Introduction to Formal Methods Chapter 04: CTL Model Checking

Roberto Sebastiani

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Ch. 04: CTL Model Checking

Outline

- CTL Model Checking: general ideas
- CTL Model Checking: a simple example
- Some theoretical issues
- CTL Model Checking: algorithms
- 5 CTL Model Checking: some examples
 - A relevant subcase: invariants
 - Exercises

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Outline



- 2) CTL Model Checking: a simple example
- 3 Some theoretical issues
- 4 CTL Model Checking: algorithms
- 5 CTL Model Checking: some examples
- 6 A relevant subcase: invariants
- 7) Exercises

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CTL Model Checking is a formal verification technique where...

• ...the system is represented as a Finite State Machine *M*:

• ...the property is expressed a CTL formula φ :

$\mathbf{AG}(p ightarrow \mathbf{AF}q)$

• ...the model checking algorithm checks whether in all initial states of M all the executions of the model satisfy the formula $(M \models \varphi)$.

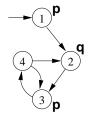
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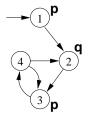
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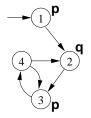
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CTL Model Checking: General Idea

Two macro-steps:

- 1 construct the set of states where the formula holds: $[\varphi] := \{ s \in S : M, s \models \varphi \}$ $([\varphi] \text{ is called the denotation of } \varphi)$
- 2 then compare with the set of initial states $L \subseteq [1,2]$
 - $I \subseteq [\varphi]$?

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```
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2 then compare with the set of initial states:

 $I \subseteq [\varphi]$?

In order to compute $[\varphi]$:

- proceed "bottom-up" on the structure of the formula, computing $[\varphi_i]$ for each subformula φ_i of $AG(p \rightarrow AFq)$:
 - [q],
 - [AFq],
 - [p],
 - $[p \rightarrow AFq],$
 - [AG(
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 - [*p* → AF*q*],
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In order to compute each $[\varphi_i]$:

- assign Propositional atoms by labeling function
- handle Boolean operators by standard set operations
- handle temporal operators AX, EX by computing pre-images
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Tableaux rules: a quote



"After all... tomorrow is another day." [Scarlett O'Hara, "Gone with the=Wind"]

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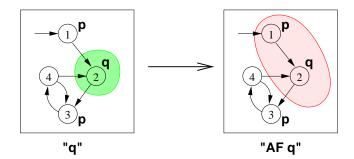
Outline



CTL Model Checking: a simple example

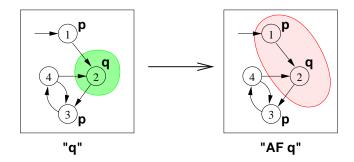
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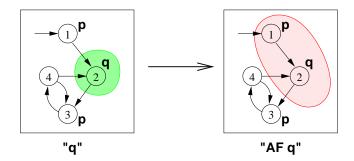
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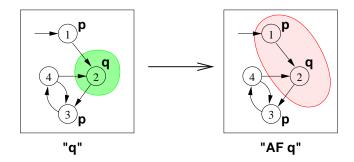
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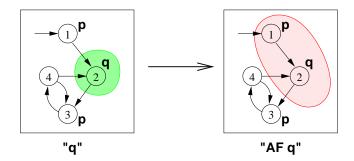


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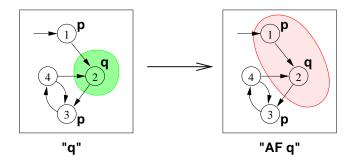
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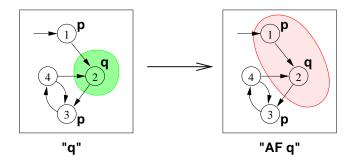
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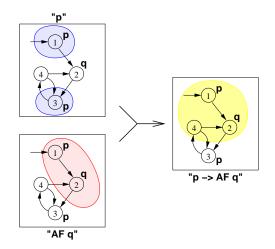
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[AFq]⁽³⁾ = [q ∨ AX(q ∨ AXq)] = {2} ∪ {1} = {1,2}
 ⇒ (fix point reached)

