Logic: Propositional Logic Tableaux

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1. Summary of last lesson

We have studied how to build tableaux, and how to use them:

- to prove satisfiability, unsatisfiability, validity of a given formula or a set of formulas. [exercises]
- to provide a model of a formula. [exercises]
- to provide a counter-example. [exercises]

Today we are going to use them to check whether $KB \models \psi$.

2. Weaker Results

- If ψ is *valid*, can we conclude it is satifiable, falsifiable or unsatisfiable?

  We can conclude ψ is *satisfiable*:

  $$\text{IF } \ \forall \mathcal{I}, \mathcal{I} \models \psi \ \text{ THEN } \ \exists \mathcal{I}, \mathcal{I} \models \psi$$

  Satisfiability is a weaker property then validity.

- If ψ is *unsatisfiable*, can we conclude it is satifiable, falsifiable or valid?

  We can conclude ψ is *falsifiable*:

  $$\text{IF } \ \forall \mathcal{I}, \mathcal{I} \not\models \psi \ \text{ THEN } \ \exists \mathcal{I}, \mathcal{I} \not\models \psi$$

  Fasiability is a weaker property then unsatisifiability.

2.1. Consequences

You are asked to prove whether ψ is valid by means of tableaux.

- If all branches of your tableaux are open, what do you conclude?

  ψ is satisfiable.

  Are you sure you cannot give a stronger answer, i.e. are you sure ψ is not valid?

  In order to check whether ψ is valid you have to look at $\neg\psi$.

  If $\neg\psi$ is unsatisfiable then ψ is also valid.

- If all branches close: ψ is unsatisfiable.

  Can you make a stronger claim?

  No this is already a strong result, there is no need to look at $\neg\psi$.

3. Properties of Entailment

- $\Theta \cup \{\phi\} \models \psi$ iff $\Theta \models \phi \to \psi$
  (Deduction Theorem)
- $\Theta \cup \{\phi\} \models \neg\psi$ iff $\Theta \cup \{\psi\} \models \neg\phi$
  (Contraposition Theorem)
- $\Theta \cup \{\phi\}$ is unsatisfiable iff $\Theta \models \neg\phi$
  (Contradiction Theorem)

4. Tableaux method: Refutation tree

Recall: $KB \models \alpha$ iff

$$M(KB) \subseteq M(\alpha)$$

in words, Knowledge Base (KB) *entails* sentence α if and only if α is **true** in all models of the KB (i.e. for all interpretations where KB is true).

We use the tableaux to search for invalidity: we negate the conclusion of the entailment and try to close the tree.

I.e. check whether

$$KB \cup \{\neg\alpha\}$$

is unsatisfiable.

If it is unsatisfiable, it means that there is no interpretation $\mathcal{I}$ s.t. $\mathcal{I} \models KB$ and $\mathcal{I} \models \neg\alpha$; in other words, for all interpretations the conclusion must be true when the premises are true. Hence, the entailment is valid.

5. Method

You have to prove: $KB \models \phi$.

You check whether $KB \cup \{\neg\phi\}$ is unsatisfiable.

1. Try to close the branch as soon as you can.
2. Once you have found a branch that is completed and does not close, you have found an interpretation that falsifies the entailment, it's a counter-example.Hence, you have proved that the entailment is not valid. You are done!

6. Examples

Recall our exercise.

Check whether the following argument is valid:

> If Paul lives in Dublin, he lives in Ireland. Paul lives in Ireland. Therefore Paul lives in Dublin.

(i) Give the keys of your formalization using PL; (ii) represent the argument formally; and (iii) Apply the truth table method to prove or disprove the validity of the argument.

Now, iv) Build a counterexample if the argumentation is not valid.

$$KB \models \alpha \qquad \{p \rightarrow i, i\} \models d$$

$$KB \cup \{\neg\alpha\} \qquad \{p \rightarrow i, i, \neg d\}$$

6.1. Exercises (I)

Are these entailments valid?

1. $\{P \vee Q, \neg P\} |= Q$
2. $\{P \rightarrow Q, Q\} |= P$
3. $\{P \rightarrow Q\} |= \neg(Q \rightarrow P)$
4. $\{P \rightarrow (\neg H \rightarrow C), P \rightarrow \neg H\} \models P \rightarrow C$
5. $\{J \vee Y, Y \rightarrow (\neg S \rightarrow C), \neg J \rightarrow S\} \models C$

6.2. Exercise (II)

You are given this text, ask to formalize it and prove it by mean of tableaux

John or Joyce or both will go to the party. If Joyce goes to the party then Clare will go unless Stephen goes. Stephen will go to the party if John does not go. Therefore, Clare will go to the party.

What do you do?

