Course Formal Methods
Module I: Automated Reasoning
Ch. 03: **Satisfiability Modulo Theories**

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Outline

1. Introduction
   - What is a Theory?
   - Satisfiability Modulo Theories
   - Motivations and Goals of SMT
2. Efficient SMT solving
   - Combining SAT with Theory Solvers
   - Theory Solvers for Theories of Interest (hints)
   - SMT for Combinations of Theories
3. Beyond Solving: Advanced SMT Functionalities
   - Proofs and Unsatisfiable Cores
   - Interpolants
   - All-SMT & Predicate Abstraction (hints)
   - SMT with Optimization (Optimization Modulo Theories)

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FOL Theories

Traditional Definition (FOL)

Given a FOL signature Σ, a Σ-Theory $\mathcal{T}$ (hereafter simply "theory") is a (possibly infinite) set of FOL closed formulas (axioms)

- Typically used to provide some intended interpretation to the symbols in the signature Σ
- FOL formulas deduces from these axioms via inference rules
- Definition used by logicians,
- Very low practical use in AR & Formal Verification

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Example: A FOL Theory of Positive Integer Numbers (aka "Peano Arithmetic", $\mathcal{P}$)

- Signature
  - (basic) unary predicate symbol: NatNum ("natural number")
  - (basic) unary function symbol: S ("successor")
  - (basic) constant symbol: 0
  - (derived) binary function symbols: +,* (infix)
  - (derived) constant symbols: 1,2,3,4,5,6,...
- Axioms
  1. $NatNum(0)$
  2. $\forall x.(NatNum(x) \rightarrow NatNum(S(x)))$
  3. $\forall x.(NatNum(x) \rightarrow (0 \neq S(x)))$
  4. $\forall x, y.((NatNum(x) \wedge NatNum(y)) \rightarrow ((x \neq y) \rightarrow (S(x) \neq S(y))))$
  5. $\forall x.(NatNum(x) \rightarrow (x = (0 + x)))$
  6. $\forall x, y.((NatNum(x) \wedge NatNum(y)) \rightarrow (S(x) + y) = S(x + y))$
  7. $1 = S(0), 2 = S(1), 3 = S(2), ...$
- Formulas deduced
  - ex: $\mathcal{P} \vdash NatNum(25)$
  - ex: $\mathcal{P} \vdash \forall x, y.((NatNum(x) \wedge NatNum(y)) \rightarrow ((x + y) = (y + x)))$

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Given a FOL signature Σ, a Σ-Theory $\mathcal{T}$ (hereafter simply "theory") is one (or more) model(s) constraining the interpretations of Σ

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    - ex: ". $<$." mapped into the arithmetical relation "less then"
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  These symbols are called interpreted
- Compliant with previous definition: model(s) satisfying all axioms
- Ad hoc "$\mathcal{T}$-aware" decision procedures for reasoning on formulas
- Very effective in practical applications

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Example: Linear Arithmetic on the Integers ($\mathcal{LIA}$)

- Domain: integer numbers
- Numerical constants interpreted as numbers
  - ex: “1”, “1346231” mapped directly into the corresponding number
- function and predicates interpreted as arithmetical operations
  - “+” as addiction, “*” as multiplication, “$<$” as less-then, . etc.
- ILP solvers used to do logical reasoning
  - ex: $(3x - 2y \leq 3) \wedge (4y - 2z < -7) \models (6x - 2z < -1)$

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Satisfiability, Validity, Entailment (Modulo a Theory $\mathcal{T}$)

Definitions

- Idea: We restrict to models satisfying $\mathcal{T}$ ("$\mathcal{T}$-models")
- A formula is satisfiable in $\mathcal{T}$ (aka "φ is $\mathcal{T}$-satisfiable") iff some model satisfying $\mathcal{T}$ satisfies also φ
  - ex: $(x < 3)$ satisfiable in $\mathcal{LIA}$
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- φ is $\mathcal{T}$-valid iff $\neg\varphi$ is $\mathcal{T}$-unsatisfiable
- $\varphi \models_{\mathcal{T}} \psi$ iff $\varphi \rightarrow \psi$ is $\mathcal{T}$-valid

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Satisfiability Modulo Theories (SMT($\mathcal{T}$))

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The problem of deciding the satisfiability of (typically quantifier-free) formulas in some decidable first-order theory $\mathcal{T}$

- $\mathcal{T}$ can also be a combination of theories $\bigcup_i \mathcal{T}_i$.

SMT($\mathcal{T}$): Theories of Interest

Some theories of interest (e.g., for formal verification)

- Equality and Uninterpreted Functions ($\mathcal{EUF}$):
  $((x = y) \wedge (y = f(z))) \rightarrow (g(x) = g(f(z)))$
- Difference logic ($\mathcal{DL}$): $((x = y) \wedge (y - z \leq 4)) \rightarrow (x - z \leq 6)$
- UTVPI ($\mathcal{UTVPI}$): $((x = y) \wedge (y - z \leq 4)) \rightarrow (x + z \leq 6)$
- Linear arithmetic over the rationals ($\mathcal{LRA}$):
  $(T_\delta \rightarrow (s_1 = s_0 + 3.4 \cdot t - 3.4 \cdot t_0)) \wedge (\neg T_\delta \rightarrow (s_1 = s_0))$
- Linear arithmetic over the integers ($\mathcal{LIA}$):
  $(x = x_l + 2^{16} x_h) \wedge (x \geq 0) \wedge (x \leq 2^{16} - 1)$
- Arrays ($\mathcal{AR}$): $(i = j) \vee read(write(a, i, e), j) = read(a, j)$
- Bit vectors ($\mathcal{BV}$):
  $x_{[16]}[15:0] = (y_{[16]}[15:8] :: z_{[16]}[7:0]) << w_{[8]}[3:0]$
- Non-Linear arithmetic over the reals ($\mathcal{NLA}(\mathbb{R})$) :
  $((c = a \cdot b) \wedge (a_1 = a - 1) \wedge (b_1 = b + 1)) \rightarrow (c = a_1 \cdot b_1 + 1)$
- ...

SMT($\mathcal{T}$): Theories of Interest

Some theories of interest (e.g., for formal verification)

- Equality and Uninterpreted Functions ($\mathcal{EUF}$):
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- Difference logic ($\mathcal{DL}$): $((x = y) \wedge (y - z \leq 4)) \rightarrow (x - z \leq 6)$
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Satisfiability Modulo Theories (SMT($\mathcal{T}$)): Example

Example: SMT($\mathcal{LIA} \cup \mathcal{EUF} \cup \mathcal{AR}$)

$$\varphi \stackrel{\text{def}}{=} (d \geq 0) \wedge (d < 1) \wedge$$
$$((f(d) = f(0)) \rightarrow (read(write(V, i, x), i + d) = x + 1))$$

- involves arithmetical, arrays, and uninterpreted function/predicate symbols, plus Boolean operators
  - Is it satisfiable?
  - No:

$$\begin{array}{ll} & \varphi \\ \Longrightarrow_{\mathcal{LIA}} & (d = 0) \\ \Longrightarrow_{\mathcal{EUF}} & (f(d) = f(0)) \\ \Longrightarrow_{Bool} & (read(write(V, i, x), i + d) = x + 1) \\ \Longrightarrow_{\mathcal{LIA}} & (read(write(V, i, x), i) = x + 1) \\ \Longrightarrow_{\mathcal{LIA}} & \neg(read(write(V, i, x), i) = x) \\ \Longrightarrow_{\mathcal{AR}} & \bot \end{array}$$

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SMT and SMT solvers

Common fact about SMT problems from various applications

SMT requires capabilities for heavy Boolean reasoning combined with capabilities for reasoning in expressive decidable F.O. theories

- SAT alone not expressive enough
- standard automated theorem proving inadequate (e.g., arithmetic)
- may involve also numerical computation (e.g., simplex)

Modern SMT solvers

- combine SAT solvers with $\mathcal{T}$-specific decision procedures (theory solvers or $\mathcal{T}$-solvers)
  - contributions from SAT, Automated Theorem Proving (ATP), formal verification (FV) and operational research (OR)

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Notational remark (1): most/all examples in $\mathcal{LRA}$

For better readability, in most/all the examples of this presentation we will use the theory of linear arithmetic on rational numbers ($\mathcal{LRA}$) because of its intuitive semantics. E.g.:

$$(\neg A_1 \vee (3x_1 - 2x_2 - 3 \leq 5)) \wedge (A_2 \vee (-2x_1 + 4x_3 + 2 = 3))$$

Nevertheless, analogous examples can be built with all other theories of interest.

Notational remark (2): “constants” vs. “variables”

- Consider, e.g., the formula:
  $(\neg A_1 \vee (3x_1 - 2x_2 - 3 \leq 5)) \wedge (A_2 \vee (-2x_1 + 4x_3 + 2 = 3))$
- How do we call A_1, A_2?:
  (*a*) Boolean/propositional variables?
  (*b*) uninterpreted 0-ary predicates?
- How do we call x_1, x_2, x_3?:
  (*a*) domain variables?
  (*b*) uninterpreted Skolem constants/0-ary uninterpreted functions?
- Hint:
  (*a*) typically used in SAT, CSP and OR communities
  (*b*) typically used in logic & ATP communities

Hereafter we call A_1, A_2 “Boolean/propositional variables” and x_1, x_2, x_3 "domain variables” (logic purists, please forgive me!)

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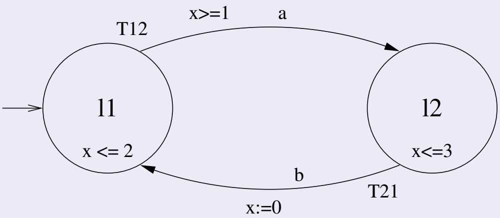
Some Motivating Applications

Interest in SMT triggered by some real-word applications

- Verification of Hybrid & Timed Systems
- Verification of RTL Circuit Designs & of Microcode
- SW Verification
- Planning with Resources
- Temporal reasoning
- Scheduling
- Compiler optimization
- ...

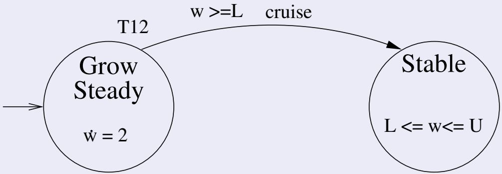
Verification of Timed Systems



- Bounded/inductive model checking of Timed Systems [6, 33, 53], ...
- Timed Automata encoded into $\mathcal{T}$-formulas:
  - discrete information (locations, transitions, events) with Boolean vars.
  - timed information (clocks, elapsed time) with differences $(t_3 - x_3 \leq 2)$, equalities $(x_4 = x_3)$ and linear constraints $(t_8 - x_8 = t_2 - x_2)$ on $\mathbb{Q}$

$\Longrightarrow$ SMT on $\mathcal{DL}(\mathbb{Q})$ or $\mathcal{LRA}$ required

Verification of Hybrid Systems ...

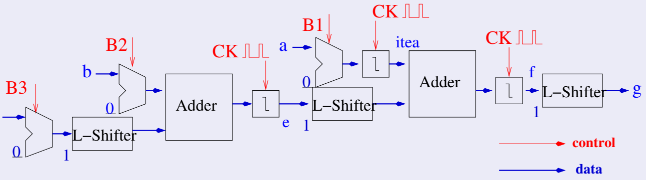


- Bounded model checking of Hybrid Systems [5],...
- Hybrid Automata encoded into $\mathcal{L}$-formulas:
  - discrete information (locs, trans., events) with Boolean vars.
  - timed information (clocks, elapsed time) with differences $(t_3 - x_3 \leq 2)$, equalities $(x_4 = x_3)$ and linear constraints $(t_8 - x_8 = t_2 - x_2)$ on $\mathbb{Q}$
  - Evolution of Physical Variables (e.g., speed, pressure) with linear $(\omega_4 = 2\omega_3)$ and non-linear constraints $(P_1 V_1 = 4 T_1)$ on $\mathbb{Q}$
- Undecidable under simple hypotheses!

$\Longrightarrow$ SMT on $\mathcal{DL}(\mathbb{Q})$, $\mathcal{LRA}$ or $\mathcal{NLA}(\mathbb{R})$ required

Verification of HW circuit designs & microcode

g = 2*f
f' = itea+2*e
itea' = ITE(B1;a;0)
e' = ITE(B2;b;0)+2*ite(B3;c;0)



- SAT/SMT-based Model Checking & Equiv. Checking of RTL designs, symbolic simulation of μ-code [24, 21, 39]
- Control paths handled by Boolean reasoning
- Data paths information abstracted into theory-specific terms
  - words (bit-vectors, integers, $\mathcal{EUF}$ vars, ...): $\underline{a}[31:0]$, a
  - word operations: ($\mathcal{BV}$, $\mathcal{EUF}$, $\mathcal{AR}$, $\mathcal{LIA}$, $\mathcal{NLA}(\mathbb{Z})$) operators) $x_{[16]}[15:0] = (y_{[16]}[15:8] :: z_{[16]}[7:0]) << w_{[8]}[3:0]$, $(a = a_L + 2^{16} a_H)$, $(m_1 = store(m_0, l_0, v_0))$, ...
- Trades heavy Boolean reasoning ($\approx 2^{64}$ factors) with $\mathcal{T}$-solving

$\Longrightarrow$ SMT on $\mathcal{BV}$, $\mathcal{EUF}$, $\mathcal{AR}$, modulo-$\mathcal{LIA}$ [$\mathcal{NLA}(\mathbb{Z})$] required

Verification of SW systems

```
...
10. i = 0;
11. acc = 0.0;
12. while (i<dim) {
13.   acc += V[i];
14.   i++;
15. }
...
```

....

$$(pc = 10) \rightarrow ((i' = 0) \wedge (pc' = 11))$$
$$(pc = 11) \rightarrow ((acc' = 0.0) \wedge (pc' = 12))$$
$$(pc = 12) \rightarrow ((i < dim) \rightarrow \wedge (pc' = 13))$$
$$(pc = 12) \rightarrow (\neg(i < dim) \rightarrow \wedge (pc' = 16))$$
$$(pc = 13) \rightarrow ((acc' = acc + read(V, i)) \wedge (pc' = 14))$$
$$(pc = 14) \rightarrow (i' = i + 1) \wedge (pc' = 15))$$
$$(pc = 15) \rightarrow (pc' = 16))$$

...

- Verification of SW code
  - BMC, K-induction, Check of proof obligations, interpolation-based model checking, symbolic simulation, concolic testing, ...

$\Longrightarrow$ SMT on $\mathcal{BV}$, $\mathcal{EUF}$, $\mathcal{AR}$, (modulo-)$\mathcal{LIA}$ [$\mathcal{NLA}(\mathbb{Z})$] required

Planning with Resources [72]

- SAT-bases planning augmented with numerical constraints
- Straightforward to encode into into SMT($\mathcal{LRA}$)

Example (sketch) [72]

$(Deliver)$	$\wedge$	`// goal`
$(MaxLoad)$	$\wedge$	`// load constraint`
$(MaxFuel)$	$\wedge$	`// fuel constraint`
$(Move \rightarrow MinFuel)$	$\wedge$	`// move requires fuel`
$(Move \rightarrow Deliver)$	$\wedge$	`// move implies delivery`
$(GoodTrip \rightarrow Deliver)$	$\wedge$	`// a good trip requires`
$(GoodTrip \rightarrow AllLoaded)$	$\wedge$	`// a full delivery`
$(MaxLoad \rightarrow (load \leq 30))$	$\wedge$	`// load limit`
$(MaxFuel \rightarrow (fuel \leq 15))$	$\wedge$	`// fuel limit`
$(MinFuel \rightarrow (fuel \geq 7 + 0.5 load))$	$\wedge$	`// fuel constraint`
$(AllLoaded \rightarrow (load = 45))$		`//`

(Disjunctive) Temporal Reasoning [69, 2]

- Temporal reasoning problems encoded as disjunctions of difference constraints

$$
\begin{array}{llll}
((x_1 - x_2 \leq 6) & \vee (x_3 - x_4 \leq -2)) & \wedge \\
((x_2 - x_3 \leq -2) & \vee (x_4 - x_5 \leq 5)) & \wedge \\
((x_2 - x_1 \leq 4) & \vee (x_3 - x_7 \leq -6)) & \wedge \\
\ldots
\end{array}
$$

- Straightforward to encode into into SMT($\mathcal{DL}$)

Goal

Provide an overview of standard “lazy” SMT:

- foundations
- SMT-solving techniques
- beyond solving: advanced SMT functionalities
- ongoing research

We do not cover related approaches like:

- Eager SAT encodings
- Rewrite-based approaches

We refer to [64, 10] for an overview and references.

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   - Theory Solvers for Theories of Interest (hints)
   - SMT for Combinations of Theories
3. Beyond Solving: Advanced SMT Functionalities
   - Proofs and Unsatisfiable Cores
   - Interpolants
   - All-SMT & Predicate Abstraction (hints)
   - SMT with Optimization (Optimization Modulo Theories)

Modern "lazy" SMT($\mathcal{T}$) solvers

A prominent "lazy" approach [42, 2, 72, 3, 8, 33] (aka "DPLL($\mathcal{T}$)")

- a CDCL SAT solver is used to enumerate truth assignments μ_i for (the Boolean abstraction of) the input formula φ
- a theory-specific solver $\mathcal{T}$*-solver* checks the $\mathcal{T}$-satisfiability of the set of $\mathcal{T}$-literals corresponding to each assignment

- Built on top of modern SAT CDCL solvers
  - benefit for free from all modern CDCL techniques (e.g., Boolean preprocessing, backjumping & learning, restarts,...)
  - benefit for free from all state-of-the-art data structures and implementation tricks (e.g., two-watched literals,...)
- Many techniques to maximize the benefits of integration [64, 10]
- Many lazy SMT tools available (Barcelogic, CVC4, MathSAT, OpenSMT, Yices, Z3, . . .)

