# Logic: First Order Logic

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# Contents

# 1. Equivalences I

How would you prove that the following equivalences hold?

| | | | |
|---|---|---|---|
| Commutativity | $\phi \vee \psi$ | $\equiv$ | $\psi \vee \phi$ |
| | $\phi \wedge \psi$ | $\equiv$ | $\psi \wedge \phi$ |
| | $\phi \leftrightarrow \psi$ | $\equiv$ | $\psi \leftrightarrow \phi$ |
| Associativity | $(\phi \vee \psi) \vee \chi$ | $\equiv$ | $\phi \vee (\psi \vee \chi)$ |
| | $(\phi \wedge \psi) \wedge \chi$ | $\equiv$ | $\phi \wedge (\psi \wedge \chi)$ |
| Idempotence | $\phi \vee \phi$ | $\equiv$ | $\phi$ |
| | $\phi \wedge \phi$ | $\equiv$ | $\phi$ |
| Absorption | $\phi \vee (\phi \wedge \psi)$ | $\equiv$ | $\phi$ |
| | $\phi \wedge (\phi \vee \psi)$ | $\equiv$ | $\phi$ |
| Distributivity | $\phi \wedge (\psi \vee \chi)$ | $\equiv$ | $(\phi \wedge \psi) \vee (\phi \wedge \chi)$ |
| | $\phi \vee (\psi \wedge \chi)$ | $\equiv$ | $(\phi \vee \psi) \wedge (\phi \vee \chi)$ |

## 1.1. Equivalences (II)

| | | | |
|---|---|---|---|
| Tautology | $\phi \vee \top$ | $\equiv$ | $\top$ |
| Unsatisfiability | $\phi \wedge \bot$ | $\equiv$ | $\bot$ |
| Negation | $\phi \vee \neg\phi$ | $\equiv$ | $\top$ |
| | $\phi \wedge \neg\phi$ | $\equiv$ | $\bot$ |
| Neutrality | $\phi \wedge \top$ | $\equiv$ | $\phi$ |
| | $\phi \vee \bot$ | $\equiv$ | $\phi$ |
| Double Negation | $\neg\neg\phi$ | $\equiv$ | $\phi$ |
| De Morgan | $\neg(\phi \vee \psi)$ | $\equiv$ | $\neg\phi \wedge \neg\psi$ |
| | $\neg(\phi \wedge \psi)$ | $\equiv$ | $\neg\phi \vee \neg\psi$ |
| Implication | $\phi \rightarrow \psi$ | $\equiv$ | $\neg\phi \vee \psi$ |

# 2. Normal Forms

*Conjunctive Normal Form* (CNF)

*conjunction* of $\underbrace{\textit{disjunctions} \text{ of } \textit{literals}}_{\textit{clauses}}$: $\bigwedge_{i=1}^{n}(\bigvee_{j=1}^{m} l_{i,j})$

E.g., $(A \vee \neg B) \wedge (B \vee \neg C \vee \neg D)$

*Disjunctive Normal Form* (DNF)

*disjunction* of $\underbrace{\textit{conjunctions} \text{ of } \textit{literals}}_{\textit{terms}}$: $\bigvee_{i=1}^{n}(\bigwedge_{j=1}^{m} l_{i,j})$

E.g., $(A \wedge B) \vee (A \wedge \neg C) \vee (A \wedge \neg D) \vee (\neg B \wedge \neg C) \vee (\neg B \wedge \neg D)$

Literals are either atoms or negations of an atom.

## 2.1. Conversion into CNF

How do we convert a formula into CNF?

1. Elimination of $\rightarrow$ and $\leftrightarrow$ by means of:
   - $A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$,
   - $A \rightarrow B \equiv \neg A \vee B$
2. push $\neg$ inwards by means of
   (a) $\neg(A \wedge B) \equiv \neg A \vee \neg B$ (De Morgan)
   (b) $\neg(A \vee B) \equiv \neg A \wedge \neg B$ (De Morgan)
   (c) $\neg\neg A \equiv A$ (double negation)
3. use the distributive law $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$ to effect the conversation to CNF.

## 2.2. Exercise

Convert the following formulas into CNF:

1. $(P \wedge Q) \leftrightarrow (R \rightarrow (P \rightarrow \neg Q))$
2. $(\neg P \wedge (\neg Q \rightarrow R)) \leftrightarrow S)$

## 2.3. Why Normal Forms?

- We can transform propositional formulas, in particular, we can construct their CNF and DNF.
- DNF tells us something as to whether a formula is satisfiable. If all disjuncts contain $\perp$ or complementary literals, then no model exists. Otherwise, the formula is satisfiable.
- CNF tells us something as to whether a formula is a tautology. If all clauses (= conjuncts) contain $\top$ or complementary literals, then the formula is a tautology. Otherwise, the formula is falsifiable.

**But**:

- the transformation into DNF or CNF is expensive (in time/space)
- it is only possible for finite sets of formulas

# 3. Tableau Calculus for PL

Which rules do we need?

$$\frac{\phi \wedge \psi}{\begin{array}{c}\phi \\ \psi\end{array}}$$

If a model satisfies a conjunction, then it also satisfies *each of* the conjuncts

$$\frac{\phi \vee \psi}{\phi \mid \psi}$$

If a model satisfies a disjunction, then it also satisfies *one of* the disjuncts. It is a non-deterministic rule, and it generates two *alternative* branches of the tableaux.

