Logic: First Order Logic

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1. How far can we go with PL?

1.1. Inference

1. Casper is bigger than John
2. John is bigger than Peter
3. Therefore, Casper is bigger then Peter.

Questions:

How would you formalize this inference in PL?

What do you need to express that cannot be expressed in PL?

Answer:

You need to express: “relations” (is bigger than) and “entities” (Casper, John, Peter)

1.2. Inference

1. Bigger(casper,john)
2. Bigger(john,peter)
3. Therefore, Bigger(casper,peter)

Question: Do you still miss something?

1.3. Exercise: Predicates and entities

Formalize:

- John is bigger or Peter is bigger than John
- If Raffaella is speaking, then Rocco is listening
- If Peter is laughing, the John is not biting him.

Predicates represent sets of objects (entities). For example:

> "is listening" is the set of all those entities that are listening —i.e. all of you!

1.4. Housing lottery problem

Housing lotteries are often used by university housing administrators to determine which students get first choice of dormitory rooms.

Consider the following problem:

1. Bob is ranked immediately ahead of Jim.
2. Jim is ranked immediately ahead of a woman who is a biology major.
3. Lisa is not near to Bob in the ranking.
4. Mary or Lisa is ranked first.

Is it true that Jim is immediately ahead of Lisa and Lisa is the last of the ranking and Mary is the first?

Questions:
How would you formalize the problem in PL?

What do you need to express that you cannot express in PL?

It would be handy to say, e.g. that *there exists* a woman who is a biology major.

1.5. Graph Coloring Problem

Recall the Graph Coloring Problem: we have to formalize our general knowledge about graphs and color assignments.

We said, first of all let $1, 2 \dots n$ be our vertices, and B (blue), R (red), G (green) ... be our colors, and let $B_i, C_{i,j}$ stand for "i is of color B" and "i is connected to j".

- **Symmetry of edges**: $C_{1,2} \leftrightarrow C_{2,1}, \dots$ (so for the other connected vertices)
- **Coloring of vertices**: $B_1 \vee G_1 \vee R_1 \dots$ (so for the colors and for the other vertices)
- **Uniqueness of colors per veritex**: $B_1 \leftrightarrow (\neg G_1 \wedge \neg R_1) \dots$ (so for the other vertices)

Secondly, we have to add the explicitly given constraints: the coloring function has to be such that no two connected vertices have the same color:

- **Explicit Constraint**: $C_{1,2} \rightarrow (B_1 \rightarrow \neg B_2) \wedge (R_1 \rightarrow \neg R_2) \wedge (G_1 \rightarrow \neg G_2) \dots$ (so for the other colors and the other vertices)

Question: Which could be a way to express these properties of colors and edges in a "few words"?

We could speak to properties that hold *for all* colors, or all vertices!

1.6. Syllogism

1. Some four-legged creatures are genus
2. All genus are herbivores
3. Therefore, some four-legged creatures are herbivores

Question:
What is the schema behind the reasoning? What you could not express in PL even if we add relations and entity?

1. Some F are G
2. All G are H
3. Therefore, some F are H

We still need *quantifiers* to express "some" and "all"!

1.7. Quantifiers

Quantifiers like truth-functional operators ($\rightarrow, \neg, \vee, \wedge$) are logical operators; but instead of indicating relationships among sentences, they express relationships among the *sets* designated by predicates.

For example, statements like

> "All A are B" assert that the set A is a subset of the set B, $A \subseteq B$;

that is

> all the members of A are also members of B.

1.8. Quantifiers

Statements like

> "Some A are B" assert that the set A shares at least one member with the B $A \cap B \neq 0$;

Note, hence "Some A are B" is considered to be different from standard usage:

- at least one member
- it does not presuppose that "not all A are B"

1.9. Variables and Quantifiers

"All A are B" can be read as saying:

For all x, *if* x is A *then* x is B.

i.e. what we said before: all the members of A are also members of B, i.e A is included in B, $A \subseteq B$.

We write this as: $\forall x.A(x) \to B(x)$

"Some A are B" can be read as saying:

For some x, x is A *and* x is B.

i.e. what we said before: there exists at least a members of A that is also a members of B.