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Basic schema: example

$\varphi =$	$\varphi^p =$
$c_1: \neg(2v_2 - v_3 > 2) \vee A_1$	$\neg B_1 \vee A_1$
$c_2: \neg A_2 \vee (v_1 - v_5 \leq 1)$	$\neg A_2 \vee B_2$
$c_3: (3v_1 - 2v_2 \leq 3) \vee A_2$	$B_3 \vee A_2$
$c_4: \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1$	$\neg B_4 \vee \neg B_5 \vee \neg A_1$
$c_5: A_1 \vee (3v_1 - 2v_2 \leq 3)$	$A_1 \vee B_3$
$c_6: (v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1$	$B_6 \vee B_7 \vee \neg A_1$
$c_7: A_1 \vee (v_3 = 3v_5 + 4) \vee A_2$	$A_1 \vee B_8 \vee A_2$

true, *false*

$$\mu^p = \{\neg B_5, B_8, B_6, \neg B_1, \neg B_3, A_1, A_2, B_2\}$$

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$\Longrightarrow$ unsatisfiable in $\mathcal{LRA}$ $\Longrightarrow$ backtrack

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$\varphi^p =$

$\neg B_1 \vee A_1$

$\neg A_2 \vee B_2$

$B_3 \vee A_2$

$\neg B_4 \vee \neg B_5 \vee \neg A_1$

$A_1 \vee B_3$

$B_6 \vee B_7 \vee \neg A_1$

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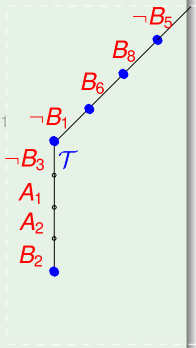
true, *false*



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true, *false*

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$\neg A_2 \vee B_2$

$B_3 \vee A_2$

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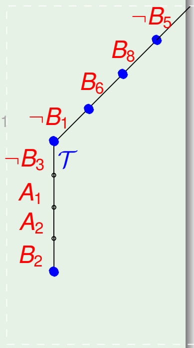
$B_6 \vee B_7 \vee \neg A_1$

$A_1 \vee B_8 \vee A_2$



$$\mu^p = \{\neg B_5, B_8, B_6, \neg B_1, \neg B_3, A_1, A_2, B_2\}$$

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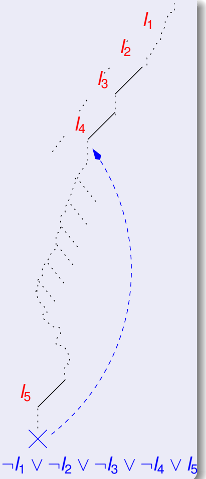
$\Longrightarrow$ unsatisfiable in $\mathcal{LRA}$ $\Longrightarrow$ backtrack

$\mathcal{T}$-Backjumping & $\mathcal{T}$-learning [47, 72, 3, 8, 33]

- Similar to Boolean backjumping & learning
- important property of $\mathcal{T}$-solver:
  - extraction of $\mathcal{T}$-conflict sets: if μ is $\mathcal{T}$-unsatisfiable, then $\mathcal{T}$*-solver* (μ) returns the subset η of μ causing the $\mathcal{T}$-unsatisfiability of μ ($\mathcal{T}$-conflict set)
- If so, the $\mathcal{T}$-conflict clause $C := \neg\eta$ is used to drive the backjumping & learning mechanism of the SAT solver
  $\Longrightarrow$ lots of search saved
- the less redundant is η, the more search is saved



$\mathcal{T}$-Backjumping & $\mathcal{T}$-learning: example

$\varphi =$

$c_1: \neg(2v_2 - v_3 > 2) \vee A_1$

$c_2: \neg A_2 \vee (v_1 - v_5 \leq 1)$

$c_3: (3v_1 - 2v_2 \leq 3) \vee A_2$

$c_4: \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1$

$c_5: A_1 \vee (3v_1 - 2v_2 \leq 3)$

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$c_8: (3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee \ldots$

$\varphi^p =$

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$\neg A_2 \vee B_2$

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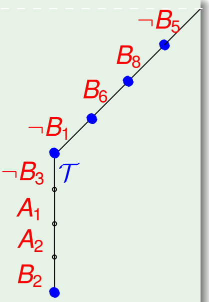
$A_1 \vee B_3$

$B_6 \vee B_7 \vee \neg A_1$

$A_1 \vee B_8 \vee A_2$

$B_5 \vee \neg B_8 \vee \neg B_2$

true, *false*



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$$\eta = \{\neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4), (v_1 - v_5 \leq 1)\}$$

$$\eta^p = \{\neg B_5, B_8, B_2\}$$

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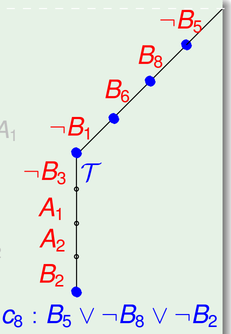

true, *false*

$\mu^p = \{\neg B_5, B_8, B_6, \neg B_1, \neg B_3, A_1, A_2, B_2\}$

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$\eta = \{\neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4), (v_1 - v_5 \leq 1)\}$

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$\mathcal{T}$-Backjumping & $\mathcal{T}$-learning: example

	$\varphi =$	$\varphi^p =$
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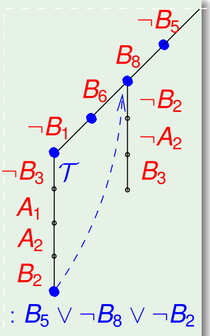
c_2 :	$\neg A_2 \vee (v_1 - v_5 \leq 1)$	$\neg A_2 \vee B_2$
c_3 :	$(3v_1 - 2v_2 \leq 3) \vee A_2$	$B_3 \vee A_2$
c_4 :	$\neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1$	$\neg B_4 \vee \neg B_5 \vee \neg A_1$
c_5 :	$A_1 \vee (3v_1 - 2v_2 \leq 3)$	$A_1 \vee B_3$
c_6 :	$(v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1$	$B_6 \vee B_7 \vee \neg A_1$
c_7 :	$A_1 \vee (v_3 = 3v_5 + 4) \vee A_2$	$A_1 \vee B_8 \vee A_2$
c_8 :	$(3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee \ldots$	$B_5 \vee \neg B_8 \vee \neg B_2$

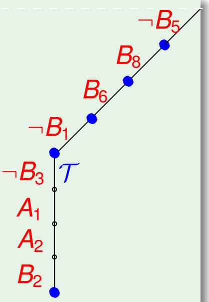
true, *false*



$$
\begin{aligned}
\mu^p &= \{\neg B_5, B_8, B_6, \neg B_1, \neg B_3, A_1, A_2, B_2\} \\
\mu &= \{\neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4), (v_2 - v_4 \leq 6), \neg(2v_2 - v_3 > 2), \\
&\quad \neg(3v_1 - 2v_2 \leq 3), (v_1 - v_5 \leq 1)\} \\
\eta &= \{\neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4), (v_1 - v_5 \leq 1)\} \\
\eta^p &= \{\neg B_5, B_8, B_2\}
\end{aligned}
$$

$\mathcal{T}$-Backjumping & $\mathcal{T}$-learning: example (2)

$\varphi =$		$\varphi^p =$
c_1 :	$\neg(2v_2 - v_3 > 2) \vee A_1$	$\neg B_1 \vee A_1$
c_2 :	$\neg A_2 \vee (v_1 - v_5 \leq 1)$	$\neg A_2 \vee B_2$
c_3 :	$(3v_1 - 2v_2 \leq 3) \vee A_2$	$B_3 \vee A_2$
c_4 :	$\neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1$	$\neg B_4 \vee \neg B_5 \vee \neg A_1$
c_5 :	$A_1 \vee (3v_1 - 2v_2 \leq 3)$	$A_1 \vee B_3$
c_6 :	$(v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1$	$B_6 \vee B_7 \vee \neg A_1$
c_7 :	$A_1 \vee (v_3 = 3v_5 + 4) \vee A_2$	$A_1 \vee B_8 \vee A_2$
c_8' :	$(3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee \ldots$	$B_5 \vee \neg B_8 \vee B_1$
c_8 :	$(3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee \ldots$	$B_5 \vee \neg B_8 \vee \neg B_2$
	true, *false*	

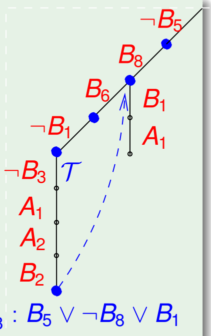


$c_8 : B_5 \vee \neg B_8 \vee \neg B_2$

$$\dfrac{\dfrac{\dfrac{\overbrace{B_5 \vee \neg B_8 \vee \neg B_2}^{c_8:\ \text{theory conflicting clause}} \quad \overbrace{\neg A_2 \vee B_2}^{c_2}}{B_5 \vee \neg B_8 \vee \neg A_2}(B_2) \quad \overbrace{B_3 \vee A_2}^{c_3}}{B_5 \vee \neg B_8 \vee B_3}(\neg A_2) \quad \overbrace{B_5 \vee B_1 \vee \neg B_3}^{c_{\mathcal{T}}}}{\underbrace{B_5 \vee \neg B_8 \vee B_1}_{c_8':\ \text{mixed Boolean+theory conflict clause}}}(B_3)$$

$\mathcal{T}$-Backjumping & $\mathcal{T}$-learning: example (2)

	$\varphi =$	$\varphi^p =$
c_1 :	$\neg(2v_2 - v_3 > 2) \vee A_1$	$\neg B_1 \vee A_1$
c_2 :	$\neg A_2 \vee (v_1 - v_5 \leq 1)$	$\neg A_2 \vee B_2$
c_3 :	$(3v_1 - 2v_2 \leq 3) \vee A_2$	$B_3 \vee A_2$
c_4 :	$\neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1$	$\neg B_4 \vee \neg B_5 \vee \neg A_1$
c_5 :	$A_1 \vee (3v_1 - 2v_2 \leq 3)$	$A_1 \vee B_3$
c_6 :	$(v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1$	$B_6 \vee B_7 \vee \neg A_1$
c_7 :	$A_1 \vee (v_3 = 3v_5 + 4) \vee A_2$	$A_1 \vee B_8 \vee A_2$
c_8' :	$(3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee \ldots$	$B_5 \vee \neg B_8 \vee B_1$
c_8 :	$(3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee \ldots$	$B_5 \vee \neg B_8 \vee \neg B_2$
	true, *false*	



$$\dfrac{\dfrac{\dfrac{\overbrace{B_5 \vee \neg B_8 \vee \neg B_2}^{c_8:\ \text{theory conflicting clause}} \qquad \overbrace{\neg A_2 \vee B_2}^{c_2}}{B_5 \vee \neg B_8 \vee \neg A_2}\,(B_2) \qquad \overbrace{B_3 \vee A_2}^{c_3}}{B_5 \vee \neg B_8 \vee B_3}\,(\neg A_2) \qquad \overbrace{B_5 \vee B_1 \vee \neg B_3}^{c_{\mathcal{T}}}}{\underbrace{B_5 \vee \neg B_8 \vee B_1}_{c_8':\ \text{mixed Boolean+theory conflict clause}}}\,(B_3)$$

$\mathcal{T}$-Backjumping & $\mathcal{T}$-learning: example (2)

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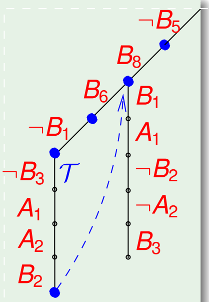
c_1 :	$\neg(2v_2 - v_3 > 2) \vee A_1$	$\neg B_1 \vee A_1$
c_2 :	$\neg A_2 \vee (v_1 - v_5 \leq 1)$	$\neg A_2 \vee B_2$
c_3 :	$(3v_1 - 2v_2 \leq 3) \vee A_2$	$B_3 \vee A_2$
c_4 :	$\neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1$	$\neg B_4 \vee \neg B_5 \vee \neg A_1$
c_5 :	$A_1 \vee (3v_1 - 2v_2 \leq 3)$	$A_1 \vee B_3$
c_6 :	$(v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1$	$B_6 \vee B_7 \vee \neg A_1$
c_7 :	$A_1 \vee (v_3 = 3v_5 + 4) \vee A_2$	$A_1 \vee B_8 \vee A_2$
c_8' :	$(3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee ...$	$B_5 \vee \neg B_8 \vee B_1$
c_8 :	$(3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee ...$	$B_5 \vee \neg B_8 \vee \neg B_2$
	true, *false*	



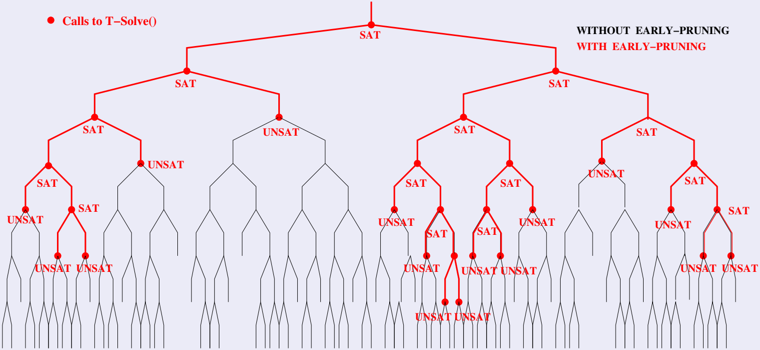
c_8' : $B_5 \vee \neg B_8 \vee B_1$

c_8 : $B_5 \vee \neg B_8 \vee \neg B_2$

$$\cfrac{\cfrac{\cfrac{\overbrace{B_5 \vee \neg B_8 \vee \neg B_2}^{c_8:\ \textit{theory conflicting clause}} \quad \overbrace{\neg A_2 \vee B_2}^{c_2}}{B_5 \vee \neg B_8 \vee \neg A_2}\ (B_2) \quad \overbrace{B_3 \vee A_2}^{c_3}}{B_5 \vee \neg B_8 \vee B_3}\ (\neg A_2) \quad \overbrace{B_5 \vee B_1 \vee \neg B_3}^{c_{\mathcal{T}}}}{\underbrace{B_5 \vee \neg B_8 \vee B_1}_{c_8':\ \textit{mixed Boolean+theory conflict clause}}}\ (B_3)$$

Early Pruning [42, 2, 72]

- Introduce a $\mathcal{T}$-satisfiability test on intermediate assignments: if $\mathcal{T}$*-solver* returns UNSAT, the procedure backtracks.
  - benefit: prunes drastically the Boolean search
  - Drawback: possibly many useless calls to $\mathcal{T}$*-solver*



Early Pruning [42, 2, 72] (cont.)

- Different strategies for interleaving Boolean search steps and $\mathcal{T}$*-solver* calls
  - Eager E.P. [72, 11, 70, 41]): invoke $\mathcal{T}$*-solver* every time a new $\mathcal{T}$-atom is added to the assignment (unit propagations included)
  - Selective E.P.: Do not call $\mathcal{T}$*-solver* if the have been added only literals which hardly cause any $\mathcal{T}$-conflict with the previous assignment (e.g., Boolean literals, disequalities $(x - y \neq 3)$, $\mathcal{T}$-literals introducing new variables $(x - z = 3)$)
  - Weakened E.P.: for intermediate checks only, use weaker but faster versions of $\mathcal{T}$*-solver* (e.g., check μ on $\mathbb{R}$ rather than on $\mathbb{Z}$): $\{(x - y \leq 4), (z - x \leq -6), (z = y), (3x + 2y - 3z = 4)\}$

Early pruning: example

$$\varphi = \{\neg(2v_2 - v_3 > 2) \vee A_1\} \wedge \\ \{\neg A_2 \vee (2v_1 - 4v_5 > 3)\} \wedge \\ \{(3v_1 - 2v_2 \leq 3) \vee A_2\} \wedge \\ \{\neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1\} \wedge \\ \{A_1 \vee (3v_1 - 2v_2 \leq 3)\} \wedge \\ \{(v_1 - v_5 \leq 1) \vee (v_5 = 5 - 3v_4) \vee \neg A_1\} \wedge \\ \{A_1 \vee (v_3 = 3v_5 + 4) \vee A_2\}.$$

$$\varphi^p = \{\neg B_1 \vee A_1\} \wedge \\ \{\neg A_2 \vee B_2\} \wedge \\ \{B_3 \vee A_2\} \wedge \\ \{\neg B_4 \vee \neg B_5 \vee \neg A_1\} \wedge \\ \{A_1 \vee B_3\} \wedge \\ \{B_6 \vee B_7 \vee \neg A_1\} \wedge \\ \{A_1 \vee B_8 \vee A_2\}.$$

- Suppose it is built the intermediate assignment:

$$\mu'^p = \neg B_1 \wedge \neg A_2 \wedge B_3 \wedge \neg B_5.$$

  corresponding to the following set of $\mathcal{T}$-literals

$$\mu' = \neg(2v_2 - v_3 > 2) \wedge \neg A_2 \wedge (3v_1 - 2v_2 \leq 3) \wedge \neg(3v_1 - v_3 \leq 6).$$

- If $\mathcal{T}$-*solver* is invoked on μ', then it returns UNSAT, and DPLL backtracks without exploring any extension of μ'.

Early pruning: remark

Incrementality & Backtrackability of $\mathcal{T}$-solvers

- With early pruning, lots of incremental calls to $\mathcal{T}$-solver:

$\mathcal{T}$-solver (μ_1)	$\Rightarrow$ Sat	Undo μ_4, μ_3, μ_2	
$\mathcal{T}$-solver $(\mu_1 \cup \mu_2)$	$\Rightarrow$ Sat	$\mathcal{T}$-solver $(\mu_1 \cup \mu_2')$	$\Rightarrow$ Sat
$\mathcal{T}$-solver $(\mu_1 \cup \mu_2 \cup \mu_3)$	$\Rightarrow$ Sat	$\mathcal{T}$-solver $(\mu_1 \cup \mu_2' \cup \mu_3')$	$\Rightarrow$ Sat
$\mathcal{T}$-solver $(\mu_1 \cup \mu_2 \cup \mu_3 \cup \mu_4)$	$\Rightarrow$ Unsat	...	

$\Longrightarrow$ Desirable features of $\mathcal{T}$-solvers:

- incrementality: $\mathcal{T}$-solver$(\mu_1 \cup \mu_2)$ reuses computation of $\mathcal{T}$-solver(μ_1) without restarting from scratch
- backtrackability (resettability): $\mathcal{T}$-solver can efficiently undo steps and return to a previous status on the stack

$\Longrightarrow$ $\mathcal{T}$-solver requires a stack-based interface

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$\mathcal{T}$-Propagation [2, 3, 41]

- strictly related to early pruning
- important property of $\mathcal{T}$*-solver*:
  - $\mathcal{T}$-deduction: when a partial assignment μ is $\mathcal{T}$-satisfiable, $\mathcal{T}$*-solver* may be able to return also an assignment η to some unassigned atom occurring in φ s.t. $\mu \models_{\mathcal{T}} \eta$.
- If so:
  - the literal η is then unit-propagated;
  - optionally, a $\mathcal{T}$-deduction clause $C := \neg\mu' \vee \eta$ can be learned, μ' being the subset of μ which caused the deduction ($\mu' \models_{\mathcal{T}} \eta$)
  - lazy explanation: compute C only if needed for conflict analysis

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Both $\mathcal{T}$-deduction clauses and $\mathcal{T}$-conflict clauses are called $\mathcal{T}$-lemmas since they are valid in $\mathcal{T}$

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$\mathcal{T}$-propagation: example

$\varphi =$

$c_1: \neg(2v_2 - v_3 > 2) \vee A_1$
$c_2: \neg A_2 \vee (v_1 - v_5 \leq 1)$
$c_3: (3v_1 - 2v_2 \leq 3) \vee A_2$
$c_4: \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1$
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$c_7: A_1 \vee (v_3 = 3v_5 + 4) \vee A_2$

$\varphi^p =$

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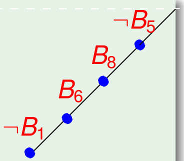

true, *false*

$$\mu^p = \{\neg B_5, B_8, B_6, \neg B_1\}$$
$$\mu = \{\underline{\neg(3v_1 - v_3 \leq 6)}, (v_3 = 3v_5 + 4), (v_2 - v_4 \leq 6), \underline{\neg(2v_2 - v_3 > 2)}\}$$
$$\models_{\mathcal{LRA}} \underbrace{\neg(3v_1 - 2v_2 \leq 3)}_{\neg B_3}$$

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c_5 :	$A_1 \vee (3v_1 - 2v_2 \le 3)$	$A_1 \vee B_3$
c_6 :	$(v_2 - v_4 \le 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1$	$B_6 \vee B_7 \vee \neg A_1$
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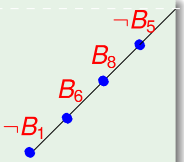

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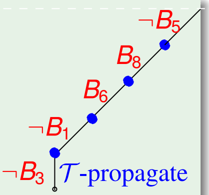

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Pure-literal filtering [72, 3, 16]

Property

If we have non-Boolean $\mathcal{T}$-atoms occurring only positively [negatively] in the original formula φ (learned clauses are not considered), we can drop every negative [positive] occurrence of them from the assignment to be checked by $\mathcal{T}$*-solver* (and from the $\mathcal{T}$-deducible ones).