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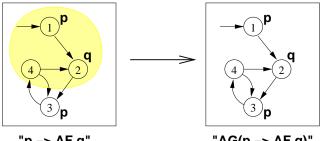
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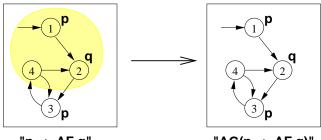
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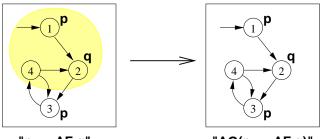
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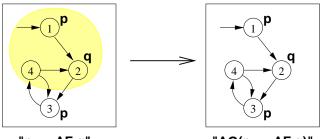
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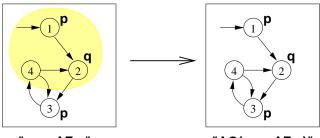
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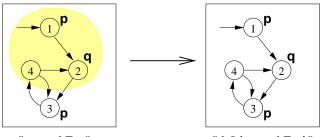
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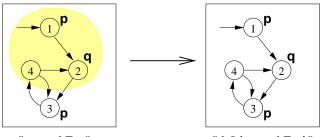
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- The set of states where the formula holds is empty
 ⇒ the initial state does not satisfy the property
 ⇒ M ⊭ AG(p → AFq)
- Counterexample: a lazo-shaped path: 1, 2, {3,4}^ω (satisfying EF(p ∧ EG¬q))

Note

Counter-example reconstruction in general is not trivial, based on intermediate sets.

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Counter-example reconstruction in general is not trivial, based on intermediate sets.

Outline

- CTL Model Checking: general ideas
- 2) CTL Model Checking: a simple example
 - Some theoretical issues
- 4) CTL Model Checking: algorithms
- 5 CTL Model Checking: some examples
- 6 A relevant subcase: invariants
- 7) Exercises

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The fixed-point theory of lattice of sets

Definition

For any finite set S, the structure (2^S, ⊆) forms a complete lattice with ∪ as join and ∩ as meet operations.

• A function $F : 2^S \mapsto 2^S$ is monotonic provided $S_1 \subseteq S_2 \Rightarrow F(S_1) \subseteq F(S_2)$.

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CTL Model Checking and Lattices

- If *M* = ⟨*S*, *I*, *R*, *L*, *AP*⟩ is a Kripke structure, then ⟨2^S, ⊆⟩ is a complete lattice
- We identify φ with its denotation [φ]

⇒ we can see logical operators as functions $F : 2^S \mapsto 2^S$ on the complete lattice $\langle 2^S, \subseteq \rangle$

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Denotation of a CTL formula φ : [φ]

Definition of $[\varphi]$

 $[\varphi] := \{ \pmb{s} \in \pmb{S} : \pmb{M}, \pmb{s} \models \varphi \}$

Recursive definition of $[\varphi]$

$$[true] = S$$

$$[false] = \{\}$$

$$[p] = \{s | p \in L(s)\}$$

$$[\neg \varphi_1] = S/[\varphi_1]$$

$$[\varphi_1 \land \varphi_2] = [\varphi_1] \cap [\varphi_2]$$

$$[EX\varphi] = \{s \mid \exists s' \in [\varphi] \ s.t. \ \langle s, s' \rangle \in R\}$$

$$[EG\beta] = \nu Z.([\beta] \cap [EXZ])$$

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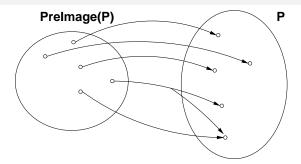
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• [**EX** φ] is said to be the Pre-image of [φ] (*Preimage*([φ]))

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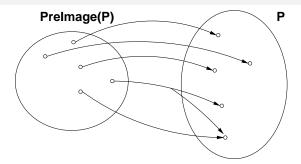
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Preimage() is monotonic: $X \subseteq X' \Longrightarrow Preimage(X) \subseteq Preimage(X')$

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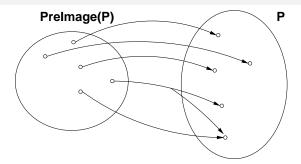
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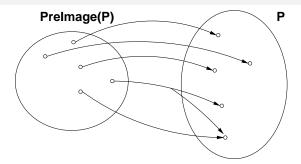
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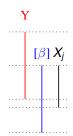
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- $\mu Z.([\beta_2] \cup ([\beta_1] \cap [\mathbf{E}XZ]))$: least fixed point of the function $F_{\beta_1,\beta_2}: 2^S \mapsto 2^S$, s.t. $F_{\beta_1,\beta_2}([\varphi]) = [\beta_2] \cup ([\beta_1] \cap Preimage([\varphi]))$ $= [\beta_2] \cup ([\beta_1] \cap \{s \mid \exists s' \in [\varphi] \ s.t. \ \langle s, s' \rangle \in R\})$
- F_{β_1,β_2} Monotonic: $a \subseteq a' \Longrightarrow F_{\beta_1,\beta_2}(a) \subseteq F_{\beta_1,\beta_2}(a')$
 - (Tarski's theorem): $\mu x.F_{\beta_1,\beta_2}(x)$ always exists
 - (Kleene's theorem): $\mu x.F_{\beta_1,\beta_2}(x)$ can be computed as the limit $\emptyset \subseteq F_{\beta_1,\beta_2}(\emptyset) \subseteq F_{\beta_1,\beta_2}(\mathcal{F}_{\beta_1,\beta_2}(\emptyset)) \subseteq \ldots$, in a finite number of steps.