## 3.1. Negation Normal Form

The given tableau calculus works only if the formula has been translated into Negation Normal Form, i.e., all the negations have been pushed inside.

Example: Build a tableau for:

$\neg(A \vee (B \wedge \neg C))$

Build a tableau for its CNF:

$(\neg A \wedge (\neg B \vee C))$

# 4. Equivalences: FOL

$$
\begin{aligned}
(\forall x.\, \phi) \wedge \psi &\equiv \forall x.\, (\phi \wedge \psi) \quad \text{if } x \text{ not free in } \psi \\
(\forall x.\, \phi) \vee \psi &\equiv \forall x.\, (\phi \vee \psi) \quad \text{if } x \text{ not free in } \psi \\
(\exists x.\, \phi) \wedge \psi &\equiv \exists x.\, (\phi \wedge \psi) \quad \text{if } x \text{ not free in } \psi \\
(\exists x.\, \phi) \vee \psi &\equiv \exists x.\, (\phi \vee \psi) \quad \text{if } x \text{ not free in } \psi \\
\forall x.\, \phi \wedge \forall x.\, \psi &\equiv \forall x.\, (\phi \wedge \psi) \\
\exists x.\, \phi \vee \exists x.\, \psi &\equiv \exists x.\, (\phi \vee \psi)
\end{aligned}
$$

$$
\begin{aligned}
\neg \forall x.\, \phi &\equiv \exists x.\, \neg \phi \\
\neg \exists x.\, \phi &\equiv \forall x.\, \neg \phi
\end{aligned}
$$

& propositional equivalences

# 5. The Prenex Normal Form

Quantifier prefix + (quantifier free) matrix

$$\forall x_1 \forall x_2 \exists x_3 \ldots \forall x_n \phi$$

1. Elimination of $\rightarrow$ and $\leftrightarrow$ by means of:
   - $A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$,
   - $(A \rightarrow B \equiv \neg A \vee B)$
2. push $\neg$ inwards by means of
   - $\neg(A \wedge B) \equiv \neg A \vee \neg B$ (De Morgan)
   - $\neg(A \vee B) \equiv \neg A \wedge \neg B$ (De Morgan)
   - $\neg\neg A \equiv A$ (double negation)
   - $\neg \forall x A(x) \equiv \exists x \neg A(x)$
   - $\neg \exists x A(x) \equiv \forall x \neg A(x)$

3. rename bound variables, if necessary
4. pull quantifiers outwards
5. use the distributive law $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$ to effect the conversation to CNF.

Renaming of variables. Let $\phi[x/t]$ be the formula $\phi$ where all occurrences of $x$ have been replaced by the term $t$.

## 5.1. Exercise

Convert the following formulas into prenex normal forms:

- $(\exists x.A(x)) \rightarrow (\forall x.B(x))$
- $\forall x(\forall y(\forall z(A(x,y,z) \wedge B(y)) \rightarrow (\forall x.C(x,z))))$
- $\forall x \forall y(A(x,y,z) \wedge \exists u C(x,u)) \rightarrow \exists v C(x,v))$
- $\exists x(S(x) \wedge \forall y(L(y) \rightarrow A(x,y)))$

## 5.2. Tableau Calculus: FOL

The completion rules for quantified formulas:

$$\frac{\forall x.\ \phi}{\begin{array}{c}\phi\{X/t\}\\ \forall x.\ \phi\end{array}}$$

If a model satisfies a universal quantified formula, the it also satisfies the formula where the quantified variable has been substituted with some term. The prescription is to use all the terms which appear in the tableaux.

$$\frac{\exists x.\ \phi}{\phi\{X/a\}}$$

If a model satisfies an existential quantified formula, then it also satisfies the formula where the quantified variable has been substituted with a fresh new *Skolem* term.

## 5.3. Example

The above set of completion rules work only if the formula has been translated into Negation Normal Form, i.e., all the negations have been pushed inside.

Build a tableau for the following formula:

$\neg(\exists x.\ (\forall y.\ (P(x) \rightarrow Q(y))))$

Build a tableau for its prenex normal form:

$\forall x.\ (\exists y.\ (P(x) \wedge \neg Q(y)))$

# 6. Summary: exercises

Take the result of the conversion of the formulas below and check by means of the tableau calculus whether they are satisfiable

1. $(P \wedge Q) \leftrightarrow (R \rightarrow (P \rightarrow \neg Q))$
2. $(\neg P \wedge (\neg Q \rightarrow R)) \rightarrow S)$
3. $\forall x \forall y((A(x, y, z) \wedge \exists u C(x, u)) \rightarrow \exists v C(x, v))$
4. $\exists x(S(x) \wedge \forall y(L(y) \rightarrow A(x, y)))$

# 7. Key Concepts

- Interpretation, Model, Domain
- Satisfiability, etc..
- Truth tables
- Tableaux

In the mid-term there will be exercises about:

1. Entailment $KB \models \phi$ in PL to be proved or refuted by means of truth tables.
2. Formalization of a simple argument in PL, and its solution by means of truth tables or tableaux
3. Evaluation of a given FOL formula in a domain/interpretation.
4. Entailment $KB \models \phi$ in FOL to be proved by means of tableaux

Send us questions/doubts by the Wednesday 13rd (20:00), we will discuss them in class on the 15th 08:30-09:30 before the mid-term (09:30-11:30).