- increases the chances of finding a model
- reduces the effort for the $\mathcal{T}$*-solver*
- eliminates unnecessary "nasty" negated literals (e.g. negative equalities like $\neg(3v_1 - 9v_2 = 3)$ in $\mathcal{LIA}$ force splitting: $(3v_1 - 9v_2 > 3) \vee (3v_1 - 9v_2 < 3)$).
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$\Longrightarrow$ Sat: $v_1 = v_2 = v_3 = 0, v_5 = -4/3$ is a solution

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- $(3v_1 - v_3 \leq -2)$ "filtered out" from μ' because it occurs only negatively in the original formula φ
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Preprocessing atoms [42, 47, 4]

Source of inefficiency:

Semantically equivalent but syntactically different atoms are not recognized to be identical [resp. one the negation of the other]

$\Longrightarrow$ they may be assigned different [resp. identical] truth values.

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Rewrite a priori trivially-equivalent atoms/literals into the same atom/literal.

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Preprocessing atoms (cont.)

- Sorting: $(v_1 + v_2 \leq v_3 + 1), (v_2 + v_1 \leq v_3 + 1), (v_1 + v_2 - 1 \leq v_3) \Longrightarrow (v_1 + v_2 - v_3 \leq 1))$;
- Rewriting dual operators: $(v_1 < v_2), (v_1 \geq v_2) \Longrightarrow (v_1 < v_2), \neg(v_1 < v_2)$
- Exploiting associativity: $(v_1 + (v_2 + v_3) = 1), ((v_1 + v_2) + v_3) = 1) \Longrightarrow (v_1 + v_2 + v_3 = 1)$;
- Factoring $(v_1 + 2.0 v_2 \leq 4.0), (-2.0 v_1 - 4.0 v_2 \geq -8.0), \Longrightarrow (0.25 v_1 + 0.5 v_2 \leq 1.0)$;
- Exploiting properties of $\mathcal{T}$: $(v_1 \leq 3), (v_1 < 4) \Longrightarrow (v_1 \leq 3)$ if $v_1 \in \mathbb{Z}$;
- ...

Preprocessing atoms (cont.)

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- ...

Preprocessing atoms (cont.)

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- Rewriting dual operators: $(v_1 < v_2), (v_1 \geq v_2) \Longrightarrow (v_1 < v_2), \neg(v_1 < v_2)$
- Exploiting associativity: $(v_1 + (v_2 + v_3) = 1), ((v_1 + v_2) + v_3) = 1) \Longrightarrow (v_1 + v_2 + v_3 = 1)$;
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Static Learning [2, 4]

- Often possible to quickly detect a priori short and "obviously unsatisfiable" pairs or triplets of literals occurring in φ.
  - mutual exclusion $\{x = 0, x = 1\}$,
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Other optimization techniques

- $\mathcal{T}$-deduced-literal filtering
- Ghost-literal filtering
- $\mathcal{T}$*-solver* layering
- $\mathcal{T}$*-solver* clustering
- ...

(see [64, 10] for an overview)

Other SAT-solving techniques for SMT?

Frequently-asked question:

Are CDCL SAT solvers the only suitable Boolean Engines for SMT?

Some previous attempts:

- Ordered Binary Decision Diagrams (OBDDs) [73, 55, 1]
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An SMT problem φ from the perspective of a SAT solver:

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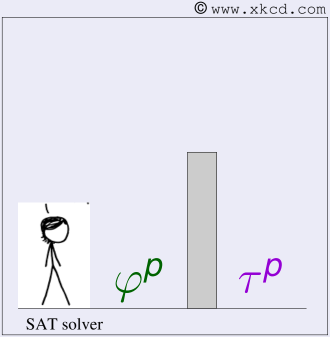

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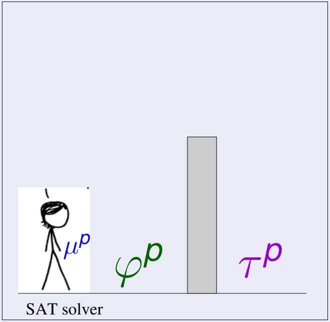

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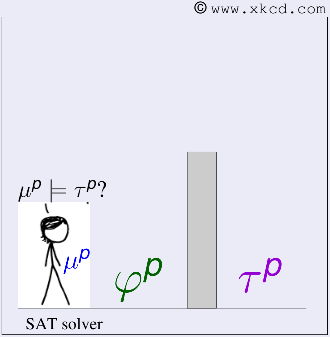

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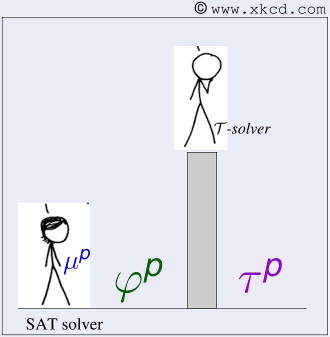

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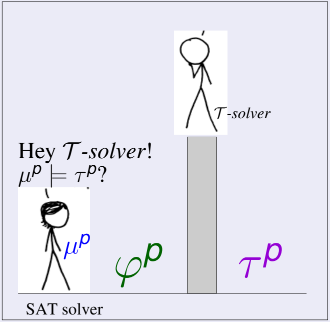

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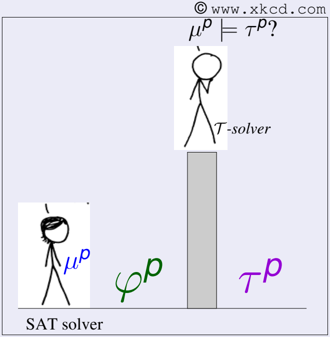

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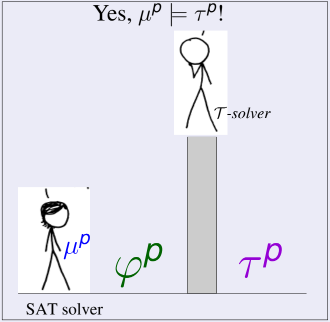

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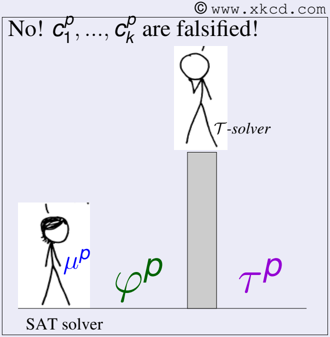

Example

φ :		φ^p :	
c_1 :	$\{A_1\}$	c_1 :	$\{A_1\}$
c_2 :	$\{\neg A_1 \vee (x - z > 4)\}$	c_2 :	$\{\neg A_1 \vee B_1\}$
c_3 :	$\{\neg A_3 \vee A_1 \vee (y \geq 1)\}$	c_3 :	$\{\neg A_3 \vee A_1 \vee B_2\}$
c_4 :	$\{\neg A_2 \vee \neg(x - z > 4) \vee \neg A_1\}$	c_4 :	$\{\neg A_2 \vee \neg B_1 \vee \neg A_1\}$
c_5 :	$\{(x - y \leq 3) \vee \neg A_4 \vee A_5\}$	c_5 :	$\{B_3 \vee \neg A_4 \vee A_5\}$
c_6 :	$\{\neg(y - z \leq 1) \vee (x + y = 1) \vee \neg A_5\}$	c_6 :	$\{\neg B_4 \vee B_5 \vee \neg A_5\}$
c_7 :	$\{A_3 \vee \neg(x + y = 0) \vee A_2\}$	c_7 :	$\{A_3 \vee \neg B_6 \vee A_2\}$
c_8 :	$\{\neg A_3 \vee (z + y = 2)\}$	c_8 :	$\{\neg A_3 \vee B_7\}$
τ :	(all possible $\mathcal{T}$-lemmas on the $\mathcal{T}$-atoms of φ)	τ^p :	
c_9 :	$\{\neg(x + y = 0) \vee \neg(x + y = 1)\}$	c_9 :	$\{\neg B_6 \vee \neg B_5\}$
c_{10} :	$\{\neg(x - z > 4) \vee \neg(x - y \leq 3) \vee \neg(y - z \leq 1)\}$	c_{10} :	$\{\neg B_1 \vee \neg B_3 \vee \neg B_4\}$
c_{11} :	$\{(x - z > 4) \vee (x - y \leq 3) \vee (y - z \leq 1)\}$	c_{11} :	$\{B_1 \vee B_3 \vee B_4\}$
c_{12} :	$\{\neg(x - z > 4) \vee \neg(x + y = 1) \vee \neg(z + y = 2)\}$	c_{12} :	$\{\neg B_1 \vee \neg B_5 \vee \neg B_7\}$
c_{13} :	$\{\neg(x - z > 4) \vee \neg(x + y = 0) \vee \neg(z + y = 2)\}$	c_{13} :	$\{\neg B_1 \vee \neg B_6 \vee \neg B_7\}$
...	...	...	...

μ_1^p : $\{A_1, B_1, \neg A_2, A_3, \neg A_4, \neg A_5, \neg B_6, B_5, B_3, B_4, B_7, \neg B_2\}$

μ_1 : $\{(x - z > 4), \neg(x + y = 0), (x + y = 1), (x - y \leq 3), (y - z \leq 1),$
$(z + y = 2), \neg(y \geq 1)\}$

satisfies φ^p, but violates both c_{10} and c_{12} in τ^p.

Example

φ :

c_1 : $\{A_1\}$

c_2 : $\{\neg A_1 \vee (x - z > 4)\}$

c_3 : $\{\neg A_3 \vee A_1 \vee (y \geq 1)\}$

c_4 : $\{\neg A_2 \vee \neg(x - z > 4) \vee \neg A_1\}$

c_5 : $\{(x - y \leq 3) \vee \neg A_4 \vee A_5\}$

c_6 : $\{\neg(y - z \leq 1) \vee (x + y = 1) \vee \neg A_5\}$

c_7 : $\{A_3 \vee \neg(x + y = 0) \vee A_2\}$

c_8 : $\{\neg A_3 \vee (z + y = 2)\}$

τ : (all possible $\mathcal{T}$-lemmas on the $\mathcal{T}$-atoms of φ)

c_9 : $\{\neg(x + y = 0) \vee \neg(x + y = 1)\}$

c_{10} : $\{\neg(x - z > 4) \vee \neg(x - y \leq 3) \vee \neg(y - z \leq 1)\}$

c_{11} : $\{(x - z > 4) \vee (x - y \leq 3) \vee (y - z \leq 1)\}$

c_{12} : $\{\neg(x - z > 4) \vee \neg(x + y = 1) \vee \neg(z + y = 2)\}$

c_{13} : $\{\neg(x - z > 4) \vee \neg(x + y = 0) \vee \neg(z + y = 2)\}$

... ...

φ^p :

c_1 : $\{A_1\}$

c_2 : $\{\neg A_1 \vee B_1\}$

c_3 : $\{\neg A_3 \vee A_1 \vee B_2\}$

c_4 : $\{\neg A_2 \vee \neg B_1 \vee \neg A_1\}$

c_5 : $\{B_3 \vee \neg A_4 \vee A_5\}$

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Exercise

Consider the following formula in the theory $\mathcal{EUF}$.

$$
\begin{aligned}
\varphi = \quad & \{(f(x) = f(f(y))) \ \vee A_2\} \quad \wedge \\
& \{\neg(h(x, f(y)) = h(g(x), y)) \ \vee \underline{\neg(h(x, g(z) = h(f(x), y)))} \ \vee \neg A_1\} \ \wedge \\
& \{A_1 \vee (h(x, y) = h(y, x))\} \quad \wedge \\
& \{\underline{(x = f(x))} \ \vee A_3 \vee \neg A_1\} \ \wedge \\
& \{\underline{\neg(w(x) = g(f(y)))} \ \vee A_1\} \wedge \\
& \{\underline{\neg A_2} \vee (w(g(x)) = w(f(x)))\} \wedge \\
& \{A_1 \vee \underline{(y = g(z))} \ \vee A_2\}
\end{aligned}
$$

and consider the partial truth assignment μ given by the underlined literals above:

$$\{\neg(w(x) = g(f(y))), \neg A_2, \neg(h(x, g(z) = h(f(x), y))), (x = f(x)), (y = g(z))\}.$$

1. Does (the Boolean abstraction of) μ propositionally satisfy (the Boolean abstraction of) φ?
2. Is μ satisfiable in $\mathcal{EUF}$?
   1. If no, find a minimal conflict set for μ and the corresponding conflict clause C.
   2. If yes, show one unassigned literal which can be deduced from μ, and show the corresponding deduction clause C.

Outline

1 Introduction
- What is a Theory?
- Satisfiability Modulo Theories
- Motivations and Goals of SMT

2 Efficient SMT solving
- Combining SAT with Theory Solvers
- Theory Solvers for Theories of Interest (hints)
- SMT for Combinations of Theories

3 Beyond Solving: Advanced SMT Functionalities
- Proofs and Unsatisfiable Cores
- Interpolants
- All-SMT & Predicate Abstraction (hints)
- SMT with Optimization (Optimization Modulo Theories)

Summary: desirable properties for $\mathcal{T}$*-solver*

- Correctness & Completeness: be correct & complete
- Time efficiency: be fast
- Incrementality & backtrackability: $\mathcal{T}$*-solver*$(\mu_1 \cup \mu_2)$ reuses computation of $\mathcal{T}$*-solver*(μ_1)
- Diagnosis capabilities: $\mathcal{T}$*-solver* able to produce conflict sets
- Deduction capabilities: $\mathcal{T}$*-solver* able to deduce assignments

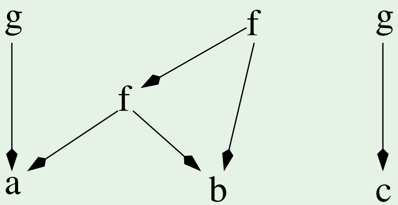
$\mathcal{T}$-solvers for Equality and Uninterpreted Functions ($\mathcal{EUF}$)

- Typically used as a "core" $\mathcal{T}$*-solver*
- $\mathcal{EUF}$ polynomial: $O(n \cdot log(n))$
- Fully incremental and backtrackable (stack-based)
- use a congruence closure data structures (E-Graphs) [36, 59, 32], based on the Union-Find data-structure for equivalence classes
- Supports efficient $\mathcal{T}$-propagation
  - Exhaustive for positive equalities
  - Incomplete for disequalities
- Supports Lazy explanations and conflict generation
  - However, minimality not guaranteed
- Supports efficient extensions
  (e.g., Integer offsets, Bit-vector slicing and concatenation)

$\mathcal{T}$-solvers for $\mathcal{EUF}$: Example

Idea (sketch): given the set of terms occurring in the formula represented as nodes in a DAG (aka term bank),

- if $(t = s)$, then merge the eq. classes of t and s
  - e.g. use the union-find data structure
- if $\forall i \in 1...k$, t_i and s_i pairwise belong to the same eq. classes, then merge the eq. classes of $f(t_1, ..., t_k)$ and $f(s_1, ..., s_k)$
- if $(t \neq s)$ and t and s belong to the same eq. class, then conflict



$$f(a, b) = a$$
$$f(f(a, b), b) = c$$
$$g(a) \neq g(c)$$

Example borrowed from [36].

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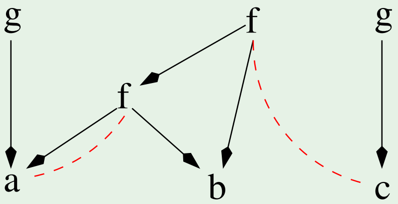

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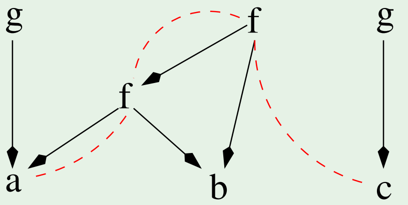

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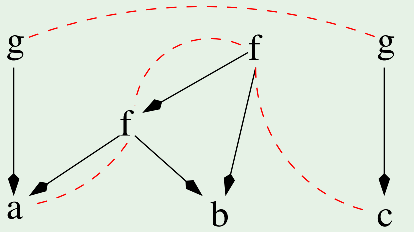
- if $\forall i \in 1...k$, t_i and s_i pairwise belong to the same eq. classes, then merge the eq. classes of $f(t_1, ..., t_k)$ and $f(s_1, ..., s_k)$
- if $(t \neq s)$ and t and s belong to the same eq. class, then conflict



$$f(a, b) = a$$
$$f(f(a, b), b) = c$$
$$g(a) \neq g(c)$$

Example borrowed from [36].

$\mathcal{T}$-solvers for $\mathcal{EUF}$: Example

Idea (sketch): given the set of terms occurring in the formula represented as nodes in a DAG (aka term bank),

- if $(t = s)$, then merge the eq. classes of t and s
  - e.g. use the union-find data structure
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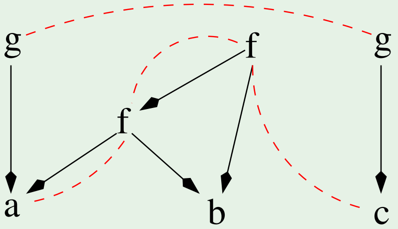

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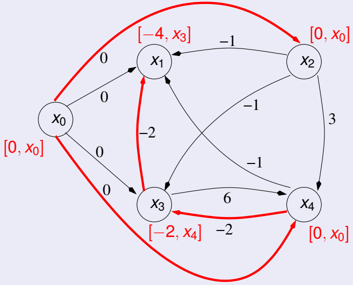
$\mathcal{T}$-solvers for Difference logic ($\mathcal{DL}$)

- $\mathcal{DL}$ polynomial: $O(\#vars \cdot \#constraints)$
- variants of the Bellman-Ford shortest-path algorithm: a negative cycle reveals a conflict [60, 31]
- Ex:

$$\{(x_1 - x_2 \le -1), (x_1 - x_4 \le -1), (x_1 - x_3 \le -2), (x_2 - x_1 \le 2), (x_3 - x_4 \le -2), (x_3 - x_2 \le -1), (x_4 - x_2 \le 3), (x_4 - x_3 \le 6)\}$$



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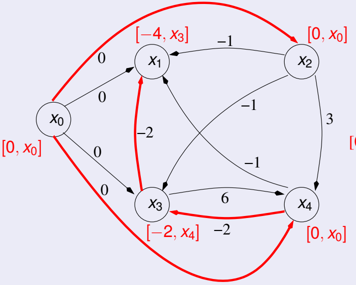

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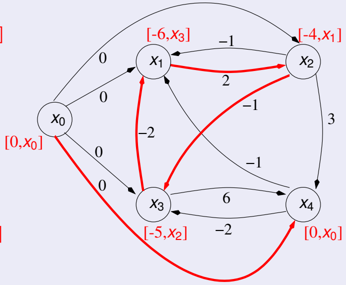

$\Longrightarrow$ Unsat

$\mathcal{T}$-solvers for Linear arithmetic over the rationals ($\mathcal{LRA}$)

- EX: $\{(s_1 - s_2 \leq 5.2), (s_1 = s_0 + 3.4 \cdot t - 3.4 \cdot t_0), \neg(s_1 = s_0)\}$
- $\mathcal{LRA}$ polynomial
- variants of the simplex LP algorithm [38]
- [38] allows for detecting conflict sets & performing $\mathcal{T}$-propagation
- strict inequalities $t < 0$ rewritten as $t + \epsilon \leq 0$, ϵ treated symbolically

$$\overset{\mathcal{B}}{\begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_N \end{bmatrix}} = \begin{bmatrix} \ldots A_{1j} \ldots \\ \vdots \\ A_{i1} \ldots A_{ij} \ldots A_{iM} \\ \vdots \\ \ldots A_{Nj} \ldots \end{bmatrix} \overset{\mathcal{N}}{\begin{bmatrix} x_{N+1} \\ \vdots \\ x_j \\ \vdots \\ x_{N+M} \end{bmatrix}};$$

Invariant: $\beta(x_j) \in [l_j, u_j] \; \forall x_j \in \mathcal{N}$

Remark: infinite precision arithmetic

In order to avoid incorrect results due to numerical errors and to overflows, all $\mathcal{T}$*-solvers* for $\mathcal{LRA}$, $\mathcal{LIA}$ and their subtheories which are based on numerical algorithms must be implemented on top of infinite-precision-arithmetic software packages.

