which Θ is true.

9. Semantics: Formally

A **truth value assignment** (or **interpretation**) of the atoms in Σ is a function $\mathcal{I}$:

$$\mathcal{I} \colon \Sigma \longrightarrow \{\mathtt{T}, \mathtt{F}\}.$$

Instead of $\mathcal{I}(a)$ we also write $a^{\mathcal{I}}$.

A formula ϕ is *satisfied* by an interpretation $\mathcal{I}$ ($\mathcal{I} \models \phi$) or is *true* under $\mathcal{I}$:

$$\begin{array}{rcl}
\mathcal{I} \models \top & & \\
\mathcal{I} \not\models \bot & & \\
\mathcal{I} \models a & \text{iff} & a^{\mathcal{I}} = \mathtt{T} \\
\mathcal{I} \models \neg\phi & \text{iff} & \mathcal{I} \not\models \phi \\
\mathcal{I} \models \phi \wedge \psi & \text{iff} & \mathcal{I} \models \phi \text{ and } \mathcal{I} \models \psi \\
\mathcal{I} \models \phi \vee \psi & \text{iff} & \mathcal{I} \models \phi \text{ or } \mathcal{I} \models \psi
\end{array}$$

$$\begin{array}{rcl}
\mathcal{I} \models \phi \rightarrow \psi & \text{iff} & \text{if } \mathcal{I} \models \phi, \text{ then } \mathcal{I} \models \psi \\
\mathcal{I} \models \phi \leftrightarrow \psi & \text{iff} & \mathcal{I} \models \phi, \text{ if and only if } \mathcal{I} \models \psi
\end{array}$$

10. Truth Tables

ϕ	$\neg\phi$
T	F
F	T
	(1)

ϕ	ψ	$\phi \wedge \psi$
T	T	T
T	F	F
F	T	F
F	F	F
		(1)

ϕ	ψ	$\phi \vee \psi$
T	T	T
T	F	T
F	T	T
F	F	F
		(1)

ϕ	ψ	$\phi \longrightarrow \psi$
T	T	T
T	F	F
F	T	T
F	F	T
		(1)

11. Model

Intuitively an interpretation is a situation: it contains the kind of things we want to speak about. It gives us two pieces of information:

- It tells us which collection of atomic propositions we are talking about (*domain*, D),
- and for each formula it gives us an appropriate *semantic value*, this is done by means of a function called *interpretation function* ($\mathcal{I}$).

Thus an interpretation is a pair: $(D, \mathcal{I})$.

An **interpretation function** (or **truth value assignment**) of the atoms in D is a function $\mathcal{I}$:

$$\mathcal{I} \colon D \to \{\mathtt{T}, \mathtt{F}\}.$$

The truth value of a complex formula depends on the truth values of its parts. We have seen that for each connective this is prescribed in a *truth table*.

An interpretation m is a *model* of a sentence α if α is *true* in m.

$$M(\alpha) \text{ is the set of all models of } \alpha.$$

i.e. the set of all the interpretations where α is true.

12. Reasoning: Entailment and Satisfiability

The main concepts to focus attention on are: (1) Entailment and (2) Satisfiability.

(1) Entailment:

$$KB \models \alpha \quad \text{iff} \quad M(KB) \subseteq M(\alpha)$$

in words, Knowledge Base (KB) *entails* sentence α if and only if

α is **true** in all models of the KB (i.e. for all interpretations where KB is true).

(2) Satisfiability:

A set of statements is *satisfiable* if the statements *can* all be true simultaneously (i.e. there is one model of all statements). Otherwise it is unsatisfiable (i.e. there is no model of all statements).

A set of statements is a *tautology* if the statements are *always* all true simultaneously (i.e. for every interpretation).

12.1. Exercise II

Check whether the following argument is satisfiable:

> If the temperature and air pressure remained constant, there was no rain. The temperature did remain constant. Therefore, if there was rain then the air pressure did not remain constant.

(i) Give the keys of your formalization using `PL`; (ii) represent the argument formally, and (iii) Apply the truth table method to prove or disprove the satisifiability of the argument.

13. Home work

- Study Chapter 1 of Kelly
- Bring at the Lab (October 20th) the solutions for the exercises.
- Today key concepts
  - Syntax of PL: atomic vs. complex formulas [exercises]
  - Semantics of PL: truth tables [exercises]
  - Formalization of an argument [exercises]
  - Interpretation function [again next time]
  - Domain [again next time]
  - Model [again next time]
  - Entailment [again next time]
  - Satisfiability [again next time]