Logic: First Order Logic

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Contents

1. Summary of Last Lesson

What did we say last time about the points below?

- motivations for using `FOL`.
- introduction to the syntax of `FOL`.
- introduction to the semantics of `FOL`.

Today, we will look at entailment in `FOL` and the tableau calculus for `FOL`.

1.1. Remarks: Variables and Quantifiers

- In the wffs $\forall x.A$ and $\exists x.A$ A is said to be the *scope* of $\forall x$ and $\exists x$.
- Do you see any difference between the following two formulas w.r.t. their variables?

  1. $\forall y.(\exists x.(P(x, y) \wedge P(y)))$
  2. $\exists x.(P(x, y) \wedge P(y))$

- When a variable occurs within the scope of a quantifier it is said to be *bound*, otherwise it is said to be *free*.
  - In 1. both x and y are bound.
  - In 2. x is bound, while y is free.
- A formula is a *sentence* if no free variable occurs in it. Hence, 1. is a sentence, and 2. is not.

1.2. Remarks: Equivalences

How do you represent in FOL the following sentences?

1. All lectures teach Logic.

   $\forall x.\texttt{Lectures}(x) \rightarrow \texttt{Teach}(x, \texttt{logic})$

2. Not all lecturers teach Logic.

   $\neg(\forall x.\texttt{Lectures}(x) \rightarrow \texttt{Teach}(x, \texttt{logic}))$

3. Some lectures teach logic

   $\exists x.\texttt{Lectures}(x) \wedge \texttt{Teach}(x, \texttt{logic})$

4. Some lectures do not teach Logic

   $\exists x.\texttt{Lectures}(x) \wedge \neg\texttt{Teach}(x, \texttt{logic})$

Question: Are any of these sentences/formulas equivalent?

Yes: 2 and 4!

Question: What do you conclude? How can you generalize this claim?

$\neg\forall x.A = \exists x \neg A$

2. Semantics of FOL: Satisfiability of formulas

A formula ϕ is satisfied by (*is true in*) an interpretation $\mathcal{I}$ under a variable assignment α,
$\mathcal{I}, \alpha \models \phi$:

$$
\begin{array}{rcl}
\mathcal{I}, \alpha \models P(t_1, \ldots, t_n) & \text{iff} & ({t_1}^{\mathcal{I},\alpha}, \ldots, {t_n}^{\mathcal{I},\alpha}) \in P^{\mathcal{I}} \\
\mathcal{I}, \alpha \models \neg\phi & \text{iff} & \mathcal{I}, \alpha \not\models \phi \\
\mathcal{I}, \alpha \models \phi \wedge \psi & \text{iff} & \mathcal{I}, \alpha \models \phi \text{ and } \mathcal{I}, \alpha \models \psi \\
\mathcal{I}, \alpha \models \phi \vee \psi & \text{iff} & \mathcal{I}, \alpha \models \phi \text{ or } \mathcal{I}, \alpha \models \psi \\
\mathcal{I}, \alpha \models \forall x.\ \phi & \text{iff} & \text{for all } \ d \in \Delta : \\
 & & \mathcal{I}, \alpha[x/d] \models \phi \\
\mathcal{I}, \alpha \models \exists x.\ \phi & \text{iff} & \text{there exists a } \ d \in \Delta : \\
 & & \mathcal{I}, \alpha[x/d] \models \phi
\end{array}
$$

2.1. Exercise

Evaluate the following wff (well formed formula) and give a counter-example if the formula is falsifiable.

$$\forall x.\mathtt{E}(x) \rightarrow \forall y.\mathtt{G}(x, y)$$

- $\Delta = \{1, 2, 3, 4, \ldots\}$, i.e. the set of positive integers.
- $\mathtt{E}^{\mathcal{I}} = \{2, 4, \ldots\}$, i.e. the set of even integers.
- $\mathtt{G}^{\mathcal{I}} = \{(1, 2), (1, 3), (1, 4), (2, 3), \ldots\}$, i.e. the relation *greater than*.

Use the same domain and interpretation, to evaluate the following formulas:

$$\exists x.\forall y.\mathtt{G}(y, x) \quad \forall y.\exists x.\mathtt{G}(y, x)$$

In the domain and interpretation above, the first formula is false and the second formula is true. What do you conclude about order of quantifiers?