$\mathcal{T}$-solvers for Linear arithmetic over the integers ($\mathcal{LIA}$)

- EX: $\{(x := x_l + 2^{16}x_h), (x \geq 0), (x \leq 2^{16} - 1)\}$
- $\mathcal{LIA}$ NP-complete
- combination of many techniques: simplex, branch&bound, cutting planes, ... [38, 44]

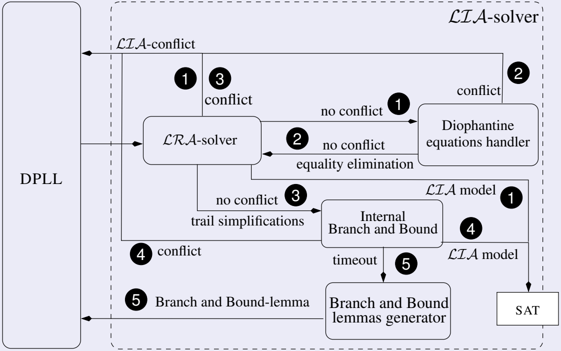

Figure courtesy of A. Griggio [44]

$\mathcal{T}$-solvers for Arrays ($\mathcal{AR}$)

- EX: $(write(A, i, v) = write(B, i, w)) \land \neg(v = w)$
- NP-complete
- congruence closure ($\mathcal{EUF}$) plus on-the-fly instantiation of array's axioms:

$$\forall a. \forall i. \forall e.\ (read(write(a, i, e), i) = e), \qquad (1)$$

$$\forall a. \forall i. \forall j. \forall e.\ ((i \neq j) \rightarrow read(write(a, i, e), j) = read(a, j)), \qquad (2)$$

$$\forall a. \forall b.\ (\forall i.(read(a, i) = read(b, i)) \rightarrow (a = b)). \qquad (3)$$

- EX:

$$\begin{array}{ll} Input: & (write(A, i, v) = write(B, i, w)) \land \neg(v = w) \\ inst.\ (1): & (read(write(A, i, v), i) = v) \\ & (read(write(B, i, w), i) = w) \\ \models_{\mathcal{EUF}} & (v = w) \\ \models_{Bool} & \bot \end{array}$$

- many strategies discussed in the literature (e.g., [36, 43, 19, 35])

$\mathcal{T}$-solvers for Bit vectors ($\mathcal{BV}$)

Bit vectors ($\mathcal{BV}$)

- EX:
  $\{(x_{[16]}[15:0] = (y_{[16]}[15:8] :: z_{[16]}[7:0]) << w_{[16]}[3:0]), ...\}$
- NP-hard
- involve complex word-level operations: word partition/concat, modulo-2^N arithmetic, shifts, bitwise-operations, multiplexers, ...
- $\mathcal{T}$-solving: combination of rewriting & simplification techniques with either:
  - final encoding into $\mathcal{LIA}$ [18, 21]
  - final encoding into SAT (lazy bit-blasting) [24, 40, 20, 39]

Eager approach

Most solvers use an eager approach for $\mathcal{BV}$ (e.g., [20]):

- Heavy preprocessing, based on rewriting rules
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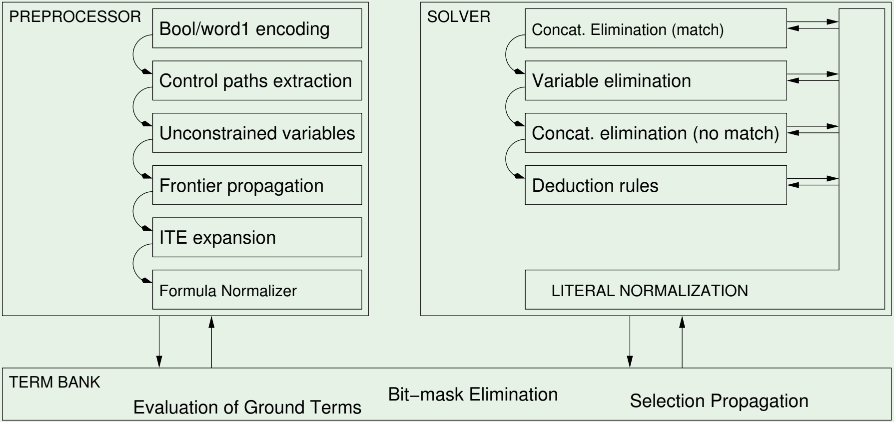


Example borrowed from [21]

$\mathcal{T}$-solvers for Bit vectors ($\mathcal{BV}$) [cont.]

Lazy bit-blasting

- Two nested SAT solvers
- bit-blast each $\mathcal{BV}$ atom ψ_i
  $\Longrightarrow \Phi \stackrel{\text{def}}{=} \bigwedge_i (A_i \leftrightarrow BB(\psi_i))$,
  A_i fresh variables labeling $\mathcal{BV}$-atoms ψ_i in φ
  $\Longrightarrow \varphi$ $\mathcal{BV}$-satisfiable iff $\varphi^p \wedge \Phi$ satisfiable
- Exploit SAT under assumptions
  - let μ^p an assignment for φ^p, s.t. $\mu^p \stackrel{\text{def}}{=} \{[\neg]A_1, ..., [\neg]A_n\}$
  - $\mathcal{T}$-solver for $\mathcal{BV}$: $SAT_{assumption}(\Phi, \mu^p)$
  - If UNSAT, generate the unsat core $\eta^p \subseteq \mu^p$

  $\Longrightarrow$ $\neg\eta^p$ used as blocking clause

Outline

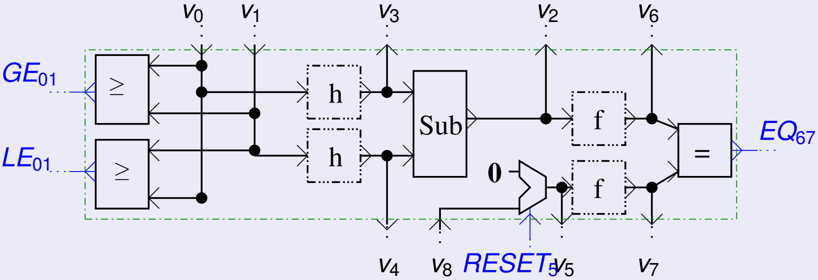
1 Introduction
- What is a Theory?
- Satisfiability Modulo Theories
- Motivations and Goals of SMT

2 Efficient SMT solving
- Combining SAT with Theory Solvers
- Theory Solvers for Theories of Interest (hints)
- SMT for Combinations of Theories

3 Beyond Solving: Advanced SMT Functionalities
- Proofs and Unsatisfiable Cores
- Interpolants
- All-SMT & Predicate Abstraction (hints)
- SMT with Optimization (Optimization Modulo Theories)

SMT for combined theories: $SMT(\bigcup_i \mathcal{T}_i)$

Problem: Many problems can be expressed as SMT problems only in combination of theories $\bigcup_i \mathcal{T}_i$ — $SMT(\bigcup_i \mathcal{T}_i)$



$\mathcal{LIA}$:	$(GE_{01} \leftrightarrow (v_0 \geq v_1)) \wedge (LE_{01} \leftrightarrow (v_0 \leq v_1)) \wedge$
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$\mathcal{EUF}$ or $\mathcal{LIA}$:	$(\neg RESET_5 \rightarrow (v_5 = v_8)) \wedge$
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SMT for combined theories: SMT($\mathcal{T}_1 \cup \mathcal{T}_2$)

- Combined theories may be much harder to decide [Pratt'77]
- Solvers have to be combined
- Standard approach for combining $\mathcal{T}_i$-*solver*'s: (deterministic) Nelson-Oppen/Shostak (N.O.) [56, 58, 67]
  - based on deduction and exchange of equalities on shared variables
  - combined $\mathcal{T}_i$-*solver*'s integrated with a SAT tool
- SMT-specific approaches: Delayed Theory Combination [14, 13] and Model-Based Theory Combination [34]
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Background: Pure Formulas

Consider two theories $\mathcal{T}_1$, $\mathcal{T}_2$ with equality and disjoint signatures Σ_1, Σ_2

- W.l.o.g. we assume all input formulas $\phi \in \mathcal{T}_1 \cup \mathcal{T}_2$ are pure.
  - A formula ϕ is pure iff every atom in ϕ is *i*-pure for some $i \in \{1, 2\}$.
  - An atom/literal in ϕ is *i*-pure if only $=$, variables and symbols from Σ_i can occur in ϕ

Purification:

Maps a formula into an equisatisfiable pure formula by labeling terms with fresh variables

$$(f(\underbrace{x + 3y}_{w}) = g(\underbrace{2x - y}_{t})) \qquad [\textit{not pure}]$$

$$\Downarrow$$

$$(w = x + 3y) \wedge (t = 2x - y) \wedge (f(w) = g(t)) \qquad [\textit{pure}]$$

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Exercise

- Purify the following $\mathcal{LIA} \cup \mathcal{EUF} \cup \mathcal{AR}$-formula (see beginning of chapter):

$$\varphi \stackrel{\text{def}}{=} (d \geq 0) \wedge (d < 1) \wedge$$
$$((f(d) = f(0)) \rightarrow (read(write(V, i, x), i + d) = x + 1))$$

Background: Interface equalities

Interface variables & equalities

- A variable v occurring in a pure formula ϕ is an interface variable iff it occurs in both 1-pure and 2-pure atoms of ϕ.
- An equality $(v_i = v_j)$ is an interface equality for ϕ iff v_i, v_j are interface variables for ϕ.
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Example:

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\end{array}$$

v_0, v_1, v_2, v_3, v_4, v_5 are interface variables, v_6, v_7, v_8 are not
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Stably-infinite Theories

A Σ-theory $\mathcal{T}$ is stably-infinite iff every quantifier-free $\mathcal{T}$-satisfiable formula is satisfiable in an infinite model of $\mathcal{T}$.

- $\mathcal{EUF}$, $\mathcal{DL}$, $\mathcal{LRA}$, $\mathcal{LIA}$ are stably-infinite
- (fixed-width) bit-vector theories are not stably-infinite

Intuition: a variable can be given an infinite amount of distinct values

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A Σ-theory $\mathcal{T}$ is convex iff, for every collection $l_1, ..., l_k, l', l''$ of literals in $\mathcal{T}$ s.t. l', l'' are in the form $(x = y)$, x, y being variables, we have that: $\{l_1, ..., l_k\} \models_{\mathcal{T}} (l' \vee l'') \iff \{l_1, ..., l_k\} \models_{\mathcal{T}} l'$ *or* $\{l_1, ..., l_k\} \models_{\mathcal{T}} l''$

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- $\mathcal{EUF}$, $\mathcal{DL}$, $\mathcal{LRA}$, $\mathcal{LIA}$ are stably-infinite
- (fixed-width) bit-vector theories are not stably-infinite

Intuition: a variable can be given an infinite amount of distinct values

Convex Theories

A Σ-theory $\mathcal{T}$ is convex iff, for every collection $l_1, ..., l_k, l', l''$ of literals in $\mathcal{T}$ s.t. l', l'' are in the form $(x = y)$, x, y being variables, we have that: $\{l_1, ..., l_k\} \models_{\mathcal{T}} (l' \vee l'') \iff \{l_1, ..., l_k\} \models_{\mathcal{T}} l'$ *or* $\{l_1, ..., l_k\} \models_{\mathcal{T}} l''$

- $\mathcal{EUF}$, $\mathcal{DL}$, $\mathcal{LRA}$ are convex
- $\mathcal{LIA}$ is not convex:
  $\{(v_0 = 0), (v_1 = 1), (v \geq v_0), (v \leq v_1)\} \models ((v = v_0) \vee (v = v_1))$,
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Intuition: non-convexity produces “case splits”

Background: Stably-infinite & Convex Theories

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$SMT(\bigcup_i \mathcal{T}_i)$ via “classic” Nelson-Oppen

Main Problem

- One predicate shared between distinct theories $\mathcal{T}_i$: equality “=”
- Given $\mu \stackrel{\text{def}}{=} \bigcup_i \mu_i$ s.t. each μ_i contains i-pure literals
  - distinct $\mathcal{T}_i$-solver can be invoked separately on each μ_i...
  - ...producing distinct $\mathcal{T}_i$-specific models $\mathcal{M}_i$
- Problem: all models must agree on interface equalities:

$$\mathcal{M}_i \models_{\mathcal{T}_i} (v_k = v_l) \;\; iff \;\; \mathcal{M}_j \models_{\mathcal{T}_j} (v_k = v_l),$$

  for every pair of shared variables v_k, v_l

Main idea

Combine two or more $\mathcal{T}_i$-*solvers* into one $(\bigcup_i \mathcal{T}_i)$-solver via Nelson-Oppen/Shostak (N.O.) combination procedure [57, 68]

- based on the deduction and exchange of equalities between shared variables/terms (interface equalities, e_{ij}s)
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Schema of N.O. combination of $\mathcal{T}$-solvers: no($\mathcal{T}_1, \mathcal{T}_2$)

For $i \in \{1, 2\}$, let $\mathcal{T}_i$ be a stably infinite theory admitting a satisfiability $\mathcal{T}_i$-solver, and μ_i a set of i-pure literals.
We want to to decide the $\mathcal{T}_1 \cup \mathcal{T}_2$-satisfiability of $\mu_1 \cup \mu_2$

- each $\mathcal{T}_i$-solver, in turn
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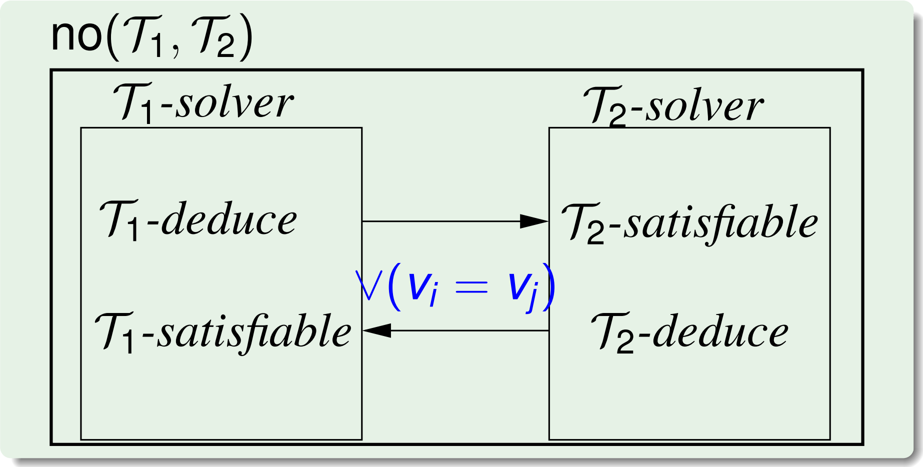

N.O. Example (Convex Theory)

$$\begin{array}{ll} \mathcal{EUF}: & (v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge \\ \mathcal{LRA}: & (v_0 \geq v_1) \wedge (v_0 \leq v_1) \wedge (v_2 = v_3 - v_4) \wedge (RESET_5 \rightarrow (v_5 = 0)) \wedge \\ Both: & (\neg RESET_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7). \end{array}$$

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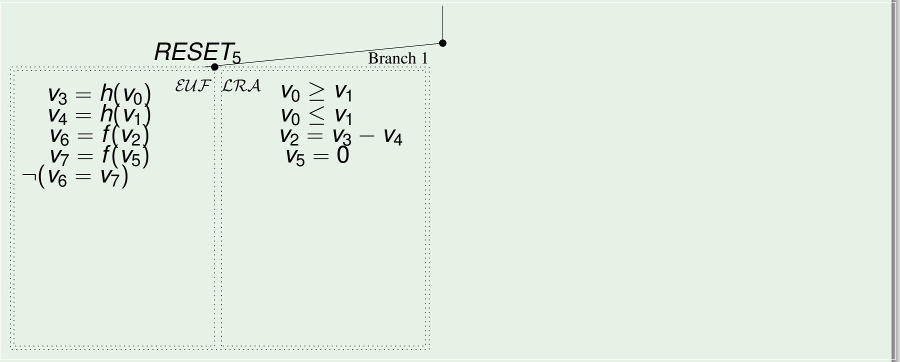

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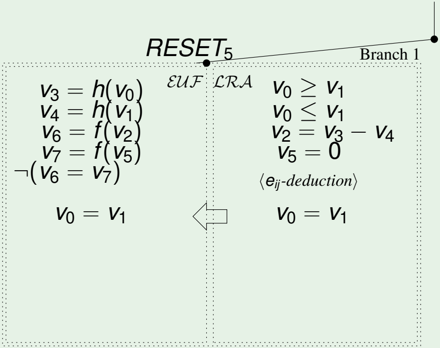

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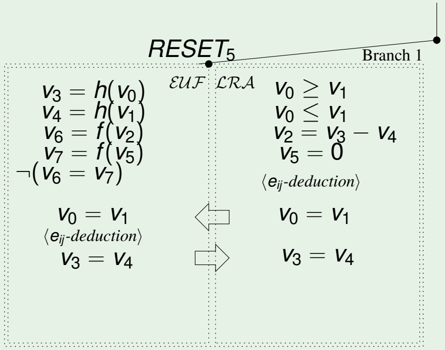

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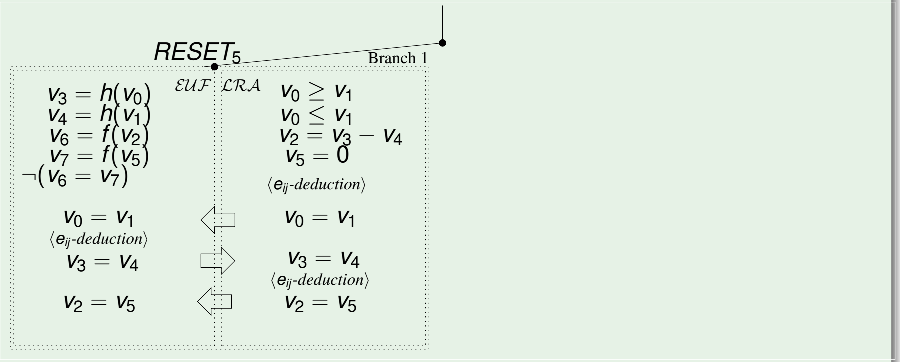
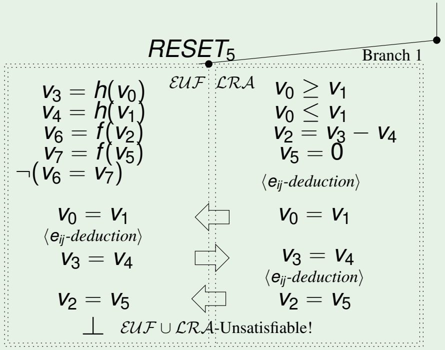


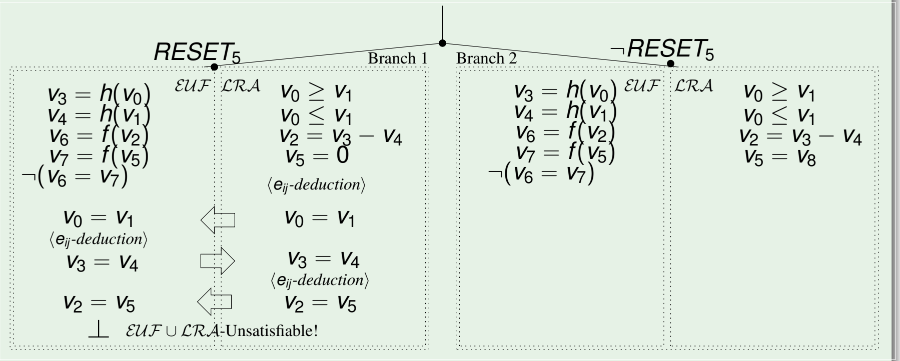
N.O. Example (Convex Theory)

$$\begin{array}{ll} \mathcal{EUF}: & (v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge \\ \mathcal{LRA}: & (v_0 \geq v_1) \wedge (v_0 \leq v_1) \wedge (v_2 = v_3 - v_4) \wedge (RESET_5 \rightarrow (v_5 = 0)) \wedge \\ Both: & (\neg RESET_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7). \end{array}$$