Theorem (Clarke & Emerson)

 $[\mathbf{E}(\beta_1\mathbf{U}\beta_2)] = \mu Z.(\ [\beta_2] \cup ([\beta_1] \cap [\mathbf{E}\mathbf{X}Z])\)$

Case EU [cont.]

- We can compute X := [E(β₁Uβ₂)] inductively as follows:

 - $X_{j+1} := F_{\beta_1,\beta_2}^{j+1}(\emptyset)) = [\beta_2] \cup ([\beta_1] \cap Preimage(X_j))$
- Noticing that $X_1 = [\beta_2]$ and $X_{j+1} \supseteq X_j$ for every $j \ge 0$, and that $([\beta_2] \cup Y) \supseteq X_j \supseteq [\beta_2] \Longrightarrow ([\beta_2] \cup Y) = (X_j \cup Y)$, we can use instead the following inductive schema:

•
$$X_1 := [\beta_2]$$

• $X_{j+1} := X_j \cup ([\beta_1] \cap Preimage(X_j))$

Case EU [cont.]

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A relevant subcase: EF

• $\mathbf{EF}\beta = \mathbf{E}(\top \mathbf{U}\beta)$

- $[\top] = S \Longrightarrow [\top] \cap Preimage(X_j) = Preimage(X_j)$
- We can compute $X := [\mathbf{EF}\beta]$ inductively as follows:
 - X₁ := [β]
 X_{i+1} := X_i ∪ Preimage(X_i)
 - $X_{j+1} := X_j \cup Preimage(X_j)$

A relevant subcase: EF

• $\mathbf{EF}\beta = \mathbf{E}(\top \mathbf{U}\beta)$

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• We can compute $X := [\mathbf{EF}\beta]$ inductively as follows:

•
$$X_1 := [\beta]$$

• $X_{i+1} := X_i \cup Preimage(X_i)$

A relevant subcase: EF

- $\mathbf{EF}\beta = \mathbf{E}(\top \mathbf{U}\beta)$
- $[\top] = S \Longrightarrow [\top] \cap Preimage(X_j) = Preimage(X_j)$
- We can compute X := [EFβ] inductively as follows:
 - $X_1 := [\beta]$
 - $X_{j+1} := X_j \cup Preimage(X_j)$

Outline

- CTL Model Checking: general ideas
- 2 CTL Model Checking: a simple example
- 3 Some theoretical issues
 - CTL Model Checking: algorithms
- 5 CTL Model Checking: some examples
- 6 A relevant subcase: invariants
- Zercises

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Assume φ written in terms of ¬, ∧, EX, EU, EG

A general M.C. algorithm (fix-point):

1. for every $\varphi_i \in Sub(\varphi)$, find $[\varphi_i]$

2. Check if $I \subseteq [\varphi]$

• Subformulas $Sub(\varphi)$ of φ are checked bottom-up

• To compute each $[\varphi_i]$: if the main operator of φ_i is a

Propositional atoms: apply labeling function

- Boolean operator: apply standard set operations
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- Subformulas Sub(φ) of φ are checked bottom-up
- To compute each [φ_i]: if the main operator of φ_i is a
 - Propositional atoms: apply labeling function
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General M.C. Procedure

```
state set Check(CTL formula \beta) {
    case \beta of
    true:
                    return S:
    false:
                    return {};
                    return {s \mid p \in L(s)};
    p:
    \neg \beta_1:
                    return S / Check(\beta_1):
    \beta_1 \wedge \beta_2:
                    return Check(\beta_1) \cap Check(\beta_2);
    \mathbf{EX}\beta_1:
                    return Prelmage(Check(\beta_1));
    EG\beta_1:
                    return Check EG(Check(\beta_1));
    \mathbf{E}(\beta_1 \mathbf{U} \beta_2):
                    return Check EU(Check(\beta_1),Check(\beta_2));
```

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PreImage

```
state_set PreImage(state_set [\beta]) {

X := \{\};

for each s \in S do

for each s' \ s.t. \ s' \in [\beta] and \langle s, s' \rangle \in R do

X := X \cup \{s\};

return X;
```

Check_EG

```
\begin{array}{l} \textbf{state\_set Check\_EG(state\_set [\beta]) } \\ X' := [\beta]; \ j := 1; \\ \textbf{repeat} \\ X' := X'; \ j := j + 1; \\ X' := X \cap PreImage(X); \\ \textbf{until } (X' = X); \\ \textbf{return } X; \\ \end{array}
```

