Introduction to Formal Methods
Chapter 08: Automata-theoretic LTL Model Checking

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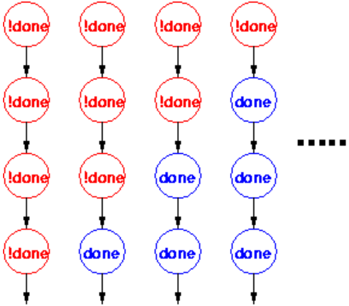
Outline

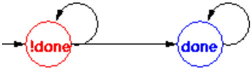
System's computations

- The behaviors (computations) of a system can be seen as sequences of assignments to propositions.

```
MODULE main
VAR  done: Boolean;
ASSIGN
   init(done):=0;
   next(done):= case
      !done: {0,1};
      done: done;
    esac;
```



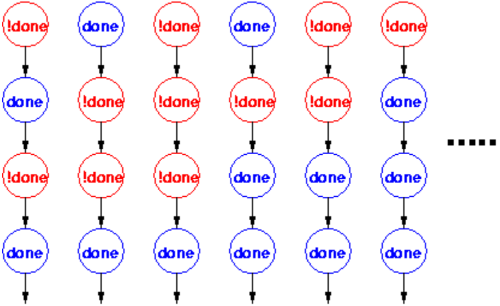
- Since the state space is finite, the set of computations can be represented by a finite automaton.

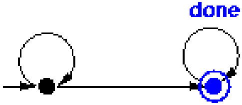


or

Correct computations

- Some computations are correct and others are not acceptable.
- We can build an automaton for the set of all acceptable computations.
- Example: eventually, done will be true forever (**FG**done).





Language Containment Problem

- Solution to the verification problem
  $\Longrightarrow$ Check if language of the system automaton is contained in the language accepted by the property automaton.
- The language containment problem is the problem of deciding if a language is a subset of another language.

$$\mathcal{L}(A_1) \subseteq \mathcal{L}(A_2) \Longleftrightarrow \mathcal{L}(A_1) \cap \overline{\mathcal{L}(A_2)} = \{\}$$

- In order to solve the language containment problem, we need to know:
  (i) how to complement an automaton,
  (ii) how to intersect two automata,
  (iii) how to check the language emptiness of an automaton.

Finite Word Languages

- An Alphabet Σ is a collection of symbols (letters).
  E.g. $\Sigma = \{a, b\}$.
- A finite word is a finite sequence of letters. (E.g. *aabb*.)
  The set of all finite words is denoted by Σ^*.
- A language U is a set of words, i.e. $U \subseteq \Sigma^*$.
  Example: Words over $\Sigma = \{a, b\}$ with equal number of a's and b's.
  (E.g. *aabb* or *abba*.)
- Language recognition problem: determine whether a word belongs to a language.
- Automata are computational devices able to solve language recognition problems.

Finite-State Automata

- Basic model of computational systems with finite memory.
- Widely applicable
  - Embedded System Controllers.
    Languages: Ester-el, Lustre, Verilog.
  - Synchronous Circuits.
  - Regular Expression Pattern Matching
    Grep, Lex, Emacs.
  - Protocols
    Network Protocols
    Architecture: Bus, Cache Coherence, Telephony,...

Notation

$a, b \in \Sigma$ finite alphabet.
$u, v, w \in \Sigma^*$ finite words.

- ϵ empty word.
- $u.v$ concatenation.
- $u^i = u.u. .u$ repeated i-times.

$U, V \subseteq \Sigma^*$ Finite word languages.

Finite-State Automata Definition

Definition

A Nondeterministic Finite-State Automaton (NFA) is $(Q, \Sigma, \delta, I, F)$ s.t.

Q Finite set of states.

Σ is a finite alphabet

$I \subseteq Q$ set of initial states.

$F \subseteq Q$ set of final states.

$\delta \subseteq Q \times \Sigma \times Q$ transition relation (edges).

We use $q \xrightarrow{a} q'$ to denote $(q, a, q') \in \delta$.

Definition

A Deterministic Finite-State Automaton (DFA) is a NFA s.t.:

$\delta : Q \times \Sigma \rightarrow Q$ is a total function

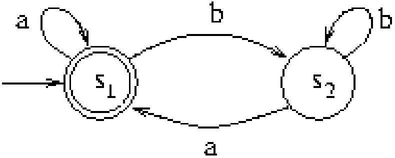
Single initial state $I = \{q_0\}$.

Regular Languages

- A run of NFA A on $u = a_0, a_1, \ldots, a_{n-1}$ is a finite sequence of states $q_0, q_1, \ldots, q_n$ s.t. $q_0 \in I$ and $q_i \xrightarrow{a_i} q_{i+1}$ for $0 \leq i < n$.
- An accepting run is one where $q_n \in F$.
- The language accepted by A is
  $\mathcal{L}(A) = \{u \in \Sigma^* \mid A \text{ has an accepting run on } u\}$
- The languages accepted by a NFA are called regular languages.

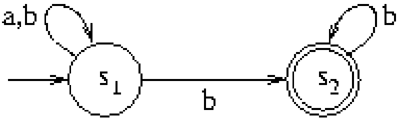
Finite-State Automata: examples

- The DFA A_1 over $\Sigma = \{a, b\}$:



Recognizes words which do not end in *b*.

- The NFA A_2 over $\Sigma = \{a, b\}$:



Recognizes words which end in *b*.

Determinisation

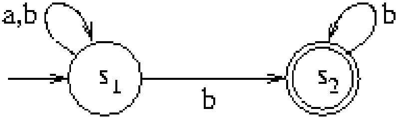
Theorem (determinisation)

Given a NFA A we can construct a DFA A' s.t. $\mathcal{L}(A) = \mathcal{L}(A')$.
Size: $|A'| = 2^{O(|A|)}$.

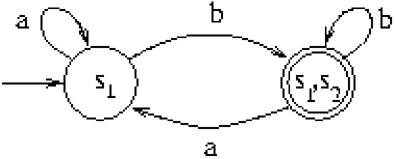
- Each state of A' corresponds to a set $\{s_1, ..., s_j\}$ of states in A ($Q' \subseteq 2^Q$), with the intended meaning that :
  - A' is in the state $\{s_1, .., s_j\}$ if A is in one of the states $s_1, ..., s_j$
- The (unique) initial state is $I' =_{def} \{s_i \mid s_i \in I\}$
- The deterministic transition relation $\delta' : 2^Q \times \Sigma \longmapsto 2^Q$ is
  - $\{s\} \xrightarrow{a} \{s_i \mid s \xrightarrow{a} s_i\}$
  - $\{s_1, ..., s_j, ..., s_n\} \xrightarrow{a} \bigcup_{j=1}^{n} \{s_i \mid s_j \xrightarrow{a} s_i\}$
- The set of final states F' is such that $\{s_1, ..., s_n\} \in F'$ iff $s_i \in F$ for some $i \in \{1, ..., n\}$

Determinisation [cont.]

- NFA A_2: Words which end in b.



- A_2 can be determinised into the automaton DA_2 below. (#States = 2^Q.)



Closure Properties

Theorem (Boolean closure)

Given NFA A_1, A_2 over Σ we can construct NFA A over Σ s.t.

- $\mathcal{L}(A) = \overline{\mathcal{L}(A_1)}$ (Complement). $|A| = 2^{O(|A_1|)}$.
- $\mathcal{L}(A) = \mathcal{L}(A_1) \cup \mathcal{L}(A_2)$ (union). $|A| = |A_1| + |A_2|$.
- $\mathcal{L}(A) = \mathcal{L}(A_1) \cap \mathcal{L}(A_2)$ (intersection). $|A| \leq |A_1| \cdot |A_2|$.