N.O. Example (Convex Theory)

$$\begin{array}{ll}
\mathcal{EUF}: & (v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge \\
\mathcal{LRA}: & (v_0 \geq v_1) \wedge (v_0 \leq v_1) \wedge (v_2 = v_3 - v_4) \wedge (RESET_5 \rightarrow (v_5 = 0)) \wedge \\
Both: & (\neg RESET_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7).
\end{array}$$



N.O. Example (Convex Theory)

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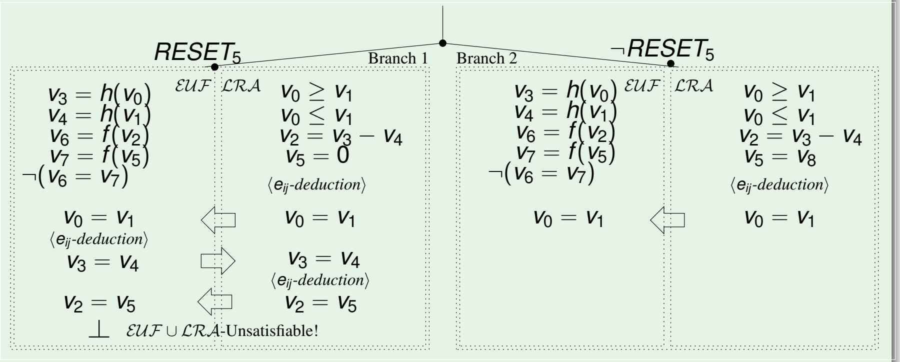

N.O. Example (Convex Theory)

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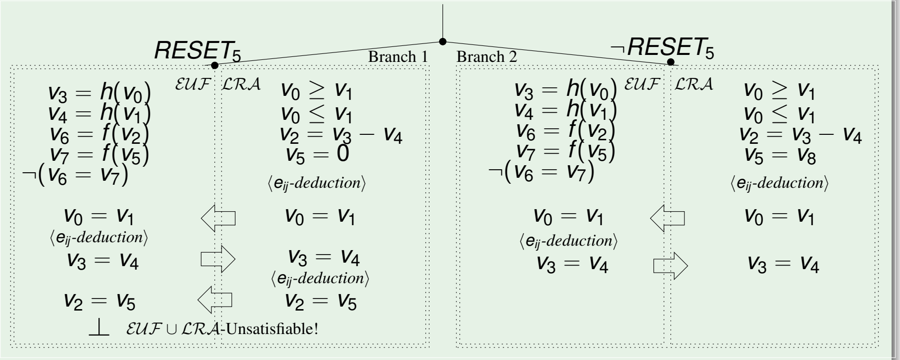
\end{array}
$$



N.O. Example (Convex Theory)

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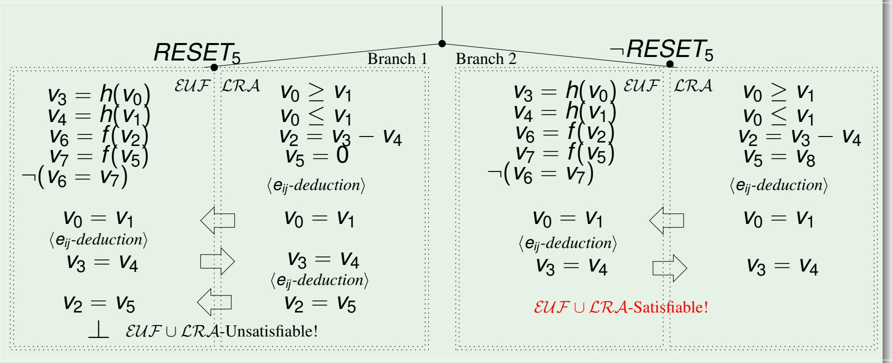
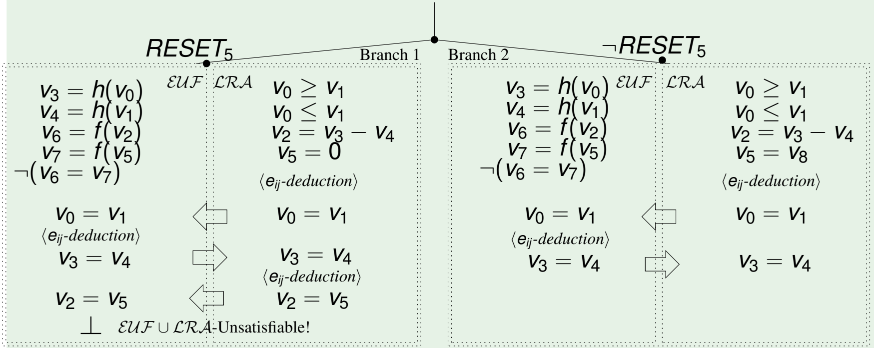
\mathcal{EUF}: & (v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge \\
\mathcal{LRA}: & (v_0 \geq v_1) \wedge (v_0 \leq v_1) \wedge (v_2 = v_3 - v_4) \wedge (RESET_5 \rightarrow (v_5 = 0)) \wedge \\
Both: & (\neg RESET_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7).
\end{array}$$



N.O.: example (convex theory) [cont.]



$\mathcal{EUF}$-conflict : $((v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge \neg(v_6 = v_7) \wedge (v_2 = v_5)) \rightarrow \bot$

$\mathcal{LRA}$-deduction : $((v_2 = v_3 - v_4) \wedge (v_5 = 0) \wedge (v_3 = v_4)) \rightarrow (v_2 = v_5)$

$\mathcal{EUF}$-deduction : $((v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_0 = v_1)) \rightarrow (v_3 = v_4)$

$\mathcal{LRA}$-deduction : $((v_0 \geq v_1) \wedge (v_0 \leq v_1)) \rightarrow (v_0 = v_1)$

$\Longrightarrow$

$\mathcal{EUF} \cup \mathcal{LRA}$-conflict : $((v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge \neg(v_6 = v_7) \wedge (v_2 = v_3 - v_4) \wedge$
$(v_5 = 0) \wedge (v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_0 \geq v_1)) \rightarrow \bot.$

Exercises

For the previous N.O. example:

- write the (minimal) clauses corresponding to each e_{ij}-deduction
- find the final conflict clauses by resolving the e_{ij}-deduction clauses

Exercises

For the previous N.O. example:

- write the (minimal) clauses corresponding to each e_{ij}-deduction
- find the final conflict clauses by resolving the e_{ij}-deduction clauses

N.O.: example (non-convex theory)

$\mu_{\mathcal{LIA}}$

$$v_1 \geq 0 \qquad v_5 = v_4 - 1$$
$$v_1 \leq 1 \qquad v_3 = 0$$
$$v_2 \geq v_6 \qquad v_4 = 1$$
$$v_2 \leq v_6 + 1$$

$\mu_{\mathcal{EUF}}$

$$\neg(f(v_1) = f(v_2))$$
$$\neg(f(v_2) = f(v_4))$$
$$f(v_3) = v_5$$
$$f(v_1) = v_6$$

N.O.: example (non-convex theory)

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$$v_2 \leq v_6 + 1$$

$\langle e_{ij}$*-deduction*$\rangle$

$$v_1 = v_3 \vee v_1 = v_4$$

$\mu_{\mathcal{EUF}}$

$$\neg(f(v_1) = f(v_2))$$
$$\neg(f(v_2) = f(v_4))$$
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N.O.: example (non-convex theory)

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$\langle e_{ij}\text{-deduction} \rangle$

$$v_1 = v_3 \vee v_1 = v_4$$

$\mu_{\mathcal{EUF}}$

$$\neg(f(v_1) = f(v_2))$$
$$\neg(f(v_2) = f(v_4))$$
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$$v_1 = v_3$$

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$$v_1 \leq 1 \qquad v_3 = 0$$
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$$v_2 \leq v_6 + 1$$

$\langle e_{ij}\text{-deduction}\rangle$

$$v_1 = v_3 \vee v_1 = v_4$$

$\Rightarrow$

$\mu_{\mathcal{EUF}}$

$$\neg(f(v_1) = f(v_2))$$
$$\neg(f(v_2) = f(v_4))$$
$$f(v_3) = v_5$$
$$f(v_1) = v_6$$

$$v_1 = v_3$$

$\langle e_{ij}\text{-deduction}\rangle$

$$v_5 = v_6$$

N.O.: example (non-convex theory)

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$$v_1 \geq 0 \qquad v_5 = v_4 - 1$$
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$\langle e_{ij}$*-deduction*$\rangle$

$$v_1 = v_3 \vee v_1 = v_4$$

$$v_5 = v_6$$

$\langle e_{ij}$*-deduction*$\rangle$

$$v_2 = v_3 \vee v_2 = v_4$$

$\mu_{\mathcal{EUF}}$

$$\neg(f(v_1) = f(v_2))$$
$$\neg(f(v_2) = f(v_4))$$
$$f(v_3) = v_5$$
$$f(v_1) = v_6$$

$$v_1 = v_3$$

$\langle e_{ij}$*-deduction*$\rangle$

$$v_5 = v_6$$

N.O.: example (non-convex theory)

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$$\neg(f(v_1) = f(v_2))$$
$$\neg(f(v_2) = f(v_4))$$
$$f(v_3) = v_5$$
$$f(v_1) = v_6$$

$\langle e_{ij}$*-deduction*$\rangle$

$$v_1 = v_3 \vee v_1 = v_4$$

⇒ $v_1 = v_3$

$\langle e_{ij}$*-deduction*$\rangle$

$v_5 = v_6$ ⇐ $v_5 = v_6$

$\langle e_{ij}$*-deduction*$\rangle$

$$v_2 = v_3 \vee v_2 = v_4$$

⇒ $v_2 = v_3$

$\bot$

N.O.: example (non-convex theory)

$\mu_{\mathcal{LIA}}$

$$v_1 \geq 0 \qquad v_5 = v_4 - 1$$
$$v_1 \leq 1 \qquad v_3 = 0$$
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$$v_2 \leq v_6 + 1$$

$\mu_{\mathcal{EUF}}$

$$\neg(f(v_1) = f(v_2))$$
$$\neg(f(v_2) = f(v_4))$$
$$f(v_3) = v_5$$
$$f(v_1) = v_6$$

$\langle e_{ij}$*-deduction*$\rangle$

$$v_1 = v_3 \vee v_1 = v_4 \quad \Rightarrow \quad v_1 = v_3$$

$\langle e_{ij}$*-deduction*$\rangle$

$$v_5 = v_6 \quad \Leftarrow \quad v_5 = v_6$$

$\langle e_{ij}$*-deduction*$\rangle$

$$v_2 = v_3 \vee v_2 = v_4 \quad \Rightarrow \quad v_2 = v_3 \;\; v_2 = v_4$$

$$\bot \qquad \bot$$

N.O.: example (non-convex theory)

$\mu_{\mathcal{LIA}}$

$v_1 \geq 0$ $\quad v_5 = v_4 - 1$

$v_1 \leq 1$ $\quad v_3 = 0$

$v_2 \geq v_6$ $\quad v_4 = 1$

$v_2 \leq v_6 + 1$

$\langle e_{ij}$-*deduction*$\rangle$

$v_1 = v_3 \vee v_1 = v_4$ ⇒

$v_5 = v_6$ ⇐

$\langle e_{ij}$-*deduction*$\rangle$

$v_2 = v_3 \vee v_2 = v_4$ ⇒

$\mu_{\mathcal{EUF}}$

$\neg(f(v_1) = f(v_2))$

$\neg(f(v_2) = f(v_4))$

$f(v_3) = v_5$

$f(v_1) = v_6$

$v_1 = v_3$ $\quad v_1 = v_4$

$\langle e_{ij}$-*deduction*$\rangle$ $\quad$ SAT!

$v_5 = v_6$

$v_2 = v_3$ $\quad v_2 = v_4$

$\bot$ $\quad \bot$

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$$v_2 \leq v_6 + 1$$

$\langle e_{ij}$*-deduction*$\rangle$

$$v_1 = v_3 \vee v_1 = v_4$$

$$v_5 = v_6$$

$\langle e_{ij}$*-deduction*$\rangle$

$$v_2 = v_3 \vee v_2 = v_4$$

$\mu_{\mathcal{EUF}}$

$$\neg(f(v_1) = f(v_2))$$
$$\neg(f(v_2) = f(v_4))$$
$$f(v_3) = v_5$$
$$f(v_1) = v_6$$

$v_1 = v_3$ | $v_1 = v_4$

$\langle e_{ij}$*-deduction*$\rangle$ | SAT!

$v_5 = v_6$

3 e_{ij}-deductions,

3 branches

$v_2 = v_3$ | $v_2 = v_4$

$\bot$ | $\bot$

$SMT(\bigcup_i \mathcal{T}_i)$ via "classic" Nelson-Oppen

Main idea

Combine two or more $\mathcal{T}_i$-*solvers* into one $(\bigcup_i \mathcal{T}_i)$-solver via Nelson-Oppen/Shostak (N.O.) combination procedure [57, 68]

- based on the deduction and exchange of equalities between shared variables/terms (interface equalities, e_{ij}s)
- important improvements and evolutions [62, 7, 36]
- drawbacks [22, 23]:
  - require (possibly expensive) deduction capabilities from $\mathcal{T}_i$-*solvers*
  - [with non-convex theories] case-splits forced by the deduction of disjunctions of e_{ij}'s
  - generate (typically long) $(\bigcup_i \mathcal{T}_i)$-lemmas, without interface equalities $\Longrightarrow$ no backjumping & learning from e_{ij}-reasoning

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  - [with non-convex theories] case-splits forced by the deduction of disjunctions of e_{ij}'s
  - generate (typically long) $(\bigcup_i \mathcal{T}_i)$-lemmas, without interface equalities $\Longrightarrow$ no backjumping & learning from e_{ij}-reasoning

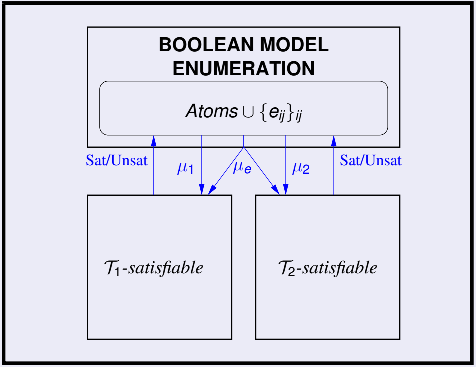
$SMT(\bigcup_i \mathcal{T}_i)$ via Delayed Theory Combination (DTC)

Main idea

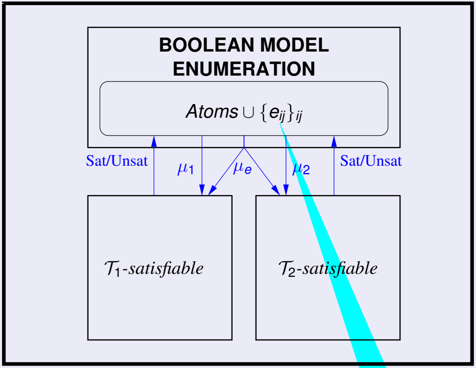
Delegate to the CDCL SAT solver part/most of the (possibly very expensive) reasoning effort on interface equalities previously due to the $\mathcal{T}_i$*-solvers* (e_{ij}-deduction, case-split). [14, 15, 23]

- based on Boolean reasoning on interface equalities via CDCL (plus $\mathcal{T}$-propagation)
- important improvements and evolutions [34, 9]
- feature wrt N.O. [22, 23]
  - do not require (possibly expensive) deduction capabilities from $\mathcal{T}_i$*-solvers*
  - with non-convex theories, case-splits on e_{ij}'s handled by SAT
  - generate $\mathcal{T}_i$-lemmas with interface equalities
    $\Longrightarrow$ backjumping & learning from e_{ij}-reasoning

DTC: Basic schema

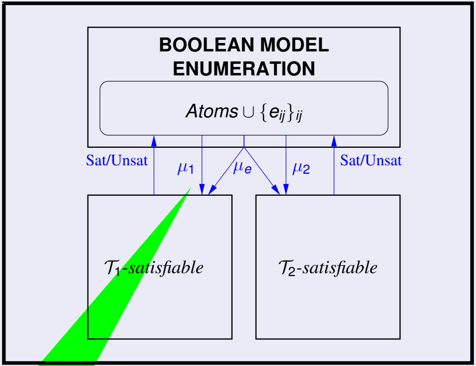


DTC: Basic schema



The boolean solver assigns values not only to atoms in $Atoms(\phi)$, but also to interface equalities $\{(v_i = v_j)\}_{ij}$:
$\mu = \mu_1 \cup \mu_2 \cup \mu_e, \quad \mu_e := \{[\neg](v_i = v_j) | v_i, v_j \in \mu_1 \cup \mu_2\}$

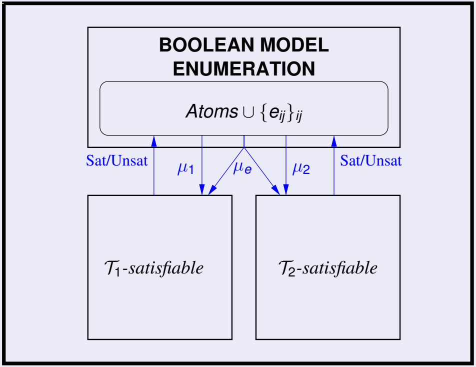
DTC: Basic schema



Each $\mathcal{T}_i$*-solver* interacts only with the boolean solver

- receives $\mu_i' := \mu_i \cup \mu_e$ from Bool
- checks the T_i-satisfiability of μ_i'

DTC: Basic schema



...until either:

- some μ propositionally satisfies ϕ and both $\mu_i' := \mu_i \cup \mu_e$ are T_i-consistent $\Longrightarrow$ (ϕ is $\mathcal{T}_1 \cup \mathcal{T}_2$-sat)
- no more assignment μ are available $\Longrightarrow$ (ϕ is $\mathcal{T}_1 \cup \mathcal{T}_2$-unsat)

DTC: enhanced schema

- CDCL-based assignment enumeration on $Atoms(\phi) \cup \{e_{ij}\}_{ij}$,
  $\Longrightarrow$ benefits of state-of-the-art SAT techniques
- Early pruning: invoke the $\mathcal{T}_i$-*solver*'s before every Boolean decision
  $\Longrightarrow$ total assignments generated only when strictly necessary
- Branching: branching on e_{ij}'s postponed
  $\Longrightarrow$ Boolean search on e_{ij}'s performed only when strictly necessary
- Theory-Backjumping & Learning: e_{ij}'s are involved in conflicts
  $\Longrightarrow$ e_{ij}'s can be assigned by unit propagation
- Theory-deduction & learning: if $\mathcal{T}_i$-*solver* deduces unassigned literals l on $Atoms(\phi) \cup \{e_{ij}\}_{ij}$
  - l is passed back to the Boolean solver, which unit-propagates it
  - the deduction $\mu' \models l$ is learned as a clause $\mu' \rightarrow l$ (deduction clause)
- ...