2.2. Remarks: Universal quantification

"Everyone in England is smart": $\forall x.\ \texttt{In}(x, \texttt{england}) \rightarrow \texttt{Smart}(x)$

$(\forall x.\ \phi)$ is equivalent to the *conjunction* of all possible *instantiations* in x of ϕ:

$$\begin{array}{ll} & \texttt{In}(\texttt{kingJohn}, \texttt{england}) \rightarrow \texttt{Smart}(\texttt{kingJohn}) \\ \wedge & \texttt{In}(\texttt{richard}, \texttt{england}) \rightarrow \texttt{Smart}(\texttt{richard}) \\ \wedge & \ldots \end{array}$$

Typically, $\rightarrow$ is the main connective with $\forall$.
Common mistake: using $\wedge$ as the main connective with $\forall$:

$$\forall x.\ \texttt{In}(x, \texttt{england}) \wedge \texttt{Smart}(x)$$

Why is a mistake? What does it mean?

it means "Everyone is in England and is smart". Note, it would entail the existence of the x.

2.3. Remarks: Existential quantification

"Someone in France is smart": $\exists x.\ In(x, france) \land Smart(x)$

$(\exists x.\ \phi)$ is equivalent to the *disjunction* of all possible *instantiations* in x of ϕ

$$
\begin{array}{ll}
 & \texttt{In}(\texttt{kingJohn}, \texttt{france}) \land \texttt{Smart}(\texttt{kingJohn}) \\
\lor & \texttt{In}(\texttt{richard}, \texttt{france}) \land \texttt{Smart}(\texttt{richard}) \\
\lor & \ldots
\end{array}
$$

Typically, $\land$ is the main connective with $\exists$.
Common mistake: using $\rightarrow$ as the main connective with $\exists$:

$$\exists x.\ \texttt{In}(x, \texttt{france}) \rightarrow \texttt{Smart}(x)$$

Why is it a mistake? What does it mean?

It would be true even if nobody is in France! It would not entail the existence of x.

3. Entailment

Entailment is defined similarly as in propositional logic.

The formula ϕ is logically implied by a formula ψ, if ϕ is true in all models of ψ (symbolically, $\psi \models \phi$):

$$\psi \models \phi \quad \text{iff} \quad \mathcal{I} \models \phi \ \text{ for all models } \mathcal{I} \ \text{ of } \psi$$

4. The Calculus

The completion rules for quantified formulas:

$$\frac{\forall x.\ \phi}{\begin{array}{c}\phi\{X/t\}\\ \forall x.\ \phi\end{array}}$$

If a model satisfies a universal quantified formula, the it also satisfies the formula where the quantified variable has been substituted with some term. The prescription is to use all the terms which appear in the tableau.

$$\frac{\exists x.\ \phi}{\phi\{X/a\}}$$

If a model satisfies an existential quantified formula, then it also satisfies the formula where the quantified variable has been substituted with a fresh new *Skolem* term.

4.1. Rules

(1) $A \land B$ A B	(2) $A \lor B$ $A \qquad B$	(3) $A \to B$ $\neg A \qquad B$
(4) $A \leftrightarrow B$ $A \land B \qquad \neg A \land \neg B$	(5) $\neg\neg A$ A	(6) $\neg(A \land B)$ $\neg A \qquad \neg B$
(7) $\neg(A \lor B)$ $\neg A$ $\neg B$	(8) $\neg(A \to B)$ A $\neg B$	(9) $\neg(A \leftrightarrow B)$ $A \land \neg B \qquad \neg A \land B$

(10) $\forall x.A(x)$ $A(t)$ where t is a term	(11) $\exists x.A(x)$ $A(t)$ where t is a term which has *not* been used in the derivation so far.
(12) $\neg\forall x(A(x))$ $\exists x(\neg A(x))$	(13) $\neg\exists x(A(x))$ $\forall x(\neg A(x))$

Finally, whenever a wff A and its negation $\neg A$ appear in a branch of a tableau, the unsatisfability is indicated in that branch and it is *closed*, i.e. it is not further extended.

4.2. Exercises

Prove that the following entailments are valid.

