Logica & Linguaggio: Tablaux

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Contents

1. Equivalences: Properties of quantifiers

Check the following equivalence, and if it's not true that the two formulas are equivalent build a model in which one formula is true and the other is false.

- $\forall x.R(x,c)$ is equivalent to $\neg\exists x.\neg R(x,c)$?
- $\exists x.R(x,c)$ is equivalent to $\neg\forall x.\neg R(x,c)$?
- $\forall x.P(x) \wedge \forall x.S(x)$ is equivalent to $\forall x(P(x) \wedge S(x))$?
- $\forall x.\forall y.\phi$ is equivalent to $\forall y.\forall x.\phi$?
- Is $\exists x.\exists y.\phi$ is equivalent to $\exists y.\exists x.\phi$?
- Is $\exists x.\forall y.\phi$ is equivalent to $\forall y.\exists x.\phi$?

  No

2. Satisfiability and Validity in FOL

Similarly as in propositional logic, a formula ϕ can be **satisfiable**, **unsatisfiable**, **falsifiable** or **valid**—the definition is in terms of the pair (I,α).

A formula ϕ is

- **satisfiable**, if there is some (I,α) that satisfies ϕ,
- **unsatisfiable**, if ϕ is not satisfiable,
- **falsifiable**, if there is some (I,α) that does not satisfy ϕ,
- **valid** (i.e., a **tautology**), if every (I,α) satisfies ϕ.

3. Exercise

Find a model in which the formula below is true.

$\exists y.(P(y) \wedge \neg Q(y)) \wedge \forall z.(\neg P(z) \vee Q(z))$

You can show with the tableaux that it's unsatifiable!

4. Tableaux

$\begin{array}{c} A \wedge B \\ A \\ B \end{array}$	$\begin{array}{ccc} & A \vee B & \\ A & & B \end{array}$	$\begin{array}{ccc} & A \rightarrow B & \\ \neg A & & B \end{array}$
$\begin{array}{ccc} & A \leftrightarrow B & \\ A \wedge B & & \neg A \wedge \neg B \end{array}$	$\begin{array}{c} \neg\neg A \\ A \end{array}$	$\begin{array}{ccc} & \neg(A \wedge B) & \\ \neg A & & \neg B \end{array}$
$\begin{array}{c} \neg(A \vee B) \\ \neg A \\ \neg B \end{array}$	$\begin{array}{c} \neg(A \rightarrow B) \\ A \\ \neg B \end{array}$	$\begin{array}{ccc} & \neg(A \leftrightarrow B) & \\ A \wedge \neg B & & \neg A \wedge B \end{array}$

(10) $\forall x.A(x)$ $A(t)$ where t is a term	(11) $\exists x.A(x)$ $A(t)$ where t is a term which has *not* been used in the derivation so far.
(12) $\neg\forall x(A(x))$ $\exists x(\neg A(x))$	(13) $\neg\exists x(A(x))$ $\forall x(\neg A(x))$

5. Weaker Results

- If ψ is *valid*, can we conclude it is satifiable, falsifiable or unsatisfiable?

  We can conclude ψ is *satisfiable*:

  $$\text{IF } \forall I, I \models \psi \text{ THEN } \exists I, I \models \psi$$

  Satisfiability is a weaker property then validity.

- If ψ is *unsatisfiable*, can we conclude it is satifiable, falsifiable or valid?

  We can conclude ψ is *falsifiable*:

  $$\text{IF } \forall I, I \not\models \psi \text{ THEN } \exists I, I \not\models \psi$$

  Falsiability is a weaker property then unsatisifiability.

5.1. Consequences

You are asked to prove whether ψ is valid by means of tableaux.

- If all branches of your tableaux are open, what do you conclude?

  ψ is satisfiable. Are you sure you cannot give a stronger answer, i.e. are you sure ψ is not valid? In order to check whether ψ is valid you have to look at $\neg\psi$. If $\neg\psi$ is unsatisfiable then ψ is also valid.

- If all branches close: ψ is unsatisfiable. Can you make a stronger claim? No this is already a strong result, there is no need to look at $\neg\psi$.

6. Heuristics

Prove $\exists x.(\exists y.(R(x,y))) \rightarrow R(a,a)$ is valid.

1.	$\neg(\exists x.(\exists y.R(x,y))) \rightarrow R(a,a))$	
2.	$\exists x.(\exists y.(R(x,y)))$	Rule (8) applied to 1.
3.	$\neg R(a,a)$	Rule (8) applied to 1.
4.	$\exists y.R(a,y)$	Rule (11) applied to 2.
5.	$R(a,a)$	Rule (11) applied to 4.

Can we conclude that the $\exists x.(\exists y.(R(x,y))) \rightarrow R(a,a)$ is valid?