DTC: example w.out $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

$\neg(f(v_1) = f(v_2))$
$\neg(f(v_2) = f(v_4))$
$f(v_3) = v_5$
$f(v_1) = v_6$

$\mu_{\mathcal{LIA}}$:

$v_1 \geq 0$ $\quad v_5 = v_4 - 1$
$v_1 \leq 1$ $\quad v_3 = 0$
$v_2 \geq v_6$ $\quad v_4 = 1$
$v_2 \leq v_6 + 1$

DTC: example w.out $\mathcal{T}$-prop. (non-convex theory)

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$\mu_{\mathcal{LIA}}$:

$v_1 \geq 0$ $\quad v_5 = v_4 - 1$
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$v_2 \geq v_6$ $\quad v_4 = 1$
$v_2 \leq v_6 + 1$

$\neg(v_1 = v_4)$

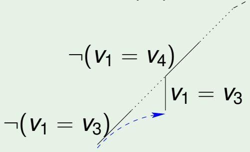
$\neg(v_1 = v_3)$

$\mathcal{LIA}$-unsat, C_{13}

$$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$

DTC: example w.out $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:	$\mu_{\mathcal{LIA}}$:	
$\neg(f(v_1) = f(v_2))$	$v_1 \geq 0$	$v_5 = v_4 - 1$
$\neg(f(v_2) = f(v_4))$	$v_1 \leq 1$	$v_3 = 0$
$f(v_3) = v_5$	$v_2 \geq v_6$	$v_4 = 1$
$f(v_1) = v_6$	$v_2 \leq v_6 + 1$	



$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$

DTC: example w.out $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

$\neg(f(v_1) = f(v_2))$

$\neg(f(v_2) = f(v_4))$

$f(v_3) = v_5$

$f(v_1) = v_6$

$\mu_{\mathcal{LIA}}$:

$v_1 \geq 0$ $\quad v_5 = v_4 - 1$

$v_1 \leq 1$ $\quad v_3 = 0$

$v_2 \geq v_6$ $\quad v_4 = 1$

$v_2 \leq v_6 + 1$

$\neg(v_1 = v_4)$

$v_1 = v_3$

$\neg(v_1 = v_3)$

$\neg(v_5 = v_6)$

$\mathcal{EUF}$-unsat, C_{56}

$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$

$C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$

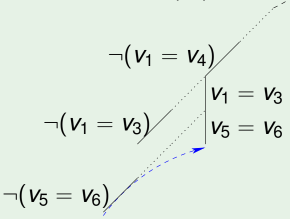
DTC: example w.out $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

$\neg(f(v_1) = f(v_2))$
$\neg(f(v_2) = f(v_4))$
$f(v_3) = v_5$
$f(v_1) = v_6$

$\mu_{\mathcal{LIA}}$:

$v_1 \geq 0$ $\quad v_5 = v_4 - 1$
$v_1 \leq 1$ $\quad v_3 = 0$
$v_2 \geq v_6$ $\quad v_4 = 1$
$v_2 \leq v_6 + 1$



$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$
$C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$

DTC: example w.out $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

$\neg(f(v_1) = f(v_2))$
$\neg(f(v_2) = f(v_4))$
$f(v_3) = v_5$
$f(v_1) = v_6$

$\mu_{\mathcal{LIA}}$:

$v_1 \geq 0$
$v_1 \leq 1$
$v_2 \geq v_6$
$v_2 \leq v_6 + 1$
$v_5 = v_4 - 1$
$v_3 = 0$
$v_4 = 1$

$\neg(v_1 = v_4)$

$v_1 = v_3$

$v_5 = v_6$

$\neg(v_1 = v_3)$

$\neg(v_5 = v_6)$

$\neg(v_2 = v_4)$

$\neg(v_2 = v_3)$

$\mathcal{LIA}$-unsat, C_{23}

$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$
$C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$
$C_{23} : (\mu''_{\mathcal{LIA}} \wedge (v_5 = v_6)) \rightarrow ((v_2 = v_3) \vee (v_2 = v_4))$

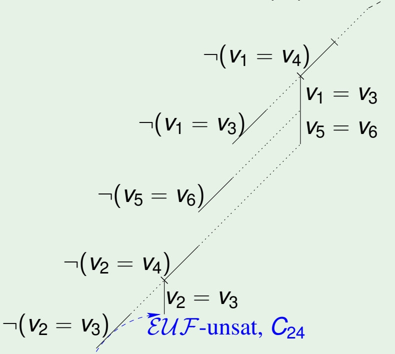
DTC: example w.out $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

$$\neg(f(v_1) = f(v_2))$$
$$\neg(f(v_2) = f(v_4))$$
$$f(v_3) = v_5$$
$$f(v_1) = v_6$$

$\mu_{\mathcal{LIA}}$:

$$v_1 \geq 0 \qquad v_5 = v_4 - 1$$
$$v_1 \leq 1 \qquad v_3 = 0$$
$$v_2 \geq v_6 \qquad v_4 = 1$$
$$v_2 \leq v_6 + 1$$



$$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$
$$C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$$
$$C_{23} : (\mu''_{\mathcal{LIA}} \wedge (v_5 = v_6)) \rightarrow ((v_2 = v_3) \vee (v_2 = v_4))$$
$$C_{24} : (\mu''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_3)) \rightarrow \bot$$

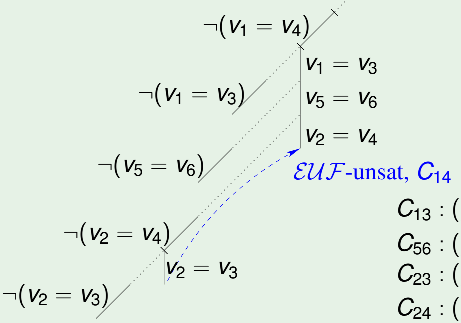
DTC: example w.out $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

$\neg(f(v_1) = f(v_2))$
$\neg(f(v_2) = f(v_4))$
$f(v_3) = v_5$
$f(v_1) = v_6$

$\mu_{\mathcal{LIA}}$:

$v_1 \geq 0$ $\quad v_5 = v_4 - 1$
$v_1 \leq 1$ $\quad v_3 = 0$
$v_2 \geq v_6$ $\quad v_4 = 1$
$v_2 \leq v_6 + 1$



$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$

$C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$

$C_{23} : (\mu''_{\mathcal{LIA}} \wedge (v_5 = v_6)) \rightarrow ((v_2 = v_3) \vee (v_2 = v_4))$

$C_{24} : (\mu''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_3)) \rightarrow \bot$

$C_{14} : (\mu'''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_4)) \rightarrow \bot$

DTC: example w.out $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

$\neg(f(v_1) = f(v_2))$
$\neg(f(v_2) = f(v_4))$
$f(v_3) = v_5$
$f(v_1) = v_6$

$\mu_{\mathcal{LIA}}$:

$v_1 \geq 0$ $\quad v_5 = v_4 - 1$
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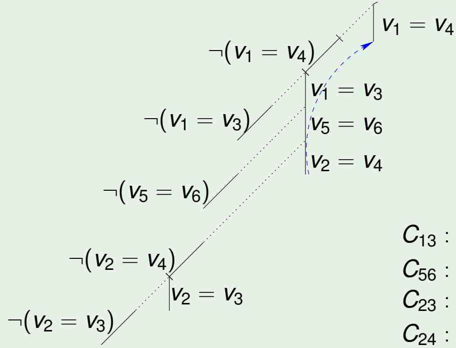

$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$
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DTC: example w.out $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

$$\neg(f(v_1) = f(v_2))$$
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$\mu_{\mathcal{LIA}}$:

$$v_1 \geq 0 \qquad v_5 = v_4 - 1$$
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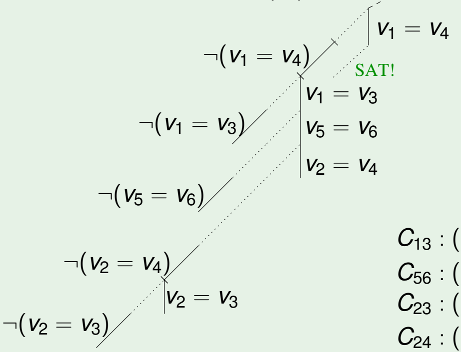

$$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$
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$$C_{24} : (\mu''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_3)) \rightarrow \bot$$
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DTC: example w.out $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

$\neg(f(v_1) = f(v_2))$
$\neg(f(v_2) = f(v_4))$
$f(v_3) = v_5$
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$\mu_{\mathcal{LIA}}$:

$v_1 \geq 0$ $\quad v_5 = v_4 - 1$
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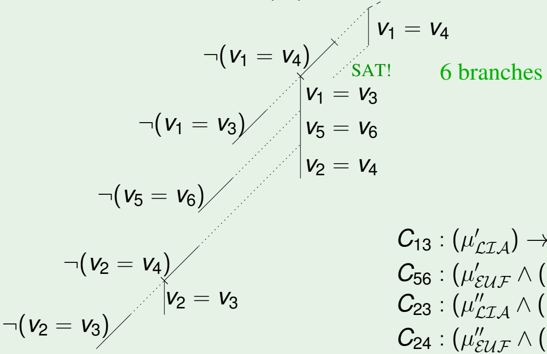
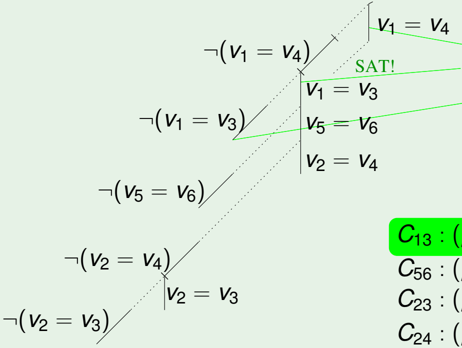
$v_2 \leq v_6 + 1$



$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$
$C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$
$C_{23} : (\mu''_{\mathcal{LIA}} \wedge (v_5 = v_6)) \rightarrow ((v_2 = v_3) \vee (v_2 = v_4))$
$C_{24} : (\mu''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_3)) \rightarrow \bot$
$C_{14} : (\mu'''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_4)) \rightarrow \bot$

DTC: example w.out $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:	$\mu_{\mathcal{LIA}}$:	
$\neg(f(v_1) = f(v_2))$	$v_1 \geq 0$	$v_5 = v_4 - 1$
$\neg(f(v_2) = f(v_4))$	$v_1 \leq 1$	$v_3 = 0$
$f(v_3) = v_5$	$v_2 \geq v_6$	$v_4 = 1$
$f(v_1) = v_6$	$v_2 \leq v_6 + 1$	



Mimics the e_{ij}-deduction
$\mu'_{\mathcal{LIA}} \models_{\mathcal{LIA}} ((v_1 = v_3) \vee (v_1 = v_4))$
and the two branches $(v_1 = v_3)$, $(v_1 = v_4)$

$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$

$C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$

$C_{23} : (\mu''_{\mathcal{LIA}} \wedge (v_5 = v_6)) \rightarrow ((v_2 = v_3) \vee (v_2 = v_4))$

$C_{24} : (\mu''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_3)) \rightarrow \bot$

$C_{14} : (\mu'''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_4)) \rightarrow \bot$

DTC: example with $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:	$\mu_{\mathcal{LIA}}$:	
$\neg(f(v_1) = f(v_2))$	$v_1 \geq 0$	$v_5 = v_4 - 1$
$\neg(f(v_2) = f(v_4))$	$v_1 \leq 1$	$v_3 = 0$
$f(v_3) = v_5$	$v_2 \geq v_6$	$v_4 = 1$
$f(v_1) = v_6$	$v_2 \leq v_6 + 1$	

DTC: example with $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:	$\mu_{\mathcal{LIA}}$:	
$\neg(f(v_1) = f(v_2))$	$v_1 \geq 0$	$v_5 = v_4 - 1$
$\neg(f(v_2) = f(v_4))$	$v_1 \leq 1$	$v_3 = 0$
$f(v_3) = v_5$	$v_2 \geq v_6$	$v_4 = 1$
$f(v_1) = v_6$	$v_2 \leq v_6 + 1$	

$\mathcal{LIA}$-deduce $(v_1 = v_4) \vee (v_1 = v_3)$, C_{13}

$$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$

DTC: example with $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

$\neg(f(v_1) = f(v_2))$
$\neg(f(v_2) = f(v_4))$
$f(v_3) = v_5$
$f(v_1) = v_6$

$\mu_{\mathcal{LIA}}$:

$v_1 \geq 0$ $\quad v_5 = v_4 - 1$
$v_1 \leq 1$ $\quad v_3 = 0$
$v_2 \geq v_6$ $\quad v_4 = 1$
$v_2 \leq v_6 + 1$

$\neg(v_1 = v_4)$

$v_1 = v_3$

$$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$

DTC: example with $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

$\neg(f(v_1) = f(v_2))$
$\neg(f(v_2) = f(v_4))$
$f(v_3) = v_5$
$f(v_1) = v_6$

$\mu_{\mathcal{LIA}}$:

$v_1 \geq 0$ $\quad v_5 = v_4 - 1$
$v_1 \leq 1$ $\quad v_3 = 0$
$v_2 \geq v_6$ $\quad v_4 = 1$
$v_2 \leq v_6 + 1$

$\neg(v_1 = v_4)$

$v_1 = v_3$ $\quad \mathcal{EUF}$-deduce $(v_5 = v_6)$, C_{56}

$v_5 = v_6$

$$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$
$$C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$$

DTC: example with $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:	$\mu_{\mathcal{LIA}}$:	
$\neg(f(v_1) = f(v_2))$	$v_1 \geq 0$	$v_5 = v_4 - 1$
$\neg(f(v_2) = f(v_4))$	$v_1 \leq 1$	$v_3 = 0$
$f(v_3) = v_5$	$v_2 \geq v_6$	$v_4 = 1$
$f(v_1) = v_6$	$v_2 \leq v_6 + 1$	

$\neg(v_1 = v_4)$

$v_1 = v_3$

$v_5 = v_6$ $\mathcal{LIA}$-deduce $(v_2 = v_4) \vee (v_2 = v_3)$, C_{23}

$$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$
$$C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$$
$$C_{23} : (\mu''_{\mathcal{LIA}} \wedge (v_5 = v_6)) \rightarrow ((v_2 = v_3) \vee (v_2 = v_4))$$

DTC: example with $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

$\neg(f(v_1) = f(v_2))$
$\neg(f(v_2) = f(v_4))$
$f(v_3) = v_5$
$f(v_1) = v_6$

$\mu_{\mathcal{LIA}}$:

$v_1 \geq 0$ $\quad v_5 = v_4 - 1$
$v_1 \leq 1$ $\quad v_3 = 0$
$v_2 \geq v_6$ $\quad v_4 = 1$
$v_2 \leq v_6 + 1$

$\neg(v_1 = v_4)$

$v_1 = v_3$

$v_5 = v_6$

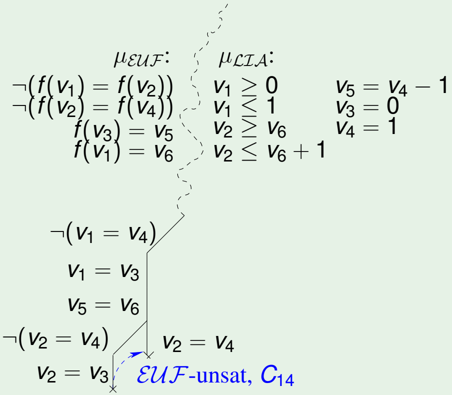
$\neg(v_2 = v_4)$

$v_2 = v_3$

$\mathcal{EUF}$-unsat, C_{24}

$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$
$C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$
$C_{23} : (\mu''_{\mathcal{LIA}} \wedge (v_5 = v_6)) \rightarrow ((v_2 = v_3) \vee (v_2 = v_4))$
$C_{24} : (\mu''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_3)) \rightarrow \bot$

DTC: example with $\mathcal{T}$-prop. (non-convex theory)



$$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$
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$$C_{24} : (\mu''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_3)) \rightarrow \bot$$
$$C_{14} : (\mu'''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_4)) \rightarrow \bot$$

DTC: example with $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

$$\neg(f(v_1) = f(v_2))$$
$$\neg(f(v_2) = f(v_4))$$
$$f(v_3) = v_5$$
$$f(v_1) = v_6$$

$\mu_{\mathcal{LIA}}$:

$$v_1 \geq 0 \qquad v_5 = v_4 - 1$$
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$$v_2 \geq v_6 \qquad v_4 = 1$$
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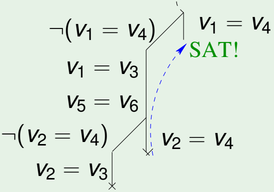

$$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$
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$$C_{24} : (\mu''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_3)) \rightarrow \bot$$
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DTC: example with $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

$\neg(f(v_1) = f(v_2))$
$\neg(f(v_2) = f(v_4))$
$f(v_3) = v_5$
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$\mu_{\mathcal{LIA}}$:

$v_1 \geq 0$ $\quad v_5 = v_4 - 1$
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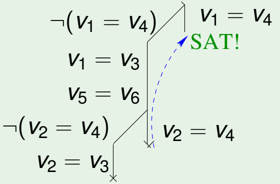

3 e_{ij}-deductions
3 branches

$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$
$C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$
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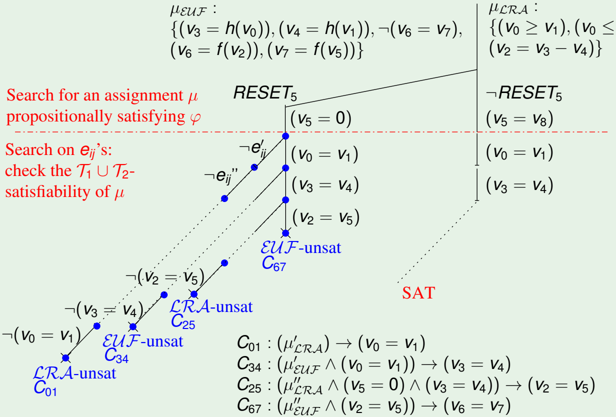
$C_{14} : (\mu'''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_4)) \rightarrow \bot$

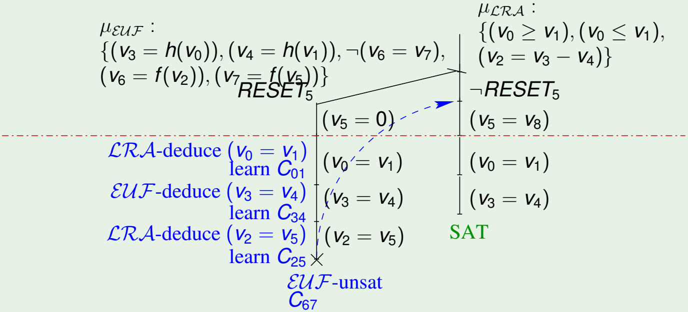
DTC: example without $\mathcal{T}$-propagation (convex theory)

$$
\begin{array}{ll}
\mathcal{EUF}: & (v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge \\
\mathcal{LRA}: & (v_0 \geq v_1) \wedge (v_0 \leq v_1) \wedge (v_2 = v_3 - v_4) \wedge (RESET_5 \rightarrow (v_5 = 0)) \wedge \\
Both: & (\neg RESET_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7).
\end{array}
$$



DTC: example with $\mathcal{T}$-propagation (convex theory)

$$
\begin{array}{ll}
\mathcal{EUF}: & (v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge \\
\mathcal{LRA}: & (v_0 \geq v_1) \wedge (v_0 \leq v_1) \wedge (v_2 = v_3 - v_4) \wedge (RESET_5 \rightarrow (v_5 = 0)) \wedge \\
Both: & (\neg RESET_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7).
\end{array}
$$



$$
\begin{array}{l}
C_{01} : (\mu'_{\mathcal{LRA}}) \rightarrow (v_0 = v_1) \\
C_{34} : (\mu'_{\mathcal{EUF}} \wedge (v_0 = v_1)) \rightarrow (v_3 = v_4) \\
C_{25} : (\mu''_{\mathcal{LRA}} \wedge (v_5 = 0) \wedge (v_3 = v_4)) \rightarrow (v_2 = v_5) \\
C_{67} : (\mu''_{\mathcal{EUF}} \wedge (v_2 = v_5)) \rightarrow (v_6 = v_7)
\end{array}
$$

DTC + Model-based heuristic (aka Model-Based Theory Combination) [34]

- Initially, no interface equalities generated
- When a model is found, check against all the possible interface equalities
  - If $\mathcal{T}_1$ and $\mathcal{T}_2$ agree on the implied equalities, then return SAT
  - Otherwise, branch on equalities implied by $\mathcal{T}_1$-model but not by $\mathcal{T}_2$-model
- "Optimistic" approach, similar to axiom instantiation

Exercises

For each of the previous DTC examples:

- write the (minimal) clauses corresponding to each e_{ij}-deduction (as clauses rather than as implications)
- compute the conflict-analysis steps leading to the backjumping steps in the figures.