Let,

- J stand for "John will go to the party"
- Y stand for "Joyce will go to the party"
- C stand for "Clare will go to the party"
- S stand for "Stephen will go to the party"

We have:

1. "John or Joyce or both will go to the party" $J \vee Y$,
2. " If Joyce goes to the party then Clare will go unless Stephen goes" $Y \rightarrow (\neg S \rightarrow C)$
3. "Stephen will go to the party if John does not go" $\neg J \rightarrow S$

How do you formally write the entailment?

$$\{J \vee Y, Y \rightarrow (\neg S \rightarrow C), \neg J \rightarrow S\} \models C$$

How do you use tableaux?

You have to prove unsatisfiability of:

$$\{J \vee Y, Y \rightarrow (\neg S \rightarrow C), \neg J \rightarrow S\} \cup \{\neg C\}$$

6.3. Exercise (III)

To solve this problem, use propositional logic and the tableaux calculus.

You are required to explicitly define in advance your own atomic propositions and their informal meaning (i.e., explain the meaning of each atomic proposition with an english sentence).

You are also required to write down, using these propositions consistently, all the statements from the text that you choose to use in the proofs of (a) and (b).

Note that to answer the questions, it may not be necessary to encode all the information provided for you in the text; make sure that you do not mention in your solutions any statements that you do not use in the proofs of (a) and (b).

For your birthday, you were given a cute chameleon. You adore it, but you don't know it well. After doing some research in the library, here's what you have discovered:

1. your chameleon can be in three moods: it can be happy, upset, or indignant; it can be in only one mood at any given time;
2. in each mood, a chameleon takes on a specific color: in particular, when a chameleon is indignant or upset it turns purple;
3. chameleon always become upset when they are hungry;
4. chameleon always become indignant when they are busy eating and you suddenly start to pet them;
5. if the food bowl is empty then you can be sure your chameleon is hungry;
6. if the food bowl is not empty and a chameleon is hungry, it immediately engages itself in busy eating;
7. when a chameleon has slept enough it becomes happy.

(a) suppose you see that the food bowl is empty: show that it follows from the data above that your chameleon is not happy;

(b) if your chameleon is not purple when you are petting it, is the food bowl empty? (show that your answer follows from the data above)

6.4. Solution (a)

Let

- happy = ”the chameleon is happy”
- upset = ”the chameleon is upset”
- indignant = ”the chameleon is indignant”
- purple = ”the chameleon is purple”
- hungry = ”the chameleon is hungry”
- busyeating = ”the chameleon is busy eating”
- pet = ”you are petting the chameleon”
- empty = ”the food bowl is empty”

We can translate some of the information from the text of the problem into the following logical sentences:

R1. $happy \rightarrow \neg upset \wedge \neg indignant$

R2. $indignant \vee upset \rightarrow purple$

R3. $hungry \rightarrow upset$

R4. $empty \rightarrow hungry$

(a) asks to prove $KB \models \neg happy$

Besides the points in 1-7, in (a) we know that food bowl is empty, hence we add *empty* to our premises. We negate the conclusion and add: *happy*.

7. Reduction to satisfiability

- A formula ϕ is satisfiable iff there is some interpretation $\mathcal{I}$ (i.e., a truth value assignment) that satisfies ϕ (i.e., ϕ is true under $\mathcal{I}$: $\mathcal{I} \models \phi$).
- Validity, equivalence, and entailment can be reduced to satisfiability:
  - ϕ is a valid (i.e., a tautology) iff
    $\neg\phi$ is not satisfiable.
  - ϕ entails ψ ($\phi \models \psi$) iff $\phi \rightarrow \psi$ is valid (*deduction theorem*).
    * $\phi \models \psi$ iff
      $\phi \wedge \neg\psi$ is not satisfiable.
  - ϕ is equivalent to ψ ($\phi \equiv \psi$) iff $\phi \leftrightarrow \psi$ is valid.
    * $\phi \equiv \psi$ iff $\phi \models \psi$ and $\psi \models \phi$
- Hence, a *sound and complete* procedure deciding satisfiability is all we need, and the *tableaux method* is a decision procedure which checks the existence of a model.

Rules (Compare them with truth tables)

$A \wedge B$ A B	$A \vee B$ $A \qquad B$	$A \rightarrow B$ $\neg A \qquad B$
$A \leftrightarrow B$ $A \wedge B \qquad \neg A \wedge \neg B$	$\neg\neg A$ A	$\neg(A \wedge B)$ $\neg A \qquad \neg B$
$\neg(A \vee B)$ $\neg A$ $\neg B$	$\neg(A \rightarrow B)$ A $\neg B$	$\neg(A \leftrightarrow B)$ $A \wedge \neg B \qquad \neg A \wedge B$

8. Home work

- Study Tableaux method
- Exercises: tableaux calculus, formalization of problems.

Next time we start with First Order Logic!!