Check_EU

```
\begin{array}{l} \textbf{state\_set Check\_EU(state\_set [\beta_1], [\beta_2]) } \\ X' := [\beta_2]; \ j := 1; \\ \textbf{repeat} \\ X := X'; \ j := j + 1; \\ X' := X \cup ([\beta_1] \cap \textit{PreImage}(X)); \\ \textbf{until } (X' = X); \\ \textbf{return } X; \\ \end{array}
```

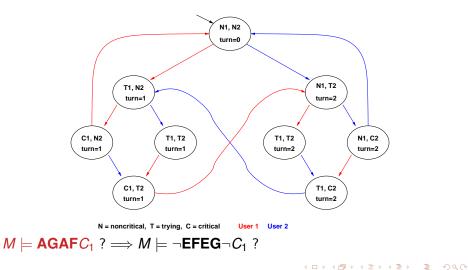
A relevant subcase: Check_EF

```
\begin{array}{l} \textbf{state\_set Check\_EF(state\_set [\beta]) } \{ \\ X' := [\beta]; \ j := 1; \\ \textbf{repeat} \\ X' := X'; \ j := j + 1; \\ X' := X \cup PreImage(X); \\ \textbf{until } (X' = X); \\ \textbf{return } X; \\ \} \end{array}
```

Outline

- CTL Model Checking: general ideas
- 2 CTL Model Checking: a simple example
- 3 Some theoretical issues
- 4) CTL Model Checking: algorithms
- 5 CTL Model Checking: some examples
 - 6 A relevant subcase: invariants
 - 7 Exercises

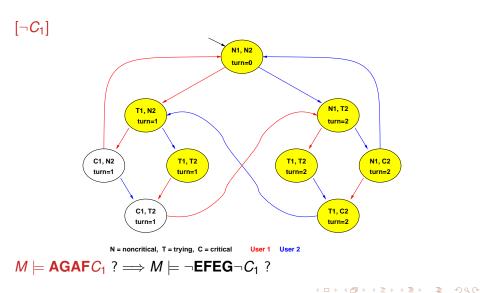
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Ch. 04: CTL Model Checking

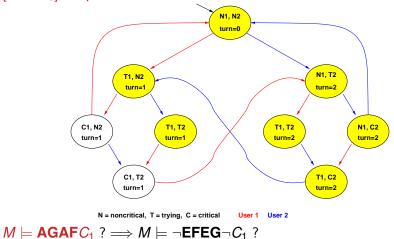
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[**EG**¬*C*₁], step 0:



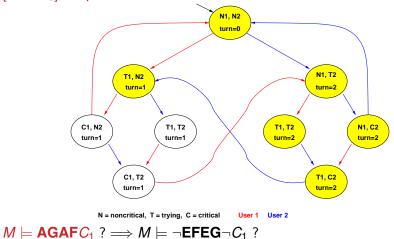
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[**EG**¬*C*₁], step 1:

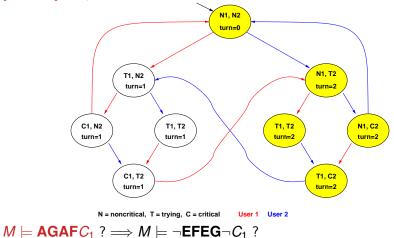


Roberto Sebastiani

Ch. 04: CTL Model Checking

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Monday 18th May, 2020 37/72

[**EG**¬*C*₁], step 2:

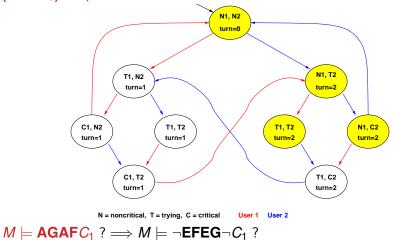


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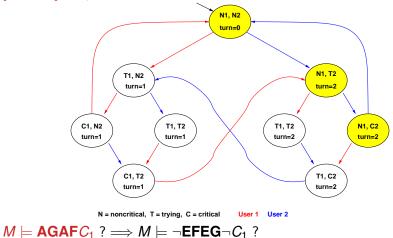


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[**EG** \neg *C*₁], step 4:



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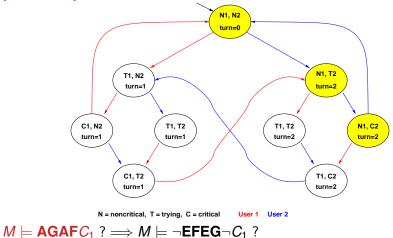
[**EG** \neg *C*₁], FIXPOINT! N1, N2 turn=0 N1. T2 T1. N2 turn=1 turn=2 C1, N2 T1, T2 T1, T2 N1, C2 turn=1 turn=1 turn=2 turn=2 C1, T2 T1, C2 turn=2 turn=1 N = noncritical, T = trying, C = critical User 1 User 2 $M \models \mathsf{AGAFC}_1 ? \Longrightarrow M \models \neg \mathsf{EFEG} \neg C_1 ?$