We write this as: $\exists x.A(x) \wedge B(x)$

1.10. Exercise

Housing lotteries are often used by university housing administrators to determine which students get first choice of dormitory rooms.

Consider the following problem:

1. Bob is ranked immediately ahead of Jim.
2. Jim is ranked immediately ahead of a woman who is a biology major.
3. Lisa is not near to Bob in the ranking.
4. Mary or Lisa is ranked first.

Formalize the statements, add statements that you would need to answer the question "is it true that Jim is listed immediately ahead of Lisa?"

1.11. Summing up: Motivations to move to FOL

- We can already do a lot with propositional logic.
- But it is unpleasant that we cannot access the *structure* of atomic sentences.
- Atomic formulas of propositional logic are *too atomic* – they are just statements which my be true or false but which have no internal structure.
- In *First Order Logic* (FOL) the atomic formulas are interpreted as statements about *relationships between objects*.

2. Syntax of full FOL

Formulas: $\phi, \psi \rightarrow$	$P(t_1, \ldots, t_n)$	atomic formulas
$\mid$	$\bot$	false
$\mid$	$\top$	true
$\mid$	$\neg\phi$	negation
$\mid$	$\phi \wedge \psi$	conjunction
$\mid$	$\phi \vee \psi$	disjunction
$\mid$	$\phi \rightarrow \psi$	implication
$\mid$	$\phi \leftrightarrow \psi$	equivalence
$\mid$	$\forall x.\ \phi$	*universal quantification*
$\mid$	$\exists x.\ \phi$	*existential quantification*

E.g. Everyone in England is smart: $\forall x.\ In(x, england) \rightarrow Smart(x)$

Someone in France is smart: $\exists x.\ In(x, france) \wedge Smart(x)$

2.1. Summary of Syntax of FOL

- Terms
  - variables
  - constants
- Literals
  - atomic formula
    - relation (predicate)
  - negation
- Well formed formulas
  - truth-functional connectives
  - existential and universal quantifiers

3. Domain and Interpretation

- Socrates, Plato, Aristotle are philosophers
- Mozart and Beethoven are musicians
- All of them are human beings
- Socrates knows Plato.
- Mozart knows Beethoven.

Which do you thing is the Domain of discourse?

What's the meaning of "knows", and of "musician", "philosopher" and "human beings"?

Are the statements below true or false in the above situation?

1. $\forall x.HumanBeings(x)$?
2. $\exists x.HumanBeings(x) \wedge Musicians(x)$?
3. $\forall x.Female(x) \rightarrow Musicians(x)$?

3.1. Semantics of FOL: intuition

- Just like in propositional logic, a (complex) FOL formula may be true (or false) with respect to a given interpretation.

- An interpretation specifies referents for
  constant symbols → **objects**
  predicate symbols → **relations**

- An atomic sentence $P(t_1, \ldots, t_n)$ is true in a given interpretation
  iff the *objects* referred to by $t_1, \ldots, t_n$
  are in the *relation* referred to by the predicate P.

- An interpretation in which a formula is true is called a *model* for the formula.

3.2. Semantic of FOL: Interpretations and Satisfaction

Interpretation: $\mathcal{I} = \langle \Delta, \cdot^{\mathcal{I}} \rangle$ where Δ is an arbitrary non-empty set and $\mathcal{I}\cdot$ is a function that maps

- individual constants to elements of Δ:
  $a^{\mathcal{I}} \in \Delta$
- n-ary predicate symbols to relation over Δ:
  $P^{\mathcal{I}} \subseteq \Delta^n$

Satisfaction of ground atoms $P(t_1, \ldots, t_n)$:

$$\mathcal{I} \models P(t_1, \ldots, t_n) \quad \text{iff} \quad ({t_1}^{\mathcal{I}}, \ldots, {t_n}^{\mathcal{I}}) \in P^{\mathcal{I}}$$