- $(P(c) \wedge T(c,e)) \vee (\neg P(c) \wedge T(e,c)) \models P(c) \rightarrow T(c,e)$
- $\exists x.\forall y.\mathtt{G}(x,y) \models \forall y.\exists x.\mathtt{G}(x,y)$

4.3. Exercises: Universal quantifier I

(a) Formalize the following problem and (b) check its validity.

All frogs are green. Everything is a frog. Therefore, Alf is green.

Let $F(x), G(x)$ and a stand for "x is a frog", "x is green", and "Alf". We may formalize the argument as:

$$\forall x.F(x) \rightarrow G(x), \forall x.F(x) \models G(a)$$

(b) Build the tableau.

4.4. Exercises: Universal Quantifiers II

Prove whether the following entailments are valid and give a counter-example (a domain and an interpretation) if they are not.

- $F(a) \rightarrow G(b), \forall x.\neg F(x) \models \neg G(b)$
- $\forall x.F(x) \rightarrow G(x), \forall x.G(x) \models F(a)$

4.5. Exercise: negated quantifiers

- $\forall x.F(x) \rightarrow G(x), \neg\exists x.G(x) \models \neg F(a)$
- $\neg\exists x.F(x) \wedge G(x) \models \neg F(a)$
- $\forall x.F(x) \rightarrow \forall x.G(x), \neg\exists x G(x) \models \exists x \neg F(x)$

4.6. Heuristics

Prove $\exists x.(\exists y.(R(x,y))) \to R(a,a)$ is valid.

1.	$\neg(\exists x.(\exists y.R(x,y))) \to R(a,a))$	
2.	$\exists x.(\exists y.(R(x,y)))$	Rule (8) applied to 1.
3.	$\neg R(a,a)$	Rule (8) applied to 1.
4.	$\exists y.R(a,y)$	Rule (11) applied to 2.
5.	$R(a,a)$	Rule (11) applied to 4.

Can we conclude that the $\exists x.(\exists y.(R(x,y))) \to R(a,a)$ is valid?

Let us interpret the above theorem as following:

- Let the natural numbers be our domain of interpretation.
- Let $R(x,y)$ stand for $x < y$

Then, $\exists x.(\exists y.(R(x,y)))$ is satisfiable. E.g. $2 < 4$.

From this it follows that $R(a,a)$ should be true as well, but it is not.

Hence, $\exists x.(\exists y.(R(x,y))) \to R(a,a)$ is not satisfiable in this interpretation, much less valid.

On line 5. Rule (11) has violeted the constraint: the term a was already used. Similarly, a could have not been used in line 4. either. Hence we don't get a contraddiction and the tableau is not closed.

Note : The same term can be used many times for universal instantiation.

$$\downarrow$$

Heuristic : When developing a semantic tableau in FOL use the existential instantiation rule (Rule 11) before the universal instantiation rule (Rule 10).

4.7. Heuristic (II)

1.	$\neg(\forall x.(A(x) \wedge B(x)) \rightarrow \forall x.A(x))$	
2.	$\forall x.(A(x) \wedge B(x))$	Rule (8) applied to 1.
3.	$\neg(\forall x.A(x))$	Rule (8) applied to 1.
4.	$\exists x.\neg A(x)$	Rule (12) applied to 3.
5.	$\neg A(a)$	Rule (11) applied to 4.
6.	$A(a) \wedge B(a)$	Rule (10) applied to 2.
7.	$A(a)$	Rule (1) applied to 6.
8.	$B(a)$	Rule (1) applied to 6.
	Closed	

Note: if we had used Rule 10 before we would have not been able to apply Rule 11.

5. Conclusion

Change in the schedule :

- 1st of December: two hours (09:30-11:30) with Rosella Gennari.
- 5th of December: one hour (17:00-18:00) with Raffaella Bernardi.
- 15th of December: three hours (08:30-11:30) with Raffaella Bernardi
- 22nd of December: one hour (08:30-09:30) to ask all your last minute doubts!(09:30-11:30): mid term (= 30% of the exam)!

NOTE: At end of the lesson some of the student proposed to have the mid term on the 15th of December. That's fine with me. You will receive a mail from Sara Mani about this change.