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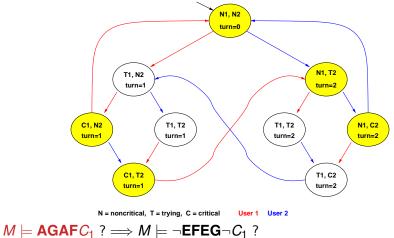
$[\mathbf{EFEG} \neg C_1], \text{STEP 0}$



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[**EFEG**¬*C*₁], STEP 1

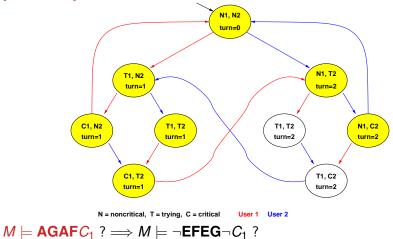


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$[\mathbf{EFEG} \neg C_1], \text{STEP 2}$



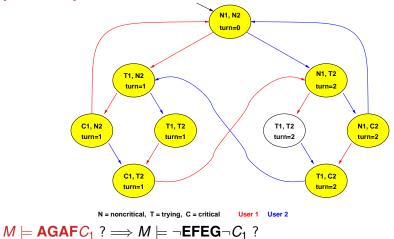
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[**EFEG** \neg *C*₁], STEP 3



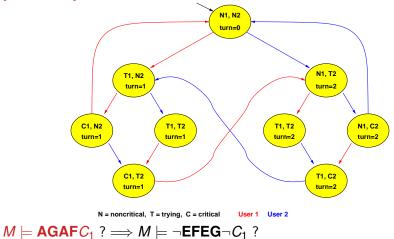
Roberto Sebastiani

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$[\mathbf{EFEG} \neg C_1], \text{STEP 4}$



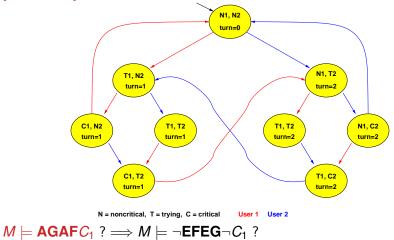
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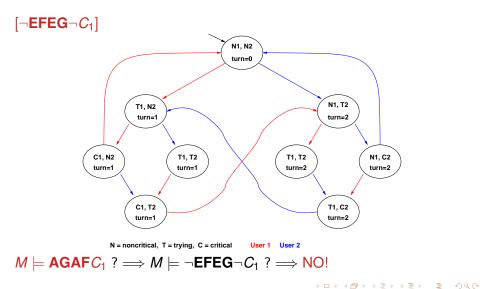
[**EFEG** \neg *C*₁], FIXPOINT!



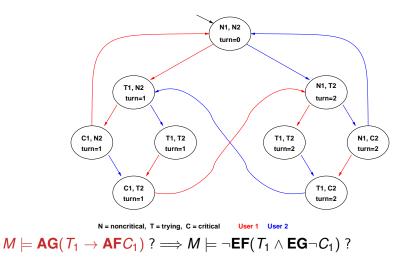
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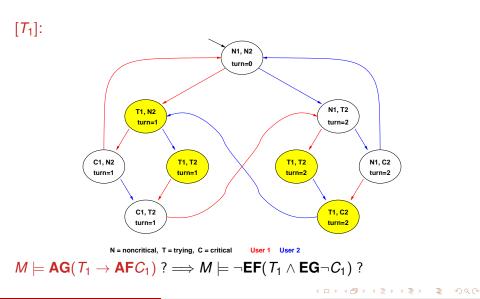


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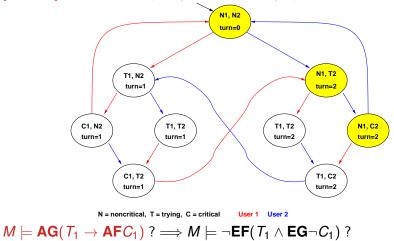
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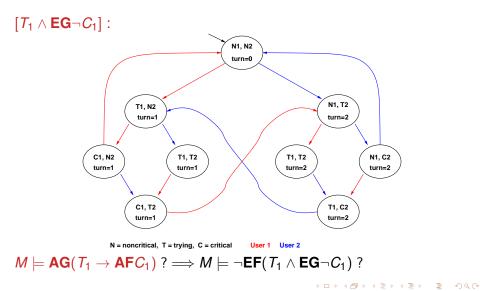
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[**EG** \neg *C*₁], STEPS 0-4: (see previous example)