Complementation of a NFA

A NFA $A = (Q, \Sigma, \delta, I, F)$ is complemented by:

- determinising it into a DFA $A' = (Q', \Sigma', \delta', I', F')$
- complementing it: $\overline{A'} = (Q', \Sigma', \delta', I', \overline{F'})$
- $|\overline{A'}| = |A'| = 2^{O(|A|)}$

Union of two NFAs

Definition: union of NFAs

Let $A_1 = (Q_1, \Sigma_1, \delta_1, I_1, F_1)$, $A_2 = (Q_2, \Sigma_2, \delta_2, I_2, F_2)$.
Then $A = A_1 \cup A_2 = (Q, \Sigma, \delta, I, F)$ is defined as follows

- $Q := Q_1 \cup Q_2$, $I := I_1 \cup I_2$, $F := F_1 \cup F_2$
- $R(s, s') := \begin{cases} R_1(s, s') \text{ if } s \in Q_1 \\ R_2(s, s') \text{ if } s \in Q_2 \end{cases}$

Theorem

- $\mathcal{L}(A) = \mathcal{L}(A_1) \cup \mathcal{L}(A_2)$
- $|A| = |A_1| + |A_2|$

Note

A is an automaton which just runs nondeterministically either A_1 or A_2

Synchronous Product Construction

Definition: product of NFAs

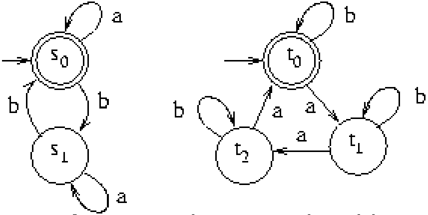
Let $A_1 = (Q_1, \Sigma, \delta_1, I_1, F_1)$ and $A_2 = (Q_2, \Sigma, \delta_2, I_2, F_2)$.
Then, $A_1 \times A_2 = (Q, \Sigma, \delta, I, F)$ where

- $Q = Q_1 \times Q_2$,
- $I = I_1 \times I_2$,
- $F = F_1 \times F_2$,
- $\langle p, q \rangle \xrightarrow{a} \langle p', q' \rangle$ iff $p \xrightarrow{a} p'$ and $q \xrightarrow{a} q'$.

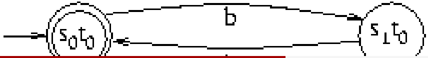
Theorem

$\mathcal{L}(A_1 \times A_2) = \mathcal{L}(A_1) \cap \mathcal{L}(A_2)$.
$|(A_1 \times A_2)| \leq |A_1| \cdot |A_2|$.

Example



- A_1 recognizes words with an even number of b's.
- A_2 recognizes words with a number of a's multiple of 3.
- The Product Automaton $A_1 \times A_2$ with $F = \{s_0, t_0\}$.



Regular Expressions

- Syntax: $\emptyset \mid \epsilon \mid a \mid reg_1.reg_2 \mid reg_1|reg_2 \mid reg^*$.
- Every regular expression *reg* denotes a language $\mathcal{L}(reg)$.
- Example: $a^*.(b|bb).a^*$. The words with either 1 *b* or 2 consecutive *b*'s.

Theorem

For every regular expression *reg* we can construct a language equivalent NFA of size $O(|reg|)$.

Theorem

For every DFA A we can construct a language equivalent regular expression $reg(A)$.

Infinite Word Languages

Modeling infinite computations of reactive systems.

- An ω-word α over Σ is an infinite sequence

  $a_0, \; a_1, \; a_2 \ldots$.

  Formally, $\alpha : \mathbb{N} \rightarrow \Sigma$.

  The set of all infinite words is denoted by Σ^ω.

- A ω-language L is collection of ω-words, i.e. $L \subseteq \Sigma^\omega$.

  Example All words over $\{a, b\}$ with infinitely many a's.

Notation:

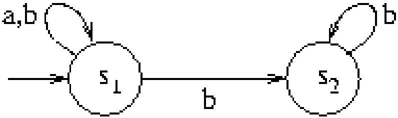
omega words $\alpha, \beta, \gamma \in \Sigma^\omega$.

omega-languages $L, L_1 \subseteq \Sigma^\omega$

For $u \in \Sigma^+$, let $u^\omega = u.u.u\ldots$

Omega-Automata

- We consider automaton running over infinite words.



- Let $\alpha = aabbbb\ldots$.
  There are several (infinite) possible runs.
  Run $\rho_1 = s_1, s_1, s_1, s_1, s_2, s_2 \ldots$
  Run $\rho_2 = s_1, s_1, s_1, s_1, s_1, s_1 \ldots$
- Acceptance Conditions: Büchi (Muller, Rabin, Street):
  Acceptance is based on states occurring infinitely often
- Notation Let $\rho \in Q^\omega$. Then,
  $$Inf(\rho) = \{s \in Q \mid \exists^\infty i \in \mathbb{N}.\ \rho(i) = s\}.$$
  (The set of states occurring infinitely many times in ρ.)

Büchi Automata

Nondeterministic Büchi Automaton

$A = (Q, \Sigma, \delta, I, F)$, where $F \subseteq Q$ is the set of accepting states.

- A run ρ of A on ω-word $\alpha = a_0, a_1, a_2, \ldots$ is an infinite sequence $\rho = q_o, q_1, q_2, \ldots$ s.t. $q_0 \in I$ and $q_i \xrightarrow{a_i} q_{i+1}$ for $0 \leq i$.
- The run ρ is accepting if

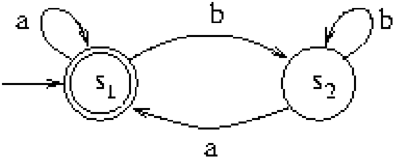
  $$Inf(\rho) \cap F \neq \emptyset.$$

- The language accepted by A

  $\mathcal{L}(A) = \{\alpha \in \Sigma^\omega \mid A \text{ has an accepting run on } \alpha\}$

Büchi Automaton: Example

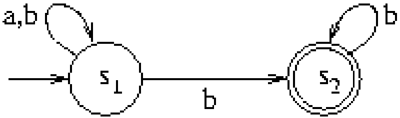
Let $\Sigma = \{a, b\}$.
Let a Deterministic Büchi Automaton (DBA) A_1 be



- With $F = \{s_1\}$ the automaton recognizes words with infinitely many *a*'s.
- With $F = \{s_2\}$ the automaton recognizes words with infinitely many *b*'s.

Büchi Automaton: Example (2)

Let a Nondeterministic Büchi Automaton (NBA) A_2 be



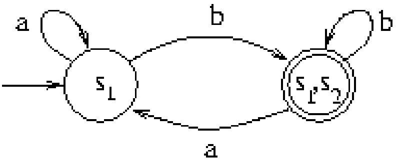
With $F = \{s_2\}$, the automaton A_2 recognizes words with finitely many a. Thus, $\mathcal{L}(A_2) = \overline{\mathcal{L}(A_1)}$.

Deterministic vs. Nondeterministic Büchi Automata

Theorem

*DBA*s are strictly less powerful than *NBA*s.

The subset construction does not work:
let DA_2 be



- DA_2 is not equivalent to A_2
  (e.g., it recognizes $(b.a)^\omega$)

Closure Properties

Theorem (union, intersection)

For the NBAs A_1, A_2 we can construct

- the NBA A s.t. $\mathcal{L}(A) = \mathcal{L}(A_1) \cup \mathcal{L}(A_2)$. $|A| = |A_1| + |A_2|$
- the NBA A s.t. $\mathcal{L}(A) = \mathcal{L}(A_1) \cap \mathcal{L}(A_2)$. $|A| \leq |A_1| \cdot |A_2| \cdot 2$.

Union of two NBAs

Definition: union of NBAs

Let $A_1 = (Q_1, \Sigma_1, \delta_1, I_1, F_1)$, $A_2 = (Q_2, \Sigma_2, \delta_2, I_2, F_2)$.
Then $A = A_1 \cup A_2 = (Q, \Sigma, \delta, I, F)$ is defined as follows

- $Q := Q_1 \cup Q_2$, $I := I_1 \cup I_2$, $F := F_1 \cup F_2$
- $R(s, s') := \begin{cases} R_1(s, s') \text{ if } s \in Q_1 \\ R_2(s, s') \text{ if } s \in Q_2 \end{cases}$

Theorem

- $\mathcal{L}(A) \;=\; \mathcal{L}(A_1) \cup \mathcal{L}(A_2)$
- $|A| = |A_1| + |A_2|$

Note

A is an automaton which just runs nondeterministically either A_1 or A_2 (same construction as with ordinary automata)

Synchronous Product of NBAs

Definition: synchronous product of NBAs

Let $A_1 = (Q_1, \Sigma, \delta_1, I_1, F_1)$ and $A_2 = (Q_2, \Sigma, \delta_2, I_2, F_2)$.
Then, $A_1 \times A_2 = (Q, \Sigma, \delta, I, F)$, where

$$Q = Q_1 \times Q_2 \times \{1, 2\}.$$
$$I = I_1 \times I_2 \times \{1\}.$$
$$F = F_1 \times Q_2 \times \{1\}.$$

$\langle p, q, 1\rangle \xrightarrow{a} \langle p', q', 1\rangle$ iff $p \xrightarrow{a} p'$ and $q \xrightarrow{a} q'$ and $p \notin F_1$.
$\langle p, q, 1\rangle \xrightarrow{a} \langle p', q', 2\rangle$ iff $p \xrightarrow{a} p'$ and $q \xrightarrow{a} q'$ and $p \in F_1$.
$\langle p, q, 2\rangle \xrightarrow{a} \langle p', q', 2\rangle$ iff $p \xrightarrow{a} p'$ and $q \xrightarrow{a} q'$ and $q \notin F_2$.
$\langle p, q, 2\rangle \xrightarrow{a} \langle p', q', 1\rangle$ iff $p \xrightarrow{a} p'$ and $q \xrightarrow{a} q'$ and $q \in F_2$.