Let us interpret the above theorem as following:

- Let the natural numbers be our domain of interpretation.
- Let $R(x,y)$ stand for $x < y$

Then, $\exists x.(\exists y.(R(x,y)))$ is satisfiable. E.g. $2 < 4$.

From this it follows that $R(a,a)$ should be true as well, but it is not.

Hence, $\exists x.(\exists y.(R(x,y))) \rightarrow R(a,a)$ is not satisfiable in this interpretation, much less valid.

On line 5. Rule (11) has violeted the constraint: the term a was already used. Similarly, a could have not been used in line 4. either. Hence we don't get a contraddiction and the tableau is not closed.

Note : The same term can be used many times for universal instantiation.

$$\downarrow$$

Heuristic : When developing a semantic tableau in `FOL` use the existential instantiation rule (Rule 11) before the universal instantiation rule (Rule 10).

6.1. Heuristic (II)

1.	$\neg(\forall x.(A(x) \wedge B(x)) \to \forall x.A(x))$	
2.	$\forall x.(A(x) \wedge B(x))$	Rule (8) applied to 1.
3.	$\neg(\forall x.A(x))$	Rule (8) applied to 1.
4.	$\exists x.\neg A(x)$	Rule (12) applied to 3.
5.	$\neg A(a)$	Rule (11) applied to 4.
6.	$A(a) \wedge B(a)$	Rule (10) applied to 2.
7.	$A(a)$	Rule (1) applied to 6.
8.	$B(a)$	Rule (1) applied to 6.
	Closed	

Note: if we had used Rule 10 before we would have not been able to apply Rule 11.

7. Exercises

Prove each of the following using semantic tableaux:

1. $\forall x.(A(x) \to (B(x) \to A(x)))$
2. $\exists x.A(x) \to A(a)$
3. $(\exists x.A(x) \land \exists x.B(x)) \to \exists x.(A(x) \land B(x))$

$\forall x.(A(x) \to (B(x) \to A(x)))$

1.	$\neg(\forall x.(A(x) \to (B(x) \to A(x))))$	
2.	$\exists x.(\neg(A(x) \to (B(x) \to A(x))))$	Rule (12) applied to 1.
3.	$\neg(A(a) \to (B(a) \to A(a)))$	Rule (8) applied to 2.
4.	$A(a)$	Rule (12) applied to 3.
5.	$\neg(B(a) \to A(a))$	Rule (8) applied to 3.
6.	$B(a)$	Rule (8) applied to 5.
7.	$\neg A(a)$	Rule (8) applied to 5.
	Closed	

2. $\exists x.(A(x)) \to A(a)$

1.	$\neg(\exists x.(A(x)) \to A(a))$	
2.	$\exists x.A(x)$	Rule (8) to line 1
3.	$\neg A(a)$	Rule (8) to line 1
4.	$A(b)$	Rule (11) to line 2

We cannot close the branch.

3. $(\exists x A(x) \wedge \exists x B(x)) \to \exists x (A(x) \wedge B(x))$

$$
\begin{array}{lcl}
1. & \neg((\exists x A(x) \wedge \exists x B(x)) \to \exists x (A(x) \wedge B(x))) & \\
2. & \exists x A(x) \wedge \exists x B(x) & \text{Rule (8) to line 1} \\
3. & \neg(\exists x (A(x) \wedge B(x))) & \text{Rule (8) to line 1} \\
4. & \forall x \neg(A(x) \wedge B(x)) & \text{Rule (13) to line 3} \\
5. & \exists x A(x) & \text{Rule (1) to line 2} \\
6. & \exists x B(x) & \text{Rule (1) to line 2} \\
7. & A(a) & \text{Rule (11) to line 5} \\
8. & B(b) & \text{Rule (11) to line 6} \\
9. & \neg(A(a) \wedge B(a)) & \text{Rule (10) to line 4} \\
 & \swarrow \quad \searrow & \\
10. & \neg A(a) \qquad \neg B(a) & \text{Rule (6) to line 9}
\end{array}
$$

We cannot close the right branch.

We could have used b in line 9 to give $\neg(A(b) \wedge B(b))$, but then the left branch could have not been closed.

8. Animal problem

Consider the following problem

1. The only animals in this house are cats.
2. Every animal that loves to look at the moon is suitable for a pet.
3. When I detest an animal, I avoid it.
4. All animals that don't prowl at night are carnivorous.
5. No cat fails to kill mice.
6. No animals ever take to me, except the ones in this house.
7. Kangaroos are not suitable for pets.
8. All animals that are carnivorous kill mice.
9. I detest animals that do not take to me.
10. Animals that prowl at night always love to look at the moon.

Is it true that I always avoid kangaroo?

(a) Represent the facts above as FOL sentences and formalize the problem. Remember to give the keys of your formalization and add the extra information you might need in order to answer the question. (b) Give the proof of your answer by means of tableaux.