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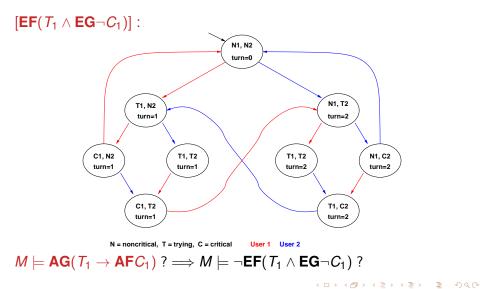
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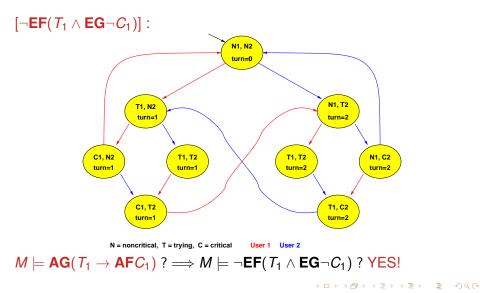
Ch. 04: CTL Model Checking

Monday 18th May, 2020



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CTL Model Checking: some examples



The property verified is...

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Homework

Apply the same process to all the CTL examples of Chapter 3.

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- Step 1: compute $[\varphi]$
 - Compute $[\varphi]$ bottom-up on the $O(|\varphi|)$ sub-formulas of φ : $O(|\varphi|)$ steps...
 - ... each requiring at most exploring O(|M|) states
 - $\Longrightarrow O(|M| \cdot |arphi|)$ steps
- Step 2: check $I \subseteq [\varphi]$: O(|M|)
- $\Longrightarrow O(|M| \cdot |\varphi|)$

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 - $\Longrightarrow O(|M| \cdot |arphi|)$ steps
- Step 2: check $I \subseteq [\varphi]$: O(|M|)
- $\implies O(|M| \cdot |\varphi|)$

Outline

- 1 CTL Model Checking: general ideas
- 2 CTL Model Checking: a simple example
- 3 Some theoretical issues
- 4 CTL Model Checking: algorithms
- 5) CTL Model Checking: some examples
- A relevant subcase: invariants
- Exercises

Ch. 04: CTL Model Checking

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- Invariant properties have the form **AG p** (e.g., **AG**¬bad)
- Checking invariants is the negation of a reachability problem:
 - is there a reachable state that is also a bad state? (AG¬bad = ¬EFbad)
- Standard M.C. algorithm reasons backward from the bad by iteratively applying PreImage computations:

 $Y' := Y \cup PreImage(Y)$

until a fixed point is reached. Then the complement is computed and *I* is checked for inclusion in the resulting set.

• Better algorithm: reasons backward from the bad by iteratively applying PreImage computations:

 $Y' := Y \cup PreImage(Y)$

until (i) it intersect [/] or (ii) a fixed point is reached

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Ch. 04: CTL Model Checking

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 $(\mathbf{AG} \neg bad = \neg \mathbf{EF} bad)$

Standard M.C. algorithm reasons backward from the bad by

• Better algorithm: reasons backward from the bad by iteratively

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Ch. 04: CTL Model Checking

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• Better algorithm: reasons backward from the *bad* by iteratively applying PreImage computations:

 $Y' := Y \cup PreImage(Y)$

until (i) it intersect [/] or (ii) a fixed point is reached

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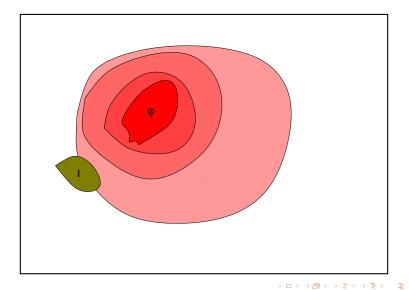
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Model Checking of Invariants [cont.]



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Alternative algorithm (often more efficient): forward checking

- Compute the set of bad states [bad]
- Compute the set of initial states I
- Compute incrementally the set of reachable states from *I* until (i) it intersect [*bad*] or (ii) a fixed point is reached
- Basic step is the (Forward) Image:

 $\mathit{Image}(Y) \stackrel{\text{\tiny def}}{=} \{ s' \mid s \in Y \textit{ and } R(s,s') \textit{ holds} \}$

• Simplest form: compute the set of reachable states.

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Alternative algorithm (often more efficient): forward checking

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- Compute incrementally the set of reachable states from *I* until (i) it intersect [*bad*] or (ii) a fixed point is reached
- Basic step is the (Forward) Image:

 $\mathit{Image}(Y) \stackrel{\text{\tiny def}}{=} \{ s' \mid s \in Y \textit{ and } R(s,s') \textit{ holds} \}$

• Simplest form: compute the set of reachable states.