Exercises

For each of the previous DTC examples:

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Exercise

Let $\mathcal{LRA}$ be the logic of linear arithmetic over the rationals and $\mathcal{EUF}$ be the logic of equality and uninterpreted functions. Consider the following pure formula φ in the combined logic $\mathcal{LRA} \cup \mathcal{EUF}$:

$$(x = 1.0) \wedge (h = 1.0) \wedge (k = 1.0) \wedge (y = 2h - k) \wedge (z < w)$$
$$(z = f(x)) \wedge (w = f(y))$$

1. Say which variables are interface variables,
2. list the interface equalities for this formula (modulo symmetry),
3. decide whether this formulas is $\mathcal{LRA} \cup \mathcal{EUF}$-satisfiable or not, using both Nelson-Oppen or Delayed Theory Combination.

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   - What is a Theory?
   - Satisfiability Modulo Theories
   - Motivations and Goals of SMT
2. Efficient SMT solving
   - Combining SAT with Theory Solvers
   - Theory Solvers for Theories of Interest (hints)
   - SMT for Combinations of Theories
3. **Beyond Solving: Advanced SMT Functionalities**
   - Proofs and Unsatisfiable Cores
   - Interpolants
   - All-SMT & Predicate Abstraction (hints)
   - SMT with Optimization (Optimization Modulo Theories)

Advanced SMT functionalities (very important in FV):

- Building proofs of $\mathcal{T}$-unsatisfiability
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Beyond Solving: advanced SAT & SMT functionalities

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Building (Resolution) Proofs of $\mathcal{T}$-Unsatisfiability

Resolution proof of $\mathcal{T}$-unsatisfiability

Very similar to building proofs with plain SAT:

- resolution proofs whose leaves are original clauses and $\mathcal{T}$-lemmas returned by the $\mathcal{T}$-*solver* (i.e., $\mathcal{T}$-conflict and $\mathcal{T}$-deduction clauses)
- built by backward traversal of implication graphs, as in CDCL SAT
- Sub-proofs of $\mathcal{T}$-lemmas can be built in some $\mathcal{T}$-specific deduction framework if requested

Important for:

- certifying $\mathcal{T}$-unsatisfiability results
- computing unsatisfiable cores
- computing interpolants

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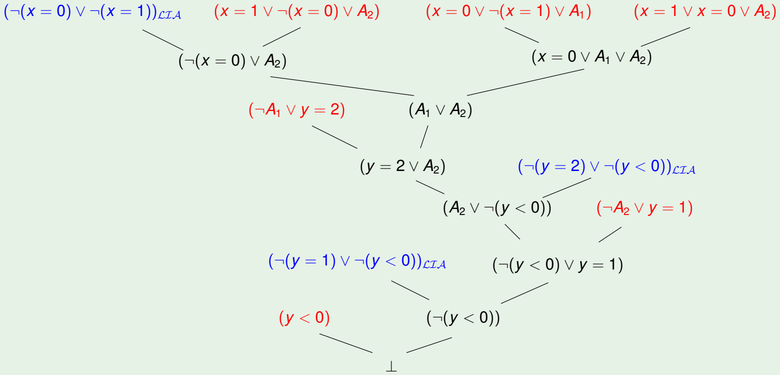
Building Proofs of $\mathcal{T}$-Unsatisfiability: example

$$(x = 0 \vee \neg(x = 1) \vee A_1) \wedge (x = 0 \vee x = 1 \vee A_2) \wedge (\neg(x = 0) \vee x = 1 \vee A_2) \wedge$$
$$(\neg A_2 \vee y = 1) \wedge (\neg A_1 \vee x + y > 3) \wedge (y < 0) \wedge (A_2 \vee x - y = 4) \wedge (y = 2 \vee \neg A_1) \wedge (x \geq 0),$$



relevant original clauses, irrelevant original clauses, $\mathcal{T}$-lemmas

Example: proof on non-strict $\mathcal{LRA}$ inequalities

- A proof of unsatisfiability for a set of non-strict $\mathcal{LRA}$ inequalities can be obtained by building a linear combination of such inequalities, each time eliminating one or more variables, until you get a contradictory inequality on constant values.
- Example:

$$\varphi \stackrel{\text{def}}{=} (0 \leq x_1 - 3x_2 + 1), (0 \leq x_1 + x_2), (0 \leq x_3 - 2x_1 - 3), (0 \leq 1 - 2x_3).$$

A proof of unsatisfiability P for φ is the following:

$$\dfrac{\dfrac{(0 \leq x_1 - 3x_2 + 1) \quad (0 \leq x_1 + x_2)}{\textsc{Comb}\ \ (0 \leq 4x_1 + 1) \text{ with coeffs 1 and 3}} \quad \dfrac{(0 \leq x_3 - 2x_1 - 3) \quad (0 \leq 1 - 2x_3)}{\textsc{Comb}\ \ (0 \leq -4x_1 - 5) \text{ with coeffs 2 and 1}}}{\textsc{Comb}\ \ (0 \leq -4) \text{ with coeffs 1 and 1}}$$

- It is possible to produce such proof from an unsatisfiable tableau in Simplex procedure for $\mathcal{LRA}$ [27, 29]
- It is straightforward to produce such proof from a negative cycle in the graph-based procedure for $\mathcal{DL}$ [27, 29]

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Extraction of $\mathcal{T}$-unsatisfiable cores

The problem

Given a $\mathcal{T}$-unsatisfiable set of clauses, extract from it a (possibly small/minimal/minimum) $\mathcal{T}$-unsatisfiable subset ($\mathcal{T}$-unsatisfiable core)

- Wide literature in SAT
- Some implementations, very few literature for SMT [26, 51]
- We recognize three approaches:
  - Proof-based approach (CVC4, MathSAT): byproduct of finding a resolution proof
  - Assumption-based approach (Yices): use extra variables labeling clauses, as in the plain Boolean case
  - Lemma-Lifting approach [26] : use an external (possibly-optimized) Boolean unsat-core extractor

The proof-based approach to $\mathcal{T}$-unsat cores

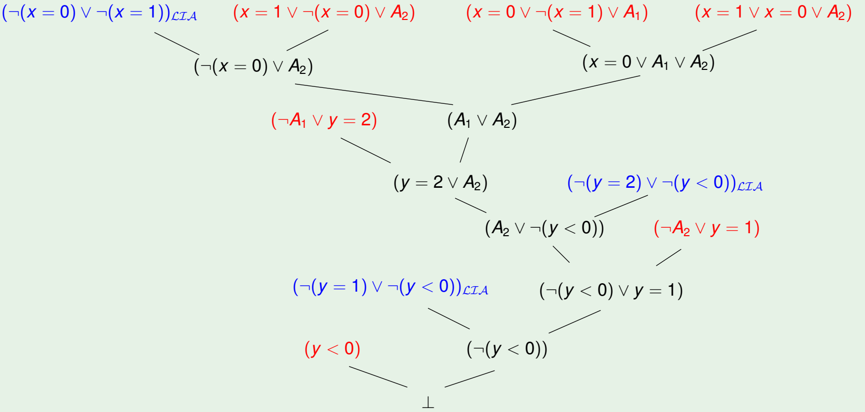
Idea (adapted from [74])

Unsatisfiable core of φ:

- in SAT: the set of leaf clauses of a resolution proof of unsatisfiability of φ
- in SMT($\mathcal{T}$): the set of leaf clauses of a resolution proof of $\mathcal{T}$-unsatisfiability of φ, minus the $\mathcal{T}$-lemmas

The proof-based approach to $\mathcal{T}$-unsat cores: example

$$(x = 0 \lor \neg(x = 1) \lor A_1) \land (x = 0 \lor x = 1 \lor A_2) \land (\neg(x = 0) \lor x = 1 \lor A_2) \land$$
$$(\neg A_2 \lor y = 1) \land (\neg A_1 \lor x + y > 3) \land (y < 0) \land (A_2 \lor x - y = 4) \land (y = 2 \lor \neg A_1) \land (x \geq 0),$$



The Assumption-based approach to $\mathcal{T}$-unsat cores

Idea (adapted from [52])

Let φ be $\bigwedge_{i=1}^{n} C_i$ s.t. φ unsatisfiable.

1. each clause C_i in φ is substituted by $\neg S_i \vee C_i$, s.t. S_i fresh "selector" variable
2. the resulting formula is checked for satisfiability under the assumption of all S_i's
3. final conflict clause at dec. level 0: $\bigvee_j \neg S_j$ $\Longrightarrow \{C_j\}_j$ is the unsat core

Extends straightforwardly to SMT($\mathcal{T}$).

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Extends straightforwardly to SMT($\mathcal{T}$).

The assumption-based approach to $\mathcal{T}$-unsat cores: Example

$$(S_1 \rightarrow (x = 0 \vee \neg(x = 1) \vee A_1)) \wedge (S_2 \rightarrow (x = 0 \vee x = 1 \vee A_2)) \wedge$$
$$(S_3 \rightarrow (\neg(x = 0) \vee x = 1 \vee A_2)) \wedge (S_4 \rightarrow (\neg A_2 \vee y = 1)) \wedge$$
$$(S_5 \rightarrow (\neg A_1 \vee x + y > 3)) \wedge (S_6 \rightarrow y < 0) \wedge$$
$$(S_7 \rightarrow (A_2 \vee x - y = 4)) \wedge (S_8 \rightarrow (y = 2 \vee \neg A_1)) \wedge (S_9 \rightarrow x \geq 0)$$

Conflict analysis (Yices 1.0.6) returns:

$$\neg S_1 \vee \neg S_2 \vee \neg S_3 \vee \neg S_4 \vee \neg S_6 \vee \neg S_7 \vee \neg S_8,$$

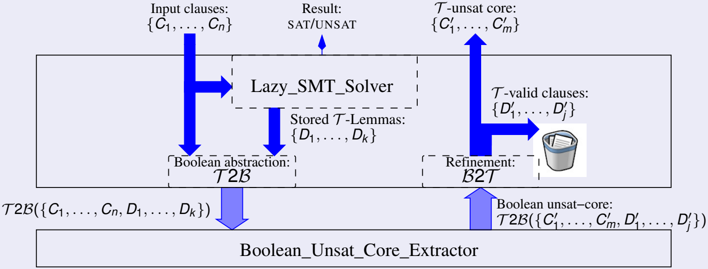
corresponding to the unsat core in red.

The lemma-lifting approach to $\mathcal{T}$-unsat cores

Idea [26, 30]

(i) The $\mathcal{T}$-lemmas D_i are valid in $\mathcal{T}$

(ii) The conjunction of φ with all the $\mathcal{T}$-lemmas $D_1, \ldots, D_k$ is propositionally unsatisfiable: $\mathcal{T}2\mathcal{B}(\varphi \wedge \bigwedge_{i=1}^{n} D_i) \models \bot$.



- interfaces with an external Boolean Unsat-core Extractor

$\Longrightarrow$ benefits for free of all state-of-the-art size-reduction techniques

The lemma-lifting approach to $\mathcal{T}$-unsat cores (cont.)

```
⟨SatValue,Clause_set⟩ T-Unsat_Core(Clause_set φ) {
   // φ is {C_1,...,C_n}
   if (Lazy_SMT_Solver(φ) == SAT)
      then return ⟨SAT,∅⟩;
   // D_1,...,D_k are the T-lemmas stored by Lazy_SMT_Solver
   ψ^p=Boolean_Core_Extractor(T2B({C_1,...,C_n,D_1,...,D_k}));
   // ψ^p is T2B({C'_1,...,C'_m,D'_1,...,D'_j});
   return ⟨UNSAT,{C'_1,...,C'_m}⟩;
}
```

The lemma-lifting approach to $\mathcal{T}$-unsat cores: example

$$(x = 0 \vee \neg(x = 1) \vee A_1) \wedge (x = 0 \vee x = 1 \vee A_2) \wedge (\neg(x = 0) \vee x = 1 \vee A_2) \wedge$$
$$(\neg A_2 \vee y = 1) \wedge (\neg A_1 \vee x + y > 3) \wedge (y < 0) \wedge (A_2 \vee x - y = 4) \wedge (y = 2 \vee \neg A_1) \wedge (x \geq 0),$$

1. The SMT solver generates the following set of $\mathcal{LIA}$-lemmas:

$$\{(\neg(x = 1) \vee \neg(x = 0)),\ (\neg(y = 2) \vee \neg(y < 0)),\ (\neg(y = 1) \vee \neg(y < 0))\}.$$

2. The following formula is passed to the external Boolean core extractor

$$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge$$
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$$(\neg B_1 \vee \neg B_0) \wedge (\neg B_6 \vee \neg B_4) \wedge (\neg B_2 \vee \neg B_4)$$

which returns the unsat core in red.

3. The unsat-core is mapped back, the three $\mathcal{T}$-lemmas are removed
$\Longrightarrow$ the final $\mathcal{T}$-unsat core (in red above).

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$$(x = 0 \vee \neg(x = 1) \vee A_1) \wedge (x = 0 \vee x = 1 \vee A_2) \wedge (\neg(x = 0) \vee x = 1 \vee A_2) \wedge$$
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Exercise

Consider the following set of clauses φ in $\mathcal{EUF}$.

$$\left\{ \begin{array}{l} (\neg(x = y) \vee (f(x) = f(y))), \\ (\neg(x = y) \vee \neg(f(x) = f(y))), \\ ((x = y) \vee (f(x) = f(y))), \\ ((x = y) \vee \neg(f(x) = f(y))) \end{array} \right\}$$

Find a minimal $\mathcal{EUF}$-unsatisfiable core.

Outline

1 Introduction
- What is a Theory?
- Satisfiability Modulo Theories
- Motivations and Goals of SMT

2 Efficient SMT solving
- Combining SAT with Theory Solvers
- Theory Solvers for Theories of Interest (hints)
- SMT for Combinations of Theories

3 Beyond Solving: Advanced SMT Functionalities
- Proofs and Unsatisfiable Cores
- Interpolants
- All-SMT & Predicate Abstraction (hints)
- SMT with Optimization (Optimization Modulo Theories)

Computing (Craig) Interpolants in SMT

Craig Interpolant

Given an ordered pair (A, B) of formulas such that $A \wedge B \models_{\mathcal{T}} \bot$, a *Craig interpolant* is a formula I s.t.:

a) $A \models_{\mathcal{T}} I$,

b) $I \wedge B \models_{\mathcal{T}} \bot$,

c) $I \preceq A$ and $I \preceq B$.

"$I \preceq A$" meaning that all non-interpreted (in $\mathcal{T}$) symbols in I occur in A (including variables)

- Important in some FV applications
- A few works presented for various theories:
  - $\mathcal{EUF}$ [54, 63], $\mathcal{DL}$ [27, 29], $\mathcal{UTVPI}$ [28, 29], $\mathcal{LRA}$ [54, 63, 27, 29], $\mathcal{LIA}$ [48, 17, 45], $\mathcal{BV}$ [49], ...

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A General Algorithm

Algorithm: Interpolant generation for SMT($\mathcal{T}$) [61, 54]

(i) Generate a resolution proof of $\mathcal{T}$-unsatisfiability $\mathcal{P}$ for $A \wedge B$.

(ii) ...

(iii) For every original leaf clause C in $\mathcal{P}$, set $I_C \stackrel{\text{def}}{=} C \downarrow B$ if $C \in A$, and $I_C \stackrel{\text{def}}{=} \top$ if $C \in B$.

(iv) For every inner node C of $\mathcal{P}$ obtained by resolution from $C_1 \stackrel{\text{def}}{=} p \vee \phi_1$ and $C_2 \stackrel{\text{def}}{=} \neg p \vee \phi_2$, set $I_C \stackrel{\text{def}}{=} I_{C_1} \vee I_{C_2}$ if p does not occur in B, and $I_C \stackrel{\text{def}}{=} I_{C_1} \wedge I_{C_2}$ otherwise.

(v) Output $I_\perp$ as an interpolant for (A, B).

"$\eta \setminus B$" [resp. "$\eta \downarrow B$"] is the set of literals in η whose atoms do not [resp. do] occur in B.

- row 2. only takes place where $\mathcal{T}$ comes in to play

$\Longrightarrow$ Reduced to the problem of finding an interpolant for two sets of $\mathcal{T}$-literals (Boolean and $\mathcal{T}$-specific component decoupled)

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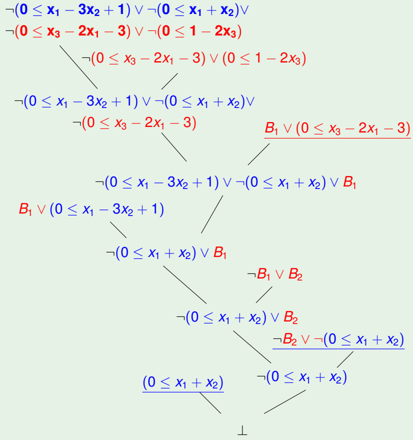
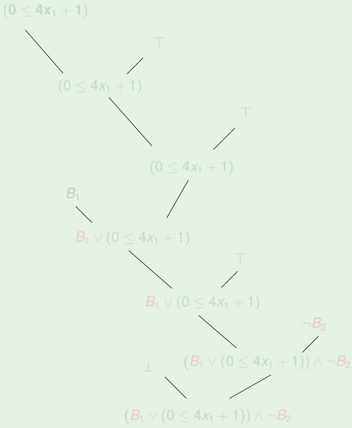
Computing Craig Interpolants in SMT: example

$A \stackrel{\text{def}}{=} (B_1 \vee (0 \leq x_1 - 3x_2 + 1)) \wedge (0 \leq x_1 + x_2) \wedge (\neg B_2 \vee \neg(0 \leq x_1 + x_2))$

$B \stackrel{\text{def}}{=} (\neg(0 \leq x_3 - 2x_1 - 3) \vee (0 \leq 1 - 2x_3)) \wedge (\neg B_1 \vee B_2) \wedge (B_1 \vee (0 \leq x_3 - 2x_1 - 3))$



original proof



interpolant proof

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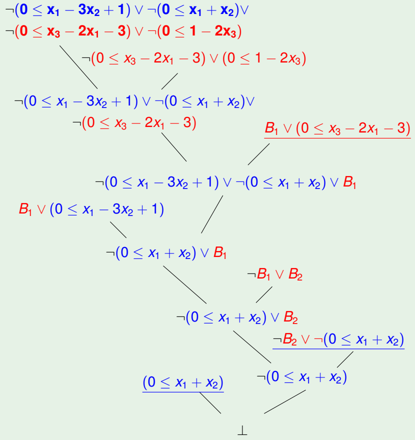

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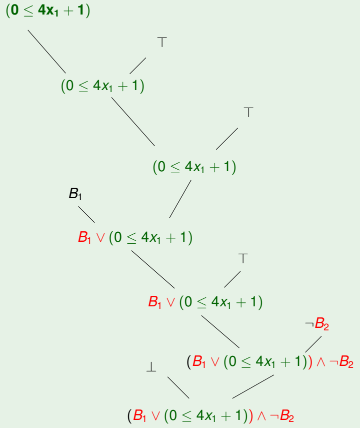

interpolant proof

McMillan's algorithm for non-strict $\mathcal{LRA}$ inequalities

$$A \stackrel{\text{def}}{=} \{(0 \le x_1 - 3x_2 + 1), (0 \le x_1 + x_2)\}$$
$$B \stackrel{\text{def}}{=} \{(0 \le x_3 - 2x_1 - 3), (0 \le 1 - 2x_3)\}.$$

A proof of unsatisfiability P for $A \wedge B$ is the following:

$$\dfrac{\dfrac{(0 \le x_1 - 3x_2 + 1) \quad (0 \le x_1 + x_2)}{\textsc{Comb}\ (0 \le 4x_1 + 1)\ \textit{with c. 1 and 3}} \qquad \dfrac{(0 \le x_3 - 2x_1 - 3) \quad (0 \le 1 - 2x_3)}{\textsc{Comb}\ (0 \le -4x_1 - 5)\ \textit{with c. 2 and 1}}}{\textsc{Comb}\ (0 \le -4)\ \textit{with c. 1 and 1}}$$

By replacing inequalities in B with $(0 \le 0)$, we obtain the proof P':

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Thus, the interpolant obtained is $(0 \le 4x_1 + 1)$.