Theorem

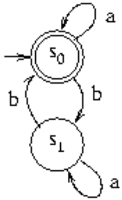
- $\mathcal{L}(A_1 \times A_2) = \mathcal{L}(A_1) \cap \mathcal{L}(A_2)$.
- $|A_1 \times A_2| \leq 2 \cdot |A_1| \cdot |A_2|$.

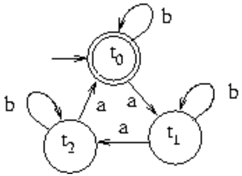
Product of NBAs: Intuition

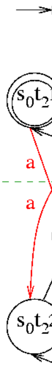
- The automaton remembers two tracks, one for each source NBA, and it points to one of the two tracks
- As soon as it goes through an accepting state of the current track, it switches to the other track
  $\Longrightarrow$ in order to visit infinitely often a state in F (i.e., F_1), it must visit infinitely often some state also in F_2
- Important subcase: If $F_2 = Q_2$, then

$$Q = Q_1 \times Q_2.$$
$$I = I_1 \times I_2.$$
$$F = F_1 \times Q_2.$$

Product of NBAs: Example







Closure Properties (2)

Theorem (complementation) [Safra, MacNaughten]

For the NBA A_1 we can construct an NBA A_2 such that $\mathcal{L}(A_2) = \overline{\mathcal{L}(A_1)}$.
$|A_2| = O(2^{|A_1| \cdot \log(|A_1|)})$.

Method: (hint)

(i) convert a Büchi automaton into a Non-Deterministic Rabin automaton
(ii) determinize and Complement the Rabin automaton
(iii) convert the Rabin automaton into a Büchi automaton.

Generalized Büchi Automaton

Definition

- A Generalized Büchi Automaton is a tuple $A := (Q, \Sigma, \delta, I, FT)$ where $FT = \langle F_1, F_2, \ldots, F_k \rangle$ with $F_i \subseteq Q$.
- A run ρ of A is accepting if $Inf(\rho) \cap F_i \neq \emptyset$ for each $1 \leq i \leq k$.

Theorem

For every Generalized Büchi Automaton we can construct a language equivalent plain Büchi Automaton.

Intuition

Let $Q' = Q \times \{1, \ldots, K\}$.
The automaton remains in phase i till it visits a state in F_i. Then, it moves to $(i + 1) mod\ K$ mode.

De-generalization of a generalized NBA

Definition: De-generalization of a generalized NBA

Let $A \stackrel{\text{def}}{=} (Q, \Sigma, \delta, I, FT)$ a generalized BA s.f. $FT \stackrel{\text{def}}{=} \{F_1, ..., F_K\}$.
Then a language-equivalent BA $A' \stackrel{\text{def}}{=} (Q', \Sigma, \delta', I', F')$ is built as follows

$$Q' = Q_1 \times \{1, ..., K\}.$$
$$I' = I \times \{1\}.$$
$$F' = F_1 \times \{1\}.$$

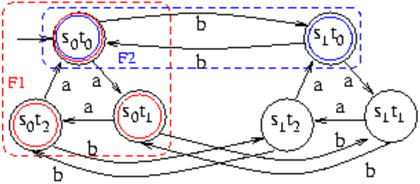
δ' is s.t., for every $i \in [1, ..., K]$:

$$\langle p, i\rangle \xrightarrow{a} \langle q, i\rangle \quad \text{iff} \quad p \xrightarrow{a} q \in \delta \quad \text{and} \quad p \notin F_i.$$
$$\langle p, i\rangle \xrightarrow{a} \langle q, (i+1) mod\ K\rangle \quad \text{iff} \quad p \xrightarrow{a} q \in \delta \quad \text{and} \quad p \in F_i.$$

Theorem

- $\mathcal{L}(A') = \mathcal{L}(A)$.
- $|A'| \leq K \cdot |A|$.

Degeneralizing a Büchi automaton: Example



Omega-regular Expressions

Definition

A language is called ω-regular if it has the form $\cup_{i=1}^{n} \; U_i.(V_i)^\omega$ where U_i, V_i are regular languages.

Theorem

A language L is ω-regular iff it is NBA-recognizable.

NFA emptiness checking

- Equivalent of finding a final state reachable from an initial state.
- It can be solved with a DFS or a BFS.
- A DFS finds a counterexample on the fly (it is stored in the stack of the procedure).
- A BFS finds a final state reachable with a shortest counterexample, but it requires a further backward search to reproduce the path.
- Complexity: $O(n)$.
- Hereafter, assume w.l.o.g. that there is only one initial state.

NFA Emptiness Checking (cont.)

// returns True if empty language, false otherwise

```
Bool DFS(NFA A) {
   stack S=I;
   Hashtable T=I;
   while S!=∅ {
      v=top(S);
      if v∈F return False
      if ∃w s.t. w∈δ(v) && T(w)==0 {
         hash(w,T);
         push(w,S);
      } else
         pop(S);
   }
   return True;
}
```

NBA emptiness checking

- Equivalent of finding an accepting cycle reachable from an initial state.
- A naive algorithm:
  (i) a DFS finds the final states f reachable from an initial state;
  (ii) for each f, a second DFS finds if it can reach f (i.e., if there exists a loop)
  - Complexity: $O(n^2)$.
- SCC-based algorithm:
  (i) Tarjan's algorithm uses a DFS to find the SCCs in linear time;
  (ii) another DFS finds if the union of non-trivial SCCs is reachable from an initial state.
  - Complexity: $O(n)$.
  - Drawbacks: it stores too much information and does not find directly a counterexample.

Double Nested DFS algorithm

- Double Nested DFS [Courcoubetis, Vardi, Wolper, Yannakakis, CAV'90]
  - two Hash tables:
    - `T1`: reachable states
    - `T2`: states reachable from a reachable final state
  - two stacks:
    - `S1`: current branch of states reachable
    - `S2`: current branch of states reachable from final state f
  - two nested DFS's:
    - `DFS1` looks for a path from an initial state to a cycle starting from an accepting state
    - `DFS2` looks for a cycle starting from an accepting state
  - It stops as soon as it finds a counterexample.
  - The counterexample is given by the stack of `DFS2` (an accepting cycle) preceded by the stack of `DFS1` (a path from an initial state to the cycle).

Double Nested DFS - First DFS

```
// returns True if empty language, false otherwise
Bool DFS1(NBA A) {
   stack S1=I; stack S2=∅;
   Hashtable T1=I; Hashtable T2=∅;
   while S1!=∅ {
      v=top(S1);
      if ∃w s.t. w∈δ(v) && T1(w)==0 {
         hash(w,T1);
         push(w,S1);
      } else {
         pop(S1);
         if (v∈F && !DFS2(v,S2,T2,A))
            return False;
   }    }
   return True;
}
```

Double Nested DFS - Second DFS

```
Bool DFS2(state f, stack & S, Hashtable & T, NBA A) {
   hash(f,T);
   S = {f}
   while S!=∅ {
      v=top(S);
      if f∈δ(v) return False;
      if ∃w s.t. w∈δ(v) && T(w)==0 {
         hash(w);
         push(w);
      } else pop(S);
   }
   return True;
}
```

Remark: `T` passed by reference, is not reset at each call of `DFS2` !