Computing Reachable states: basic

State_Set Compute_reachable() { $Y' := I; Y := \emptyset; j := 1;$ while $(Y' \neq Y)$ j := j + 1; Y := Y'; $Y' := Y \cup Image(Y);$ } return Y; }

Y=reachable

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Computing Reachable states: advanced

```
State_Set Compute_reachable() {

Y := F := I; j := 1;

while (F \neq \emptyset)

j := j + 1;

F := Image(F) \setminus Y;

Y := Y \cup F;

}

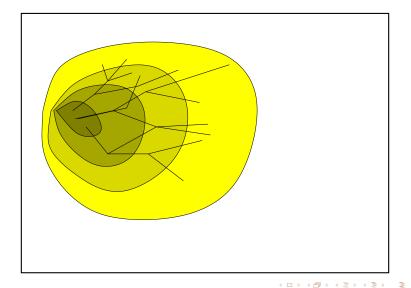
return Y;

}
```

```
Y=reachable;F=frontier (new)
```

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Computing Reachable states [cont.]



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Checking of Invariant Properties: basic

```
bool Forward Check EF(State Set BAD) {
    Y := I; Y' := \emptyset; i := 1;
   while (Y' \neq Y) and (Y' \cap BAD) = \emptyset
        i := i + 1:
        Y := Y':
        Y' := Y \cup Image(Y);
   if (Y' \cap BAD) \neq \emptyset // counter-example
        return true
   else
                         // fixpoint reached
        return false
```

Y=reachable;

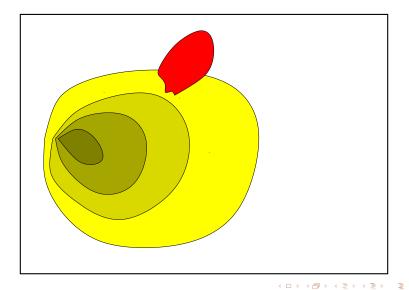
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Checking of Invariant Properties: advanced

```
bool Forward Check EF(State Set BAD) {
    Y := F := I; i := 1;
   while (F \neq \emptyset) and (F \cap BAD) = \emptyset
        i := i + 1;
         F := Image(F) \setminus Y;
         Y := Y \cup F:
    if (F \cap BAD) \neq \emptyset // counter-example
         return true
    else
                         // fixpoint reached
         return false
```

```
Y=reachable;F=frontier (new)
```

Checking of Invariant Properties [cont.]



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Checking of Invariants: Counterexamples

- if layer *n* intersects with the bad states, then the property is violated
- a counterexample can be reconstructed proceeding backwards
 (i) select any state of BAD ∩ F[n] (we know it is satisfiable), call it t[n]
 - (ii) compute *Preimage*(*t*[*n*]), i.e. the states that can result in *t*[*n*] in one step
 - (iii) compute $Preimage(t[n]) \cap F[n-1]$, and select one state t[n-1]
- iterate (i)-(iii) until the initial states are reached
- *t*[0], *t*[1],..., *t*[*n*] is our counterexample

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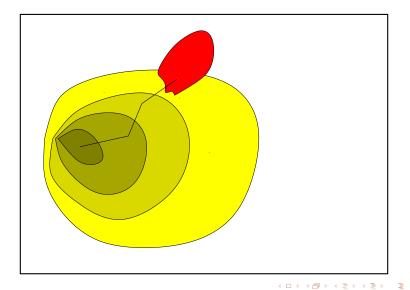
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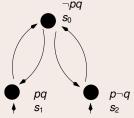
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Outline

- CTL Model Checking: general ideas
- 2 CTL Model Checking: a simple example
- 3 Some theoretical issues
- 4 CTL Model Checking: algorithms
- 5 CTL Model Checking: some examples
- A relevant subcase: invariants
- 2 Exercises

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Consider the Kripke Model *M* below, and the CTL property $\varphi \stackrel{\text{def}}{=} \mathbf{AG}((p \land q) \rightarrow \mathbf{EG}q)$.

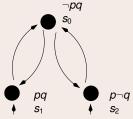


(a) Rewrite φ into an equivalent formula φ' expressed in terms of EX, EG, EU/EF only.

(b) Compute bottom-up the denotations of all subformulas of φ' . (Ex: $[p] = \{s_1, s_2\}$)

(c) As a consequence of point (b), say whether $M \models \varphi$ or not.

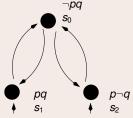
Consider the Kripke Model *M* below, and the CTL property $\varphi \stackrel{\text{def}}{=} \mathbf{AG}((p \land q) \rightarrow \mathbf{EG}q)$.