3.3. Examples

$$
\begin{aligned}
\Delta &= \{d_1, \ldots, d_n\} \\
\mathtt{a}^{\mathcal{I}} &= d_1 \\
\mathtt{b}^{\mathcal{I}} &= d_2 \\
\mathtt{Block}^{\mathcal{I}} &= \{d_1\} \\
\mathtt{Red}^{\mathcal{I}} &= \Delta \\
\mathcal{I} &\models \mathtt{Red(b)} \\
\mathcal{I} &\not\models \mathtt{Block(b)}
\end{aligned}
$$

$$
\begin{aligned}
\Delta &= \{1, 2, 3, \ldots\} \\
\mathtt{1}^{\mathcal{I}} &= 1 \\
\mathtt{2}^{\mathcal{I}} &= 2 \\
&\vdots \\
\mathtt{Even}^{\mathcal{I}} &= \{2, 4, 6, \ldots\} \\
\mathtt{succ}^{\mathcal{I}} &= \{(1 \mapsto 2), (2 \mapsto 3), \ldots\} \\
\mathcal{I} &\not\models \mathtt{Even}(3) \\
\mathcal{I} &\models \mathtt{Even}(\mathtt{succ}(3))
\end{aligned}
$$

3.4. Exercise

Given the Domain: $\Delta = \{b, c\}$ where b stands for "George Bush" and c for "Clinton". Given P and S that stand for "the class of all the twentieth-century US Presidents", and "the class of saxophone players", respectively. Given:

- $P^{\mathcal{I}} = \{b, c\}$
- $S^{\mathcal{I}} = \{c\}$

Is the following formula true?

$$\forall x.P(x) \rightarrow S(x)$$

motivate your answer.

- $P(c) \rightarrow S(c)$ is true, since $c \in P$ and $c \in S$
- $P(b) \rightarrow S(b)$ is false, since $b \in P$ but $b \notin S$

3.5. Semantics of FOL: Variable Assignments

V set of all variables. Function $\alpha\colon V \to \Delta$.

Notation: $\alpha[x/d]$ is identical to α except for the variable x.

Interpretation of terms *under* $\mathcal{I}, \alpha$:

$$\begin{aligned} x^{\mathcal{I},\alpha} &= \alpha(x) \\ a^{\mathcal{I},\alpha} &= a^{\mathcal{I}} \end{aligned}$$

Satisfiability of atomic formulas:

$$\mathcal{I}, \alpha \models P(t_1, \ldots, t_n) \quad \text{iff} \quad ({t_1}^{\mathcal{I},\alpha}, \ldots, {t_n}^{\mathcal{I},\alpha}) \in P^{\mathcal{I}}$$

3.6. Semantics of FOL: Satisfiability of formulas

A formula ϕ is satisfied by (*is true in*) an interpretation $\mathcal{I}$ under a variable assignment α,
$\mathcal{I}, \alpha \models \phi$:

$$
\begin{array}{rcl}
\mathcal{I}, \alpha \models P(t_1, \ldots, t_n) & \text{iff} & ({t_1}^{\mathcal{I},\alpha}, \ldots, {t_n}^{\mathcal{I},\alpha}) \in P^{\mathcal{I}} \\
\mathcal{I}, \alpha \models \neg\phi & \text{iff} & \mathcal{I}, \alpha \not\models \phi \\
\mathcal{I}, \alpha \models \phi \wedge \psi & \text{iff} & \mathcal{I}, \alpha \models \phi \text{ and } \mathcal{I}, \alpha \models \psi \\
\mathcal{I}, \alpha \models \phi \vee \psi & \text{iff} & \mathcal{I}, \alpha \models \phi \text{ or } \mathcal{I}, \alpha \models \psi \\
\mathcal{I}, \alpha \models \forall x.\ \phi & \text{iff} & \text{for all } d \in \Delta : \\
 & & \mathcal{I}, \alpha[x/d] \models \phi \\
\mathcal{I}, \alpha \models \exists x.\ \phi & \text{iff} & \text{there exists a } d \in \Delta : \\
 & & \mathcal{I}, \alpha[x/d] \models \phi
\end{array}
$$

3.7. Example

Find a model of the formula:

$$\exists y.\ [\ P(y) \wedge \neg Q(y)\ ] \wedge \forall z.\ [\ P(z) \vee Q(z)\ ]$$

$\Delta = \{a, b\}$
$P^{\mathcal{I}} = \{a\}$
$Q^{\mathcal{I}} = \{b\}$

4. Summing up

Exercise with

- FOL language.
- FOL domain and models

Next time we will look at Tableau Calculus for FOL.