Double nested DFS: intuition

DFS1 invokes DFS2 on each $f_1, ..., f_n$ only after popping it (postorder):

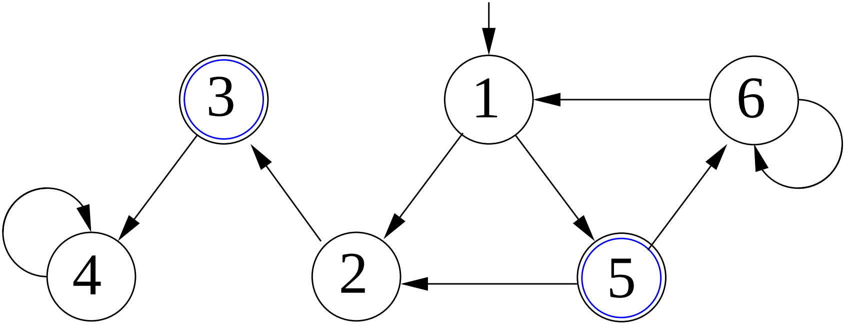
- suppose $DFS2$ is invoked on f_j before than on f_i

$\Longrightarrow$ f_i not reachable from (any state s which is reachable from) f_j

- If during $DFS2(f_i, ...)$ it is encountered a state S which has already been explored by $DFS2(f_j, ...)$ for some f_j,
  - can we reach f_i from S?
  - No, because f_i is not reachable from f_j!

$\Longrightarrow$ it is safe to backtrack.

Double Nested DFS: example



T1

S1

T2

S2

Automata-Theoretic LTL Model Checking

- Let M be a Kripke model and ψ be an LTL formula

$$
\begin{aligned}
& M \models \mathbf{A}\psi \quad \text{(CTL}^*\text{)} \\
\Longleftrightarrow\ & M \models \psi \quad \text{(LTL)} \\
\Longleftrightarrow\ & \mathcal{L}(M) \subseteq \mathcal{L}(\psi) \\
\Longleftrightarrow\ & \mathcal{L}(M) \cap \overline{\mathcal{L}(\psi)} = \emptyset \\
\Longleftrightarrow\ & \mathcal{L}(M) \cap \mathcal{L}(\neg\psi) = \emptyset \\
\Longleftrightarrow\ & \mathcal{L}(A_M) \cap \mathcal{L}(A_{\neg\psi}) = \emptyset \\
\Longleftrightarrow\ & \mathcal{L}(A_M \times A_{\neg\psi}) = \emptyset
\end{aligned}
$$

- A_M is a Büchi Automaton equivalent to M (which represents all and only the executions of M)
- $A_{\neg\psi}$ is a Büchi Automaton which represents all and only the paths that satisfy $\neg\psi$ (do not satisfy ψ)

$\Longrightarrow$ $A_M \times A_{\neg\psi}$ represents all and only the paths appearing in M and not in ψ.

Automata-Theoretic LTL M.C. (dual version)

- Let M be a Kripke model and $\varphi \stackrel{\text{def}}{=} \neg\psi$ be an LTL formula

$$
\begin{aligned}
& M \models \mathbf{E}\varphi \\
\Longleftrightarrow\ & M \not\models \mathbf{A}\neg\varphi \\
\Longleftrightarrow\ & ... \\
\Longleftrightarrow\ & \mathcal{L}(A_M \times A_\varphi) \neq \emptyset
\end{aligned}
$$

- A_M is a Büchi Automaton equivalent to M (which represents all and only the executions of M)
- A_φ is a Büchi Automaton which represents all and only the paths that satisfy φ

$\Longrightarrow$ $A_M \times A_\varphi$ represents all and only the paths appearing in both A_M and A_φ.

Automata-Theoretic LTL Model Checking

Four steps:

(i) Compute A_M

(ii) Compute A_φ

(iii) Compute the product $A_M \times A_\varphi$

(iv) Check the emptiness of $\mathcal{L}(A_M \times A_\varphi)$

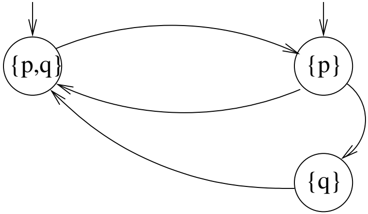
Computing an NBA A_M from a Kripke Structure M

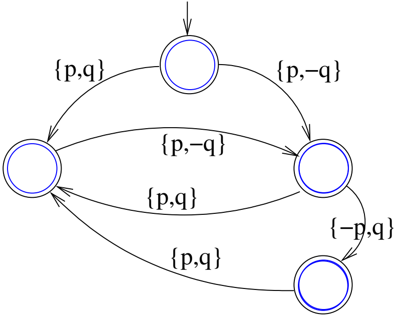
- Transform a Kripke structure $M = \langle S, S_0, R, L, AP\rangle$ into an NBA $A_M = \langle Q, \Sigma, \delta, I, F\rangle$ s.t.:
  - States: $Q := S \cup \{init\}$, *init* being a new initial state
  - Alphabet: $\Sigma := 2^{AP}$
  - Initial State: $I := \{init\}$
  - Accepting States: $F := Q = S \cup \{init\}$
  - Transitions:

$$\delta : \quad q \xrightarrow{a} q' \text{ iff } (q, q') \in R \text{ and } L(q') = a$$
$$init \xrightarrow{a} q \text{ iff } q \in S_0 \text{ and } L(q) = a$$

- $\mathcal{L}(A_M) = \mathcal{L}(M)$
- $|A_M| = |M| + 1$

Computing a NBA A_M from a Kripke Structure M: Example

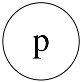




$\Longrightarrow$ Substantially, add one initial state, move labels from states to incoming edges, set all states as accepting states

Labels on Kripke Structures and BA's - Remark

Note that the labels of a Büchi Automaton are different from the labels of a Kripke Structure. Also graphically, they are interpreted differently:



- in a Kripke Structure, it means that *p* is true and all other propositions are false;
- in a Büchi Automaton, it means that *p* is true and all other propositions are irrelevant ("don't care"), i.e. they can be either true or false.

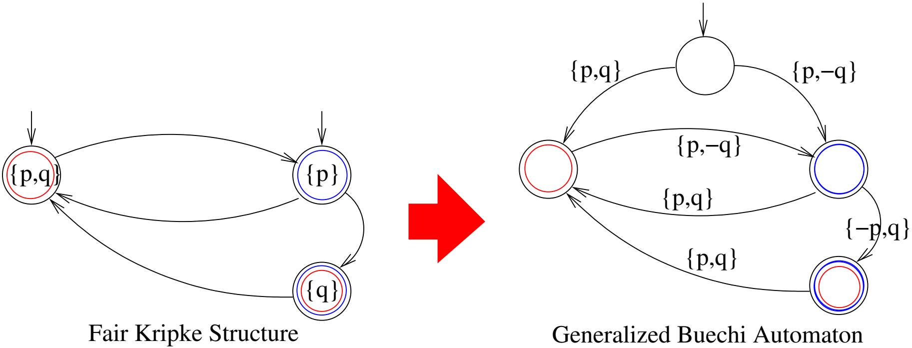
Computing a NBA A_M from a Fair Kripke Structure M

- Transforming a fair K.S. $M = \langle S, S_0, R, L, AP, FT \rangle$, $FT = \{F_1, ..., F_n\}$, into a generalized NBA $A_M = \langle Q, \Sigma, \delta, I, FT' \rangle$ s.t.:
  - States: $Q := S \cup \{init\}$, *init* being a new initial state
  - Alphabet: $\Sigma := 2^{AP}$
  - Initial State: $I := \{init\}$
  - Accepting States: $FT' := FT$
  - Transitions:

$$\delta : \quad \begin{array}{l} q \stackrel{a}{\longrightarrow} q' \text{ iff } (q, q') \in R \textit{ and } L(q') = a \\ init \stackrel{a}{\longrightarrow} q \text{ iff } q \in S_0 \textit{ and } L(q) = a \end{array}$$

- $\mathcal{L}(A_M) = \mathcal{L}(M)$
- $|A_M| = |M| + 1$

Computing a (Generalized) BA A_M from a Fair Kripke Structure M: Example



$\Longrightarrow$ Substantially, add one initial state, move labels from states to incoming edges, set fair states as accepting states

Translation problem

Problem

Given an LTL formula ϕ, find a Büchi Automaton that accepts the same language of ϕ.

- It is a fundamental problem in LTL model checking (in other words, every model checking algorithm that verifies the correctness of an LTL formula translates it in some sort of finite-state machine).
- We will translate an LTL formula into a Generalized Büchi Automata (GBA).