McMillan's algorithm for non-strict $\mathcal{LRA}$ inequalities

$$A \stackrel{\text{def}}{=} \{(0 \le x_1 - 3x_2 + 1), (0 \le x_1 + x_2)\}$$
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A proof of unsatisfiability P for $A \wedge B$ is the following:

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Example: Interpolation Algorithms for Difference Logic

An inference-based algorithm [54]

$$A \stackrel{\text{def}}{=} \{(0 \le x_1 - x_2 + 1), (0 \le x_2 - x_3), (0 \le x_4 - x_5 - 1)\}$$

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$\Longrightarrow$ Interpolant: $(0 \le x_1 - x_3 + x_4 - x_5)$ (not in $\mathcal{DL}$, and weaker).

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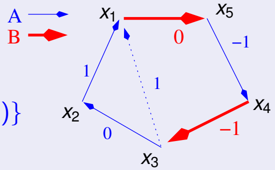
Example: Interpolation Algorithms for Difference Logic

A graph-based algorithm [27, 29]

$$A \stackrel{\text{def}}{=} \{\overbrace{(0 \le x_1 - x_2 + 1), (0 \le x_2 - x_3)}^{\textit{Chord}:\ (0 \le x_1 - x_3 + 1)}, (0 \le x_4 - x_5 - 1)\}$$

$$B \stackrel{\text{def}}{=} \{(0 \le x_5 - x_1), (0 \le x_3 - x_4 - 1)\}.$$



$\Longrightarrow$ Interpolant: $(0 \le x_1 - x_3 + 1) \wedge (0 \le x_4 - x_5 - 1)$ (still in $\mathcal{DL}$)

Exercise

Consider the following formulas in difference logic ($\mathcal{DL}$):

$$
\begin{aligned}
\varphi_1 \stackrel{\text{def}}{=} \; & (x_2 - x_3 \leq -4) \quad \wedge \\
& (x_3 - x_4 \leq -6) \quad \wedge \\
& (x_5 - x_6 \leq 4) \quad \wedge \\
& (x_6 - x_1 \leq 2) \quad \wedge \\
& (x_6 - x_7 \leq -2) \quad \wedge \\
& (x_7 - x_8 \leq 1)
\end{aligned}
$$

$$
\begin{aligned}
\varphi_2 \stackrel{\text{def}}{=} \; & (x_4 - x_9 \leq 2) \quad \wedge \\
& (x_9 - x_5 \leq 0) \quad \wedge \\
& (x_1 - x_2 \leq 1)
\end{aligned}
$$

which are such that $\varphi_1 \wedge \varphi_2 \models_{\mathcal{DL}} \bot$. Compute an interpolant for $\langle \varphi_1, \varphi_2 \rangle$, using both methods presented in previous slides.

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   - What is a Theory?
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   - Motivations and Goals of SMT
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   - Combining SAT with Theory Solvers
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All-SAT/All-SMT (hints)

- All-SAT: enumerate all truth assignments satisfying φ
- All-SMT: enumerate all $\mathcal{T}$-satisfiable truth assignments propositionally satisfying φ
- All-SMT over an “important” subset of atoms $\Gamma \stackrel{\text{def}}{=} \{\gamma_i\}_i$: enumerate all assignments over Γ which can be extended to $\mathcal{T}$-satisfiable truth assignments propositionally satisfying φ
  $\Longrightarrow$ can compute predicate abstraction
- Algorithms:
  - BCLT [50]
    each time a $\mathcal{T}$-satisfiable assignment $\{l_1, ..., l_n\}$ is found, perform conflict-driven backjumping as if the restricted clause $(\bigvee_i \neg l_i) \downarrow \Gamma$ belonged to the clause set
  - MathSAT/NuSMV [25]
    As above, plus the Boolean search of the SMT solver is driven by an OBDD.

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Predicate Abstraction

Predicate abstraction

if $\varphi(\mathbf{v})$ is a SMT formula over the domain variables $\mathbf{v} \stackrel{\text{def}}{=} \{v_j\}_j$, $\{\gamma_i\}_i$ is a set of "relevant" predicates over $\mathbf{v}$, and $\mathbf{P} \stackrel{\text{def}}{=} \{P_i\}_i$ a set of fresh Boolean labels, then:

$$
\begin{aligned}
& \mathit{PredAbs}_{\mathbf{P}}(\varphi) \\
\stackrel{\text{def}}{=} \ & \exists \mathbf{v}.(\ \varphi(\mathbf{v}) \wedge \bigwedge_i P_i \leftrightarrow \gamma_i(\mathbf{v})\ ) \\
= \ & \bigvee \left\{ \mu \mid \begin{array}{c} \mu \text{ truth assignment on } \mathbf{P} \\ \text{s.t. } \mu \wedge \varphi \wedge \bigwedge_i (P_i \leftrightarrow \gamma_i) \text{ is } \mathcal{T}\text{-satisfiable} \end{array} \right\}
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- projection of φ over (the Boolean abstraction of) the set $\{\gamma_i\}_i$.
- important step in FV: extracts finite-state abstractions from a infinite state space

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Predicate Abstraction: example

$$\begin{aligned} \varphi &\stackrel{\text{def}}{=} (v_1 + v_2 > 12) \\ \gamma_1 &\stackrel{\text{def}}{=} (v_1 + v_2 = 2) \\ \gamma_2 &\stackrel{\text{def}}{=} (v_1 - v_2 < 10) \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} PreAbs(\varphi)_{\{P_1,P_2\}} &\stackrel{\text{def}}{=} \exists\, v_1\ v_2\ .\ \begin{pmatrix} (v_1 + v_2 > 12) & \wedge \\ (P_1 \leftrightarrow (v_1 + v_2 = 2)) & \wedge \\ (P_2 \leftrightarrow (v_1 - v_2 < 10)) & \end{pmatrix} \\ &= (\neg P_1 \wedge \neg P_2) \vee (\neg P_1 \wedge P_2) \\ &= \neg P_1 . \end{aligned}$$

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Optimization Modulo Theories: General Case

Ingredients: $\langle \varphi, cost \rangle$

- a SMT formula φ in some background theory $\mathcal{T} = \mathcal{T}_{\preceq} \cup \bigcup_i \mathcal{T}_i$
  - $\bigcup_i \mathcal{T}_i$ may be empty
  - $\mathcal{T}_{\preceq}$ has a predicate $\preceq$ representing a total order
- a $\mathcal{T}_{\preceq}$-variable/term "*cost*" occurring in φ

Optimization Modulo $\mathcal{T}_{\preceq} \cup \bigcup_i \mathcal{T}_i$ (OMT($\mathcal{T}_{\preceq} \cup \bigcup_i \mathcal{T}_i$))

The problem of finding a model $\mathcal{M}$ for φ whose value of *cost* is minimum according to $\preceq$.

- maximization is dual

Note

The cost term can be rewritten as a variable

$$\langle \varphi, term \rangle \implies \langle \varphi \wedge (cost = term), cost \rangle, \quad cost \text{ fresh}$$

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Optimization Modulo Theories with $\mathcal{LA}$ costs

Ingredients

- an SMT formula φ on $\mathcal{LA} \cup \mathcal{T}$
  - $\mathcal{LA}$ can be $\mathcal{LRA}$, $\mathcal{LIA}$ or a combination of both
  - $\mathcal{T} \stackrel{\text{def}}{=} \bigcup_i \mathcal{T}_i$, possibly empty
  - $\mathcal{LA}$ and $\mathcal{T}_i$ Nelson-Oppen theories (i.e. signature-disjoint infinite-domain theories)
- a $\mathcal{LA}$ variable [term] "*cost*" occurring in φ
- (optionally) two constant numbers lb (lower bound) and ub (upper bound) s.t. lb $\leq$ *cost* $<$ ub (lb, ub may be $\mp\infty$)

Optimization Modulo Theories with $\mathcal{LA}$ costs (OMT($\mathcal{LA} \cup \mathcal{T}$))

Find a model for φ whose value of *cost* is minimum.

- maximization dual

We first restrict to the case $\mathcal{LA} = \mathcal{LRA}$ and $\bigcup_i \mathcal{T}_i = \{\}$ (OMT($\mathcal{LRA}$)).

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Solving OMT($\mathcal{LRA}$) [65, 66]

General idea

Combine standard SMT and LP minimization techniques.

Offline Schema

- Minimizer: based on the Simplex $\mathcal{LRA}$-solver by [37]
  - Handles strict inequalities
- Search Strategies:
  - Linear-Search strategy
  - Mixed Linear/Binary strategy

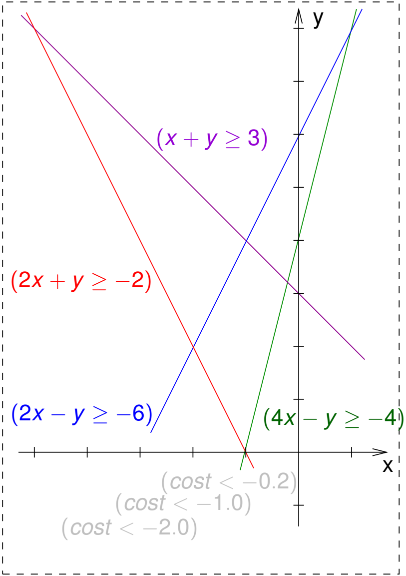
A toy example (linear search)

[w. pure-literal filt. $\Longrightarrow$ partial assignments]

- OMT($\mathcal{LRA}$) problem:

$$
\begin{array}{rll}
\varphi & \stackrel{\text{def}}{=} & (\neg A_1 \vee (2x + y \geq -2)) \\
& \wedge & (\;\; A_1 \vee (x + y \geq 3)) \\
& \wedge & (\neg A_2 \vee (4x - y \geq -4)) \\
& \wedge & (\;\; A_2 \vee (2x - y \geq -6)) \\
& \wedge & (cost < -0.2) \\
& \wedge & (cost < -1.0) \\
& \wedge & (cost < -2.0) \\
cost & \stackrel{\text{def}}{=} & x
\end{array}
$$

- $\mu = \left\{ \begin{array}{l} A_1, \neg A_1, \quad A_2, \neg A_2, \\ (4x - y \geq -4), \\ (x + y \geq 3), \\ (2x + y \geq -2), \\ (2x - y \geq -6) \\ (cost < -0.2) \\ (cost < -1.0) \\ (cost < -2.0) \end{array} \right\}$



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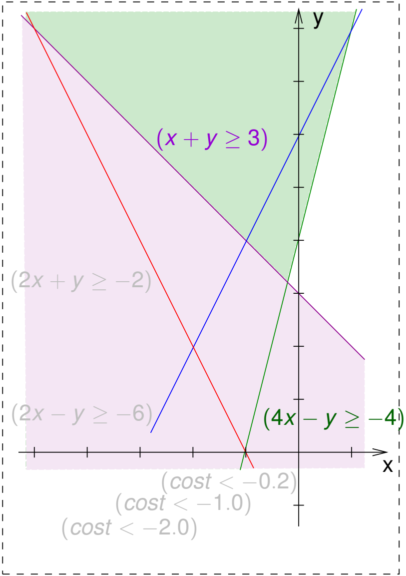
$\Longrightarrow$ SAT, $min = -0.2$



A toy example (linear search)

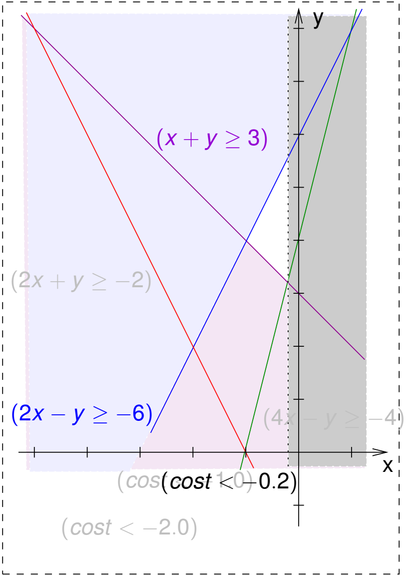
[w. pure-literal filt. $\Longrightarrow$ partial assignments]

- OMT($\mathcal{LRA}$) problem:

$$\begin{array}{rcl} \varphi & \stackrel{\text{def}}{=} & (\neg A_1 \vee (2x + y \geq -2)) \\ & \wedge & (\ A_1 \vee (x + y \geq 3)) \\ & \wedge & (\neg A_2 \vee (4x - y \geq -4)) \\ & \wedge & (\ A_2 \vee (2x - y \geq -6)) \\ & \wedge & (cost < -0.2) \\ & \wedge & (cost < -1.0) \\ & \wedge & (cost < -2.0) \\ cost & \stackrel{\text{def}}{=} & x \end{array}$$

- $\mu = \left\{ \begin{array}{l} A_1, \neg A_1, \ \ A_2, \neg A_2, \\ (4x - y \geq -4), \\ (x + y \geq 3), \\ (2x + y \geq -2), \\ (2x - y \geq -6) \\ (cost < -0.2) \\ (cost < -1.0) \\ (cost < -2.0) \end{array} \right\}$

$\Longrightarrow$ SAT, $min = -1.0$



A toy example (linear search)

[w. pure-literal filt. $\Longrightarrow$ partial assignments]

- OMT($\mathcal{LRA}$) problem:

$$\begin{array}{rl} \varphi \stackrel{\text{def}}{=} & (\neg A_1 \vee (2x + y \geq -2)) \\ \wedge & (\;\; A_1 \vee (x + y \geq 3)) \\ \wedge & (\neg A_2 \vee (4x - y \geq -4)) \\ \wedge & (\;\; A_2 \vee (2x - y \geq -6)) \\ \wedge & (cost < -0.2) \\ \wedge & (cost < -1.0) \\ \wedge & (cost < -2.0) \\ cost \stackrel{\text{def}}{=} & x \end{array}$$

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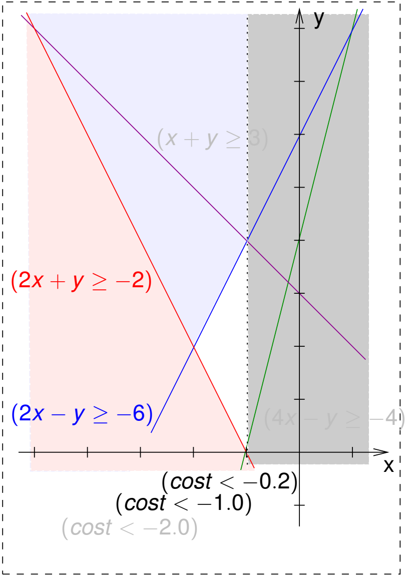
$\Longrightarrow$ SAT, $min = -2.0$



A toy example (linear search)

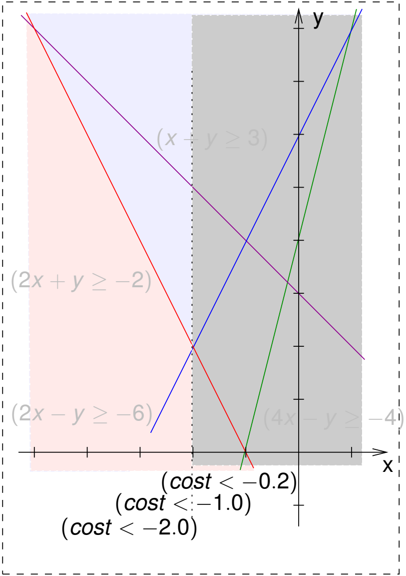
[w. pure-literal filt. $\Longrightarrow$ partial assignments]

- OMT($\mathcal{LRA}$) problem:

$$\begin{array}{rll} \varphi & \stackrel{\text{def}}{=} & (\neg A_1 \vee (2x + y \geq -2)) \\ & \wedge & (\ \ A_1 \vee (x + y \geq 3)) \\ & \wedge & (\neg A_2 \vee (4x - y \geq -4)) \\ & \wedge & (\ \ A_2 \vee (2x - y \geq -6)) \\ & \wedge & (cost < -0.2) \\ & \wedge & (cost < -1.0) \\ & \wedge & (cost < -2.0) \\ cost & \stackrel{\text{def}}{=} & x \end{array}$$

- $\mu = \left\{ \begin{array}{l} A_1, \neg A_1, \quad A_2, \neg A_2, \\ (4x - y \geq -4), \\ (x + y \geq 3), \\ (2x + y \geq -2), \\ (2x - y \geq -6) \\ (cost < -0.2) \\ (cost < -1.0) \\ (cost < -2.0) \end{array} \right\}$

$\Longrightarrow$ UNSAT, $min = -2.0$



Offline Schema: Mixed Linear/Binary-Search Strategy

Input: $\langle \varphi, \mathit{cost}, \text{lb}, \text{ub} \rangle$ // lb can be $-\infty$, ub can be $+\infty$

$\text{l} \leftarrow \text{lb}; \text{u} \leftarrow \text{ub}; \mathcal{M} \leftarrow \emptyset; \varphi \leftarrow \varphi \cup \{\neg(\mathit{cost} < \text{lb}), (\mathit{cost} < \text{ub})\};$

while $(\text{l} < \text{u})$ **do**

Offline Schema: Mixed Linear/Binary-Search Strategy

Input: $\langle \varphi, cost, \mathrm{lb}, \mathrm{ub} \rangle$ // lb can be $-\infty$, ub can be $+\infty$

$\mathrm{l} \leftarrow \mathrm{lb}$; $\mathrm{u} \leftarrow \mathrm{ub}$; $\mathcal{M} \leftarrow \emptyset$; $\varphi \leftarrow \varphi \cup \{\neg(cost < \mathrm{lb}), (cost < \mathrm{ub})\}$;

while $(\mathrm{l} < \mathrm{u})$ **do**

  if (BinSearchMode()) **then** // Binary-search Mode

  else // Linear-search Mode

Offline Schema: Mixed Linear/Binary-Search Strategy

Input: $\langle \varphi, cost, \text{lb}, \text{ub} \rangle$ // lb can be $-\infty$, ub can be $+\infty$

$\text{l} \leftarrow \text{lb}$; $\text{u} \leftarrow \text{ub}$; $\mathcal{M} \leftarrow \emptyset$; $\varphi \leftarrow \varphi \cup \{\neg(cost < \text{lb}), (cost < \text{ub})\}$;

while $(\text{l} < \text{u})$