- (a) Rewrite φ into an equivalent formula φ' expressed in terms of **EX**, **EG**, **EU**/**EF** only. [Solution: $\varphi' = \neg \mathbf{EF} \neg ((\neg p \lor \neg q) \lor \mathbf{EG}q) = \neg \mathbf{EF}((p \land q) \land \neg \mathbf{EG}q)$]
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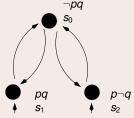
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- (a) Rewrite φ into an equivalent formula φ' expressed in terms of **EX**, **EG**, **EU**/**EF** only. [Solution: $\varphi' = \neg \mathbf{EF} \neg ((\neg p \lor \neg q) \lor \mathbf{EG}q) = \neg \mathbf{EF}((p \land q) \land \neg \mathbf{EG}q)$]
- (b) Compute bottom-up the denotations of all subformulas of φ' . (Ex: $[p] = \{s_1, s_2\}$) [Solution:

(c) As a consequence of point (b), say whether $M \models \varphi$ or not.

Consider the Kripke Model *M* below, and the CTL property $\varphi \stackrel{\text{def}}{=} \mathbf{AG}((p \land q) \to \mathbf{EG}q)$.



- (a) Rewrite φ into an equivalent formula φ' expressed in terms of **EX**, **EG**, **EU**/**EF** only. [Solution: $\varphi' = \neg \mathbf{EF} \neg ((\neg p \lor \neg q) \lor \mathbf{EG}q) = \neg \mathbf{EF}((p \land q) \land \neg \mathbf{EG}q)$]
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Consider the Kripke Model *M* below, and the CTL property $AG(AFp \rightarrow AFq)$.



(a) Rewrite φ into an equivalent formula $\varphi' \exp^{\hat{s}_i}$ espective of EX, EG, EU/EF only.

(b) Compute bottom-up the denotations of all subformulas of φ' . (Ex: $[p] = \{s_1, s_2\}$)

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Consider the Kripke Model *M* below, and the CTL property $AG(AFp \rightarrow AFq)$.



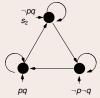
(a) Rewrite φ into an equivalent formula φ' expressed in terms of **EX**, **EG**, **EU**/**EF** only. [Solution:

 $\varphi' = \mathsf{AG}(\mathsf{AF}\rho \to \mathsf{AF}q) = \neg \mathsf{EF} \neg (\neg \mathsf{EG} \neg \rho \to \neg \mathsf{EG} \neg q) = \neg \mathsf{EF}(\neg \mathsf{EG} \neg \rho \land \mathsf{EG} \neg q)]$

(b) Compute bottom-up the denotations of all subformulas of φ' . (Ex: $[p] = \{s_1, s_2\}$)

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Consider the Kripke Model *M* below, and the CTL property $AG(AFp \rightarrow AFq)$.



(a) Rewrite φ into an equivalent formula φ' expressed in terms of EX, EG, EU/EF only. [Solution:

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Consider the Kripke Model *M* below, and the CTL property $AG(AFp \rightarrow AFq)$.



(a) Rewrite φ into an equivalent formula $\varphi' \exp^{i \theta}$ espectives of EX, EG, EU/EF only. [Solution:

 $\varphi' = \mathsf{AG}(\mathsf{AF}p \to \mathsf{AF}q) = \neg \mathsf{EF} \neg (\neg \mathsf{EG} \neg p \to \neg \mathsf{EG} \neg q) = \neg \mathsf{EF}(\neg \mathsf{EG} \neg p \land \mathsf{EG} \neg q)]$

(b) Compute bottom-up the denotations of all subformulas of φ' . (Ex: $[p] = \{s_1, s_2\}$) [Solution:

 $\begin{bmatrix} p \end{bmatrix} = \{s_0\} \quad [\neg q] = \{s_1\} \\ [\neg p] = \{s_1, s_2\} \quad [EG\neg q] = \{s_1\} \\ [EG\neg p] = \{s_1, s_2\} \quad [\neg EG\neg p \land EG\neg q] = \{\} \\ [\neg EG\neg p] = \{s_0\} \quad [EF(\neg EG\neg p \land EG\neg q)] = \{\} \\ [q] = \{s_0, s_2\} \quad [\neg EF(\neg EG\neg p \land EG\neg q)] = \{s_0, s_1, s_2\} \\ (c) \text{ As a consequence of point } (b), \text{ say whether } M \models \varphi \text{ or not.} \\ [Solution: Yes, \{s_0, s_1, s_2\} \subseteq [\varphi'].] \\ \hline \text{Roberto Sebastiani} \qquad Ch. 04: CTL Model Checking \qquad Monday 18^{th} May, 2020 \qquad 72/72 \\ \hline \end{array}$