Exponential Translation

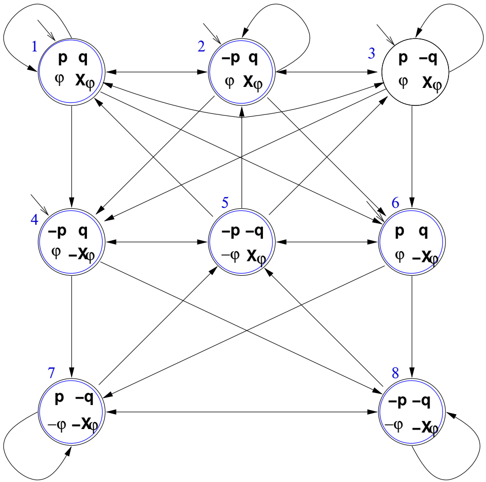
- From φ, create a fair Kripke model, like in chapter 7.
- Convert it into a (Generalized) Büchi Automaton

Remark

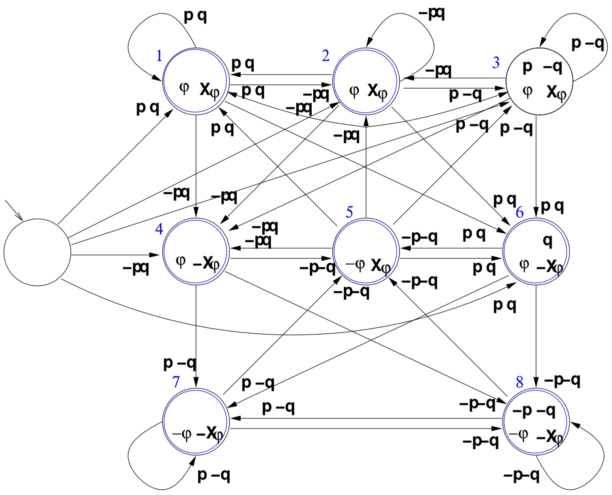
Inefficient: up to $2^{EL(\varphi)}$ states.

- Kripke models require total truth assignments to state variables

Example



Example



LTL Negative Normal Form (NNF)

- Every LTL formula φ can be written into an equivalent formula φ' using only the operators $\wedge$, $\vee$, **X**, **U**, **R** on propositional literals.
- Done by pushing negations down to literal level:

$$
\begin{array}{lcl}
\neg(\varphi_1 \vee \varphi_2) & \Longrightarrow & (\neg\varphi_1 \wedge \neg\varphi_2) \\
\neg(\varphi_1 \wedge \varphi_2) & \Longrightarrow & (\neg\varphi_1 \vee \neg\varphi_2) \\
\neg\mathbf{X}\varphi_1 & \Longrightarrow & \mathbf{X}\neg\varphi_1 \\
\neg(\varphi_1\mathbf{U}\varphi_2) & \Longrightarrow & (\neg\varphi_1\mathbf{R}\neg\varphi_2) \\
\neg(\phi_1\mathbf{R}\phi_2) & \Longrightarrow & (\neg\phi_1\mathbf{U}\neg\phi_2)
\end{array}
$$

  $\Longrightarrow$ the resulting formula is expressed in terms of $\vee$, $\wedge$, X, **U**, **R** and literals (Negative Normal Form, NNF).
  - encoding linear if a DAG representation is used
- In the construction of A_φ we now assume that φ is in NNF.

On-the-fly Construction of A_φ (Intuition)

Apply recursively the following steps:

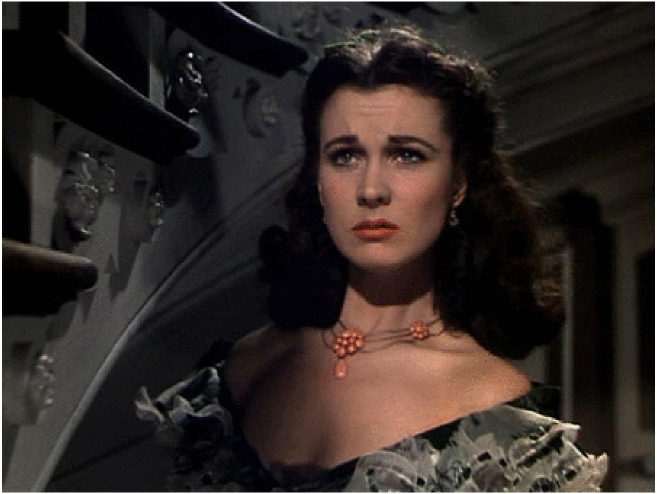
Step 1: Apply the tableau expansion rules to φ

$\psi_1 \mathbf{U} \psi_2 \Longrightarrow \psi_2 \vee (\psi_1 \wedge \mathbf{X}(\psi_1 \mathbf{U} \psi_2))$

$\psi_1 \mathbf{R} \psi_2 \Longrightarrow \psi_2 \wedge (\psi_1 \vee \mathbf{X}(\psi_1 \mathbf{R} \psi_2))$

until we get a Boolean combination of elementary subformulas of φ
(An elementary formula is a proposition or a **X**-formula.)

Tableaux rules: a quote



“After all... tomorrow is another day."
[Scarlett O’Hara, “Gone with the Wind”]

On-the-fly Construction of A_φ (Intuition) [cont.]

Step 2: Convert all formulas into Disjunctive Normal Form, and then push the conjunctions inside the next:

$$\varphi \implies \bigvee_i (\bigwedge_j l_{ij} \wedge \bigwedge_k \mathbf{X}\psi_{ik}) \implies \bigvee_i (\bigwedge_j l_{ij} \wedge \mathbf{X} \bigwedge_k \psi_{ik}).$$

- Each disjunct $(\overbrace{\bigwedge_j l_{ij}}^{labels} \wedge \overbrace{\mathbf{X} \bigwedge_k \psi_{ik}}^{next\ part})$ represents a state:
  - the conjunction of literals $\bigwedge_j l_{ij}$ represents a set of labels in Σ (e.g., if $Vars(\varphi) = \{p, q, r\}$, $p \wedge \neg q$ represents the two labels $\{p, \neg q, r\}$ and $\{p, \neg q, \neg r\}$)
  - $\mathbf{X} \bigwedge_k \psi_{ik}$ represents the next part of the state (obligations for the successors)
- N.B., if no next part occurs, $\mathbf{X}\top$ is implicitly assumed

On-the-fly Construction of A_φ (Intuition) [cont.]

Step 3: For every state S_i represented by $(\bigwedge_j l_{ij} \wedge \mathbf{X} \overbrace{\bigwedge_k \psi_{ik}}^{\varphi_i})$

- label the incoming edges of S_i with $\bigwedge_j l_{ij}$
- mark that the state S_i satisfies φ
- apply recursively steps 1-2-3 to $\varphi_i \stackrel{\text{def}}{=} \bigwedge_k \psi_{ik}$,
  - rewrite φ_i into $\bigvee_{i'}(\bigwedge_j l'_{i'j} \wedge \mathbf{X} \bigwedge_k \psi'_{i'k})$
  - from each disjunct $(\bigwedge_j l'_{i'j} \wedge \mathbf{X} \bigwedge_k \psi'_{i'k})$ generate a new state $S_{ii'}$ (if not already present) and label it as satisfying $\varphi_i \stackrel{\text{def}}{=} \bigwedge_k \psi_{ik}$
- draw an edge from S_i to all states $S_{ii'}$ which satisfy $\bigwedge_k \psi_{ik}$
- (if no next part occurs, $\mathbf{X}\top$ is implicitly assumed, so that an edge to a “true” node is drawn)

On-the-fly Construction of A_φ (Intuition) [cont.]

On-the-fly Construction of A_φ (Intuition) [cont.]

When the recursive applications of steps 1-3 has terminated and the automata graph has been built, then apply the following:

Step 4: For every $\psi_i \mathbf{U} \varphi_i$, for every state q_j, mark q_j with F_i iff $(\psi_i \mathbf{U} \varphi_i) \notin q_j$ *or* $\varphi_i \in q_j$
(If there is no U-subformulas, then mark all states with F_1
—i.e., $FT \stackrel{\text{def}}{=} \{Q\}$).

On-the-fly Construction of A_ϕ - State

- Henceforth, a state is represented by a tuple $s := \langle \lambda, \chi, \sigma \rangle$ where:
  - λ is the set of labels
  - χ is the next part, i.e. the set of X-formulas satisfied by s
  - σ is the set of the subformulas of ϕ satisfied by s (necessary for the fairness definition)
- Given a set of LTL formulas $\Psi \stackrel{\text{def}}{=} \{\psi_1, ..., \psi_k\}$, we define $Cover(\Psi) \stackrel{\text{def}}{=} Expand(\Psi, \langle \emptyset, \emptyset, \emptyset \rangle)$ to be the set of initial states of the Buchi automaton representing $\bigwedge_j \psi_j$.
  - Combines steps 1. and 2. of previous slides
  - Expand() defined recursively as follows

On-the-fly Construction of A_ϕ - Expand

Given a set of formulas Φ to expand and a state s, we define the set of states $Expand(\Phi, s)$ recursively as follows:

- if $\Phi = \emptyset$, $Expand(\Phi, s) = \{s\}$
- if $\bot \in \Phi$, $Expand(\Phi, s) = \emptyset$
- if $\top \in \Phi$ and $s = \langle \lambda, \chi, \sigma \rangle$,
  $Expand(\Phi, s) = Expand(\Phi \backslash \{\top\}, \langle \lambda, \chi, \sigma \cup \{\top\} \rangle)$
- if $l \in \Phi$ and $s = \langle \lambda, \chi, \sigma \rangle$, l propositional literal
  $Expand(\Phi, s) = Expand(\Phi \backslash \{l\}, \langle \lambda \cup \{l\}, \chi, \sigma \cup \{l\} \rangle)$
  (add l to the labels of s and to set of satisfied formulas)
- if $\mathbf{X}\psi \in \Phi$ and $s = \langle \lambda, \chi, \sigma \rangle$,
  $Expand(\Phi, s) = Expand(\Phi \backslash \{X\psi\}, \langle \lambda, \chi \cup \{\psi\}, \sigma \cup \{\mathbf{X}\psi\} \rangle)$
  (add ψ to the next part of s and $\mathbf{X}\psi$ to set of satisfied formulas)
- if $\psi_1 \wedge \psi_2 \in \Phi$ and $s = \langle \lambda, \chi, \sigma \rangle$,
  $Expand(\Phi, s) =$
  $Expand(\Phi \cup \{\psi_1, \psi_2\} \backslash \{\psi_1 \wedge \psi_2\}, \langle \lambda, \chi, \sigma \cup \{\psi_1 \wedge \psi_2\} \rangle)$
  (process both ψ_1 and ψ_2 and add $\psi_1 \wedge \psi_2$ to σ)

On-the-fly Construction of A_ϕ - Expand

- if $\psi_1 \vee \psi_2 \in \Phi$ and $s = \langle \lambda, \chi, \sigma \rangle$,
  $$Expand(\Phi, s) = Expand(\Phi \cup \{\psi_1\} \backslash \{\psi_1 \vee \psi_2\}, \langle \lambda, \chi, \sigma \cup \{\psi_1 \vee \psi_2\} \rangle) \cup Expand(\Phi \cup \{\psi_2\} \backslash \{\psi_1 \vee \psi_2\}, \langle \lambda, \chi, \sigma \cup \{\psi_1 \vee \psi_2\} \rangle)$$
  (split s in two copies, process ψ_2 on the first, ψ_1 on the second, add $\psi_1 \vee \psi_2$ to σ)
- if $\psi_1 \mathbf{U} \psi_2 \in \Phi$ and $s = \langle \lambda, \chi, \sigma \rangle$,
  $$Expand(\Phi, s) = Expand(\Phi \cup \{\psi_1\} \backslash \{\psi_1 \mathbf{U} \psi_2\}, \langle \lambda, \chi \cup \{\psi_1 \mathbf{U} \psi_2\}, \sigma \cup \{\psi_1 \mathbf{U} \psi_2\} \rangle) \cup Expand(\Phi \cup \{\psi_2\} \backslash \{\psi_1 \mathbf{U} \psi_2\}, \langle \lambda, \chi, \sigma \cup \{\psi_1 \mathbf{U} \psi_2\} \rangle)$$
  (split s in two copies and process ψ_1 on the first, ψ_2 on the second, add $\psi_1 \mathbf{U} \psi_2$ to σ)
- if $\psi_1 \mathbf{R} \psi_2 \in \Phi$ and $s = \langle \lambda, \chi, \sigma \rangle$,
  $$Expand(\Phi, s) = Expand(\Phi \cup \{\psi_2\} \backslash \{\psi_1 \mathbf{R} \psi_2\}, \langle \lambda, \chi \cup \{\psi_1 \mathbf{R} \psi_2\}, \sigma \cup \{\psi_1 \mathbf{R} \psi_2\} \rangle) \cup Expand(\Phi \cup \{\psi_1, \psi_2\} \backslash \{\psi_1 \mathbf{R} \psi_2\}, \langle \lambda, \chi, \sigma \cup \{\psi_1 \mathbf{R} \psi_2\} \rangle)$$
  (split s in two copies and process ψ_1 on the first, ψ_2 on the second, add $\psi_1 \mathbf{R} \psi_2$ to σ)

On-the-fly Construction of A_ϕ - Expand

Two relevant subcases: $\mathbf{F}\psi \stackrel{\text{def}}{=} \top\mathbf{U}\psi$ and $\mathbf{G}\psi \stackrel{\text{def}}{=} \bot\mathbf{R}\psi$

- if $\mathbf{F}\psi \in \Phi$ and $s = \langle\lambda, \chi, \sigma\rangle$,
  $Expand(\Phi, s) = Expand(\Phi \backslash \{\mathbf{F}\psi\}, \langle\lambda, \chi \cup \{\mathbf{F}\psi\}, \sigma \cup \{\mathbf{F}\psi\}\rangle)$
  $\cup Expand(\Phi \cup \{\psi\} \backslash \{\mathbf{F}\psi\}, \langle\lambda, \chi, \sigma \cup \{\mathbf{F}\psi\}\rangle)$
- if $\mathbf{G}\psi \in \Phi$ and $s = \langle\lambda, \chi, \sigma\rangle$,
  $Expand(\Phi, s) = Expand(\Phi \cup \{\psi\} \backslash \{\mathbf{G}\psi\}, \langle\lambda, \chi \cup \{\mathbf{G}\psi\}, \sigma \cup \{\mathbf{G}\psi\}\rangle)$
  Note: $Expand(\Phi \cup \{\bot, \psi\} \backslash \{\mathbf{G}\psi\}, ...) = \emptyset$

Definition of A_ϕ

Given a set of LTL formulas Ψ, we define $Cover(\Psi) \stackrel{\text{def}}{=} Expand(\Psi, \langle \emptyset, \emptyset, \emptyset \rangle)$.

For an LTL formula ϕ, we construct a Generalized NBA $A_\phi = (Q, Q_0, \Sigma, L, T, FT)$ as follows:

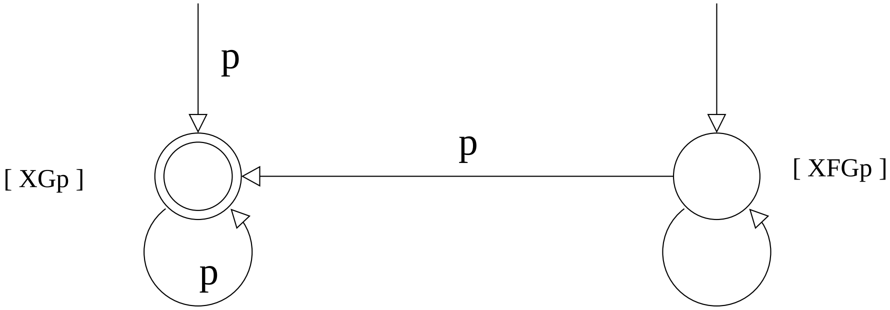
- $\Sigma \;=\; 2^{vars(\phi)}$
- Q is the smallest set such that
  - $Cover(\{\phi\}) \subseteq Q$
  - if $\langle \lambda, \chi, \sigma \rangle \in Q$, then $Cover(\chi) \in Q$
- $Q_0 = Cover(\{\phi\})$.
- $L(\langle \lambda, \chi, \sigma \rangle) = \{a \in \Sigma | a \models \lambda\}$
- $(s, s') \in T$ iff, $s = \langle \lambda, \chi, \sigma \rangle$ and $s' \in Cover(\chi)$
- $FT = \langle F_1, F_2, ..., F_k \rangle$ where, for all $(\psi_i \mathbf{U} \phi_i)$ occurring positively in ϕ, $F_i \;=\; \{\langle \lambda, \chi, \sigma \rangle \in Q \;\mid\; (\psi_i \mathbf{U} \phi_i) \notin \sigma \;\; or \;\; \phi_i \in \sigma\}$.
  (If there is no U-subformulas, then $FT \stackrel{\text{def}}{=} \{Q\}$).

Example: $\phi = \mathbf{FG}p$

- $$\begin{aligned} & Cover(\{\mathbf{FG}p\}) \\ & = Expand(\{\mathbf{FG}p\}, \langle \emptyset, \emptyset, \emptyset \rangle) \\ & = Expand(\emptyset, \langle \emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\} \rangle) \cup Expand(\{\mathbf{G}p\}, \langle \emptyset, \emptyset, \{\mathbf{FG}p\} \rangle) \\ & = \{\langle \emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\} \rangle\} \cup Expand(\{p\}, \langle \emptyset, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p\} \rangle) \\ & = \{\langle \emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\} \rangle\} \cup Expand(\emptyset, \langle \{p\}, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p, p\} \rangle) \\ & = \{\langle \emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\} \rangle, \langle \{p\}, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p, p\} \rangle\} \end{aligned}$$
- $$\begin{aligned} Cover(\{\mathbf{G}p\}) & = Expand(\{\mathbf{G}p\}, \langle \emptyset, \emptyset, \emptyset \rangle) \\ & = Expand(\{p\}, \langle \emptyset, \{\mathbf{G}p\}, \{\mathbf{G}p\} \rangle) \\ & = Expand(\emptyset, \langle \{p\}, \{\mathbf{G}p\}, \{\mathbf{G}p, p\} \rangle) \\ & = \{\langle \{p\}, \{\mathbf{G}p\}, \{\mathbf{G}p, p\} \rangle\} \end{aligned}$$
- Optimization:
  merge $\langle \{p\}, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p, p\} \rangle$ and $\langle \{p\}, \{\mathbf{G}p\}, \{\mathbf{G}p, p\} \rangle$

Example: $\phi = \mathbf{FG}p$

- Call $s_1 = \langle \emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\}\rangle$, $s_2 = \langle \{p\}, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p, p\}\rangle$
- $Q = \{s_1, s_2\}$
- $Q_0 = \{s_1, s_2\}$.
- $T: \quad s_1 \rightarrow \{s_1, s_2\},$
  $\quad\quad s_2 \rightarrow \{s_2\}$
- $FT = \langle F_1 \rangle$ where $F_1 = \{s_2\}$.

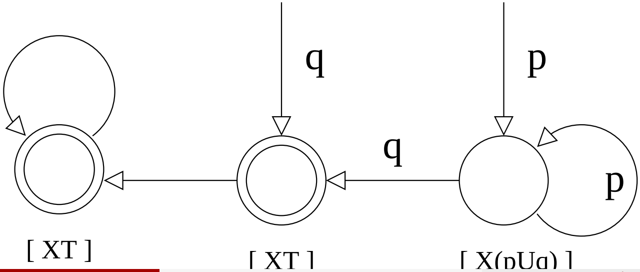


Example: $\phi = p\mathbf{U}q$

- $$\begin{aligned} & Cover(\{p\mathbf{U}q\}) \\ = \ & Expand(\{p\mathbf{U}q\}, \langle\emptyset, \emptyset, \emptyset\rangle) \\ = \ & Expand(\{p\}, \langle\emptyset, \{p\mathbf{U}q\}, \{p\mathbf{U}q\}\rangle) \cup Expand(\{q\}, \langle\emptyset, \emptyset, \{p\mathbf{U}q\}\rangle) \\ = \ & Expand(\emptyset, \langle\{p\}, \{p\mathbf{U}q\}, \{p\mathbf{U}q, p\}\rangle) \cup Expand(\emptyset, \langle\{q\}, \emptyset, \{p\mathbf{U}q, q \\ = \ & \{\langle\{p\}, \{p\mathbf{U}q\}, \{p\mathbf{U}q, p\}\rangle\} \cup \{\langle\{q\}, \{\top\}, \{p\mathbf{U}q, q\}\rangle\} \end{aligned}$$
- $Cover(\{\top\}) = \{\langle\emptyset, \{\top\}, \{\top\}\rangle\}$

Example: $\phi = p\mathbf{U}q$

- Let $s_1 =_{def} \langle \{p\}, \{p\mathbf{U}q\}, \{p\mathbf{U}q, p\} \rangle$, $s_2 =_{def} \langle \{q\}, \{\top\}, \{p\mathbf{U}q, q\} \rangle$, $s_3 =_{def} \langle \emptyset, \{\top\}, \{\top\} \rangle$.
- $Q = \{s_1, s_2, s_3\}$,
- $Q_0 = \{s_1, s_2\}$,
- $T: \quad s_1 \rightarrow \{s_1, s_2\}$,
  $s_2 \rightarrow \{s_3\}$
  $s_3 \rightarrow \{s_3\}$
- $FT = \langle F_1 \rangle$ where $F_1 = \{s_2, s_3\}$.



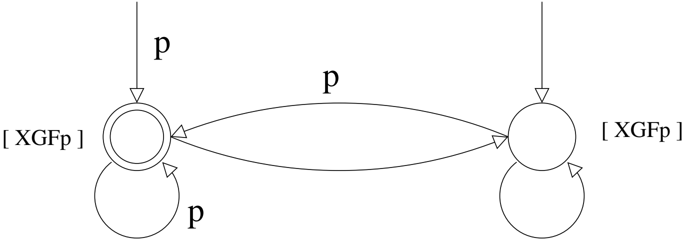
Example: $\phi = \mathbf{GF}p$

$$
\begin{array}{rl}
& \mathit{Cover}(\{\mathbf{GF}p\}) \\
= & E(\{\mathbf{GF}p\}, \langle \emptyset, \emptyset, \emptyset \rangle) \\
= & E(\{\mathbf{F}p\}, \langle \emptyset, \{\mathbf{GF}p\}, \{\mathbf{GF}p\} \rangle) \\
= & E(\{\}, \langle \emptyset, \{\mathbf{GF}p, \mathbf{F}p\}, \{\mathbf{GF}p, \mathbf{F}p\} \rangle) \cup E(\{p\}, \langle \{\}, \{\mathbf{GF}p\}, \{\mathbf{GF}p, \mathbf{F}p\} \rangle) \\
= & E(\{\}, \langle \emptyset, \{\mathbf{GF}p, \mathbf{F}p\}, \{\mathbf{GF}p, \mathbf{F}p\} \rangle) \cup E(\{\}, \langle \{p\}, \{\mathbf{GF}p\}, \{\mathbf{GF}p, \mathbf{F}p, p\} \rangle \\
= & \{\langle \emptyset, \{\mathbf{GF}p, \mathbf{F}p\}, \{\mathbf{GF}p, \mathbf{F}p\} \rangle\} \cup \{\langle \{p\}, \{\mathbf{GF}p\}, \{\mathbf{GF}p, \mathbf{F}p, p\} \rangle\}
\end{array}
$$

Note: $\mathbf{GF}p \wedge \mathbf{F}p \Longleftrightarrow \mathbf{GF}p$, s.t. $\mathit{Cover}(\mathbf{GF}p \wedge \mathbf{F}p) = \mathit{Cover}(\mathbf{GF}p)$

Example: $\mathbf{GF}p$

- Let $s_1 =_{def} \langle \{p\}, \{\mathbf{GF}p\}, \{\mathbf{GF}p, \mathbf{F}p, p\} \rangle$,
  $s_2 =_{def} \langle \emptyset, \{\mathbf{GF}p, \mathbf{F}p\}, \{\mathbf{GF}p, \mathbf{F}p\} \rangle$,
- $Q = \{s_1, s_2\}$,
- $Q_0 = \{s_1, s_2\}$,
- $T: \quad s_1 \to \{s_1, s_2\},$
  $\quad\quad s_2 \to \{s_1, s_2\}$
- $FT = \langle F_1 \rangle$ where $F_1 = \{s_1\}$.



NBAs of disjunctions of formulas

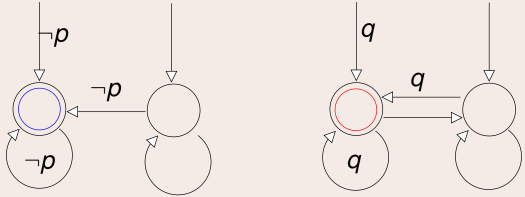
Remark

If $\varphi \stackrel{\text{def}}{=} (\varphi_1 \vee \varphi_2)$ and $A_{\varphi_1}, A_{\varphi_2}$ are NBAs encoding φ_1 and φ_2 resp., then $\mathcal{L}(\varphi) = \mathcal{L}(\varphi_1) \cup \mathcal{L}(\varphi_2)$, so that $A_\varphi \stackrel{\text{def}}{=} A_{\varphi_1} \cup A_{\varphi_2}$ is an NBA encoding φ

- A_φ non necessarily the smallest/best NBA encoding φ

Example

Let $\varphi \stackrel{\text{def}}{=} (\mathbf{GF}p \rightarrow \mathbf{GF}q)$, i.e., $\varphi \equiv (\mathbf{FG}\neg p \vee \mathbf{GF}q)$.
Then $A_{\mathbf{FG}\neg p} \cup A_{\mathbf{GF}q}$ encodes φ:



Suggested Exercises:

- Find an NBA encoding:
  - p
  - $\mathbf{F}p$
  - $\mathbf{G}p$
  - $p\mathbf{R}q$
  - $(\mathbf{GF}p \wedge \mathbf{GF}q) \rightarrow \mathbf{G}r$

Automata-Theoretic LTL Model Checking: complexity

Four steps:

(i) Compute A_M: $|A_M| = O(|M|)$

(ii) Compute A_φ: $|A_\varphi| = O(2^{|\varphi|})$

(iii) Compute the product $A_M \times A_\varphi$:
$|A_M \times A_\varphi| = |A_M| \cdot |A_\varphi| = O(|M| \cdot 2^{|\varphi|})$

(iv) Check the emptiness of $\mathcal{L}(A_M \times A_\varphi)$: $O(|A_M \times A_\varphi|) = O(|M| \cdot 2^{|\varphi|})$

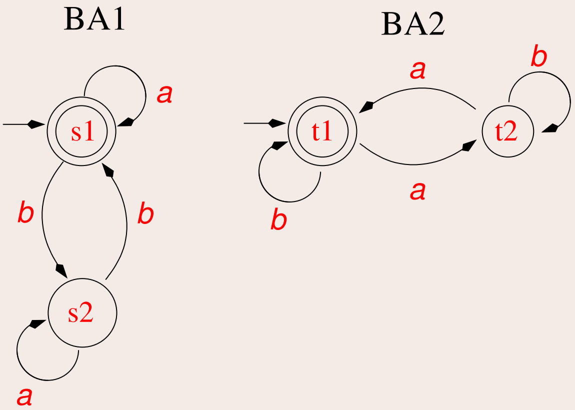
$\Longrightarrow$ the complexity of LTL M.C. grows linearly wrt. the size of the model M and exponentially wrt. the size of the property φ

Final Remarks

- Büchi automata are in general more expressive than LTL!
  $\Longrightarrow$ Some tools (e.g., Spin) allow specifications to be expressed directly as NBAs
  $\Longrightarrow$ complementation of NBA important!
- for every LTL formula, there are many possible equivalent NBAs
  $\Longrightarrow$ lots of research for finding "the best" conversion algorithm
- performing the product and checking emptiness very relevant
  $\Longrightarrow$ lots of techniques developed (e.g., partial order reduction)
  $\Longrightarrow$ lots on ongoing research

Ex: Product of Büchi automata

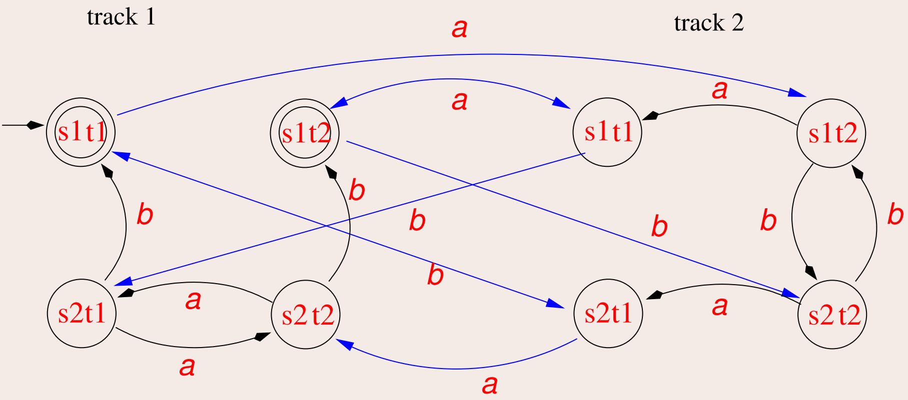
Given the following two Büchi automata (doubly-circled states represent accepting states, *a*, *b* are labels):



Write the product Büchi automaton $BA1 \times BA2$.

Ex: Product of Büchi automata

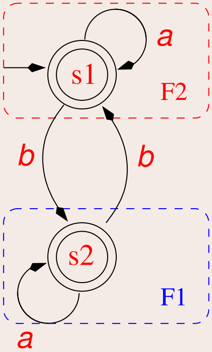
[Solution: The product is:



]

Ex: De-generalization of Büchi Automata

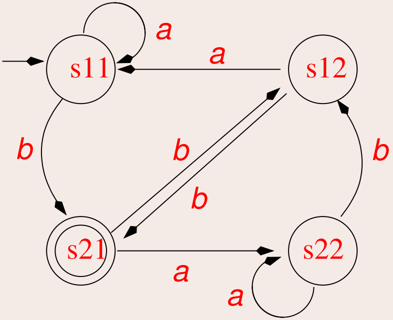
Given the following generalized Büchi automaton $A \stackrel{\text{def}}{=} \langle Q, \Sigma, \delta, I, FT \rangle$, with two sets of accepting states $FT \stackrel{\text{def}}{=} \{F1, F2\}$ s.t. $F1 \stackrel{\text{def}}{=} \{s2\}$, $F2 \stackrel{\text{def}}{=} \{s1\}$:



convert it into an equivalent plain Büchi automaton.

Ex: De-generalization of Büchi Automata

[Solution: The result is:



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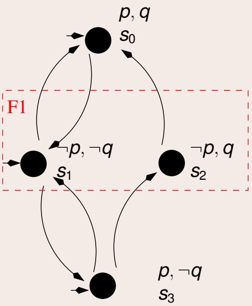
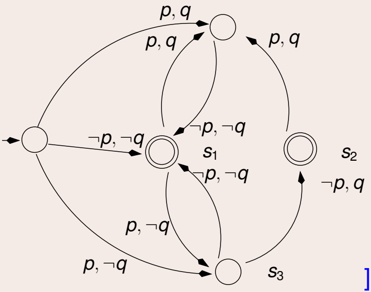
Ex: From Kripke models to Büchi automata

Given the following fair Kripke model M, convert it into an equivalent Buchi automaton.



[Solution:



]

Ex: Construction of Büchi Automata

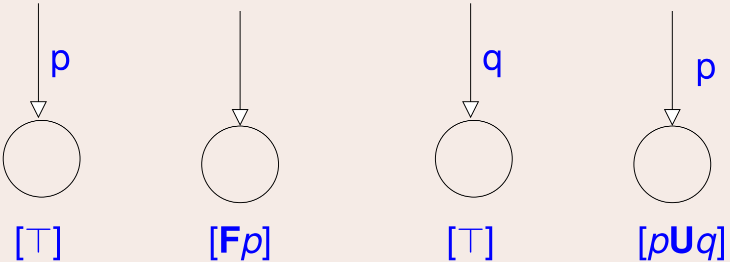
Consider the LTL formula $\varphi \stackrel{\text{def}}{=} (\mathbf{G}\neg p) \rightarrow (p\mathbf{U}q)$.

(a) rewrite φ into Negative Normal Form

[Solution: $(\mathbf{G}\neg p) \rightarrow (p\mathbf{U}q) \Longrightarrow (\neg\mathbf{G}\neg p) \vee (p\mathbf{U}q) \Longrightarrow (\mathbf{F}p) \vee (p\mathbf{U}q)$]

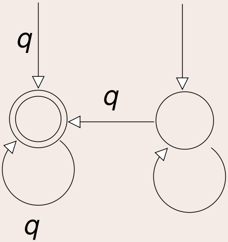
(b) find the initial states of a corresponding Buchi automaton (for each state, define the labels of the incoming arcs and the “next” section.)

[Solution: Applying tableaux rules we obtain: $p \vee \mathbf{XF}p \vee q \vee (p \wedge \mathbf{X}(p\mathbf{U}q))$, which is already in disjunctive normal form. This correspond to the following four initial states:



Ex: Büchi automaton

Given the following Büchi automaton BA (doubly-circled states represent accepting states):



Say which of the following sentences are true and which are false.

(*a*) BA accepts all and only the paths verifying **GF**q. [Solution: false]
(*b*) BA accepts all and only the paths verifying **FG**q. [Solution: true]
(*c*) BA accepts only paths verifying **F**q, but not all of them. [Solution: true]
(*d*) BA accepts all the paths verifying **F**q, but not only them. [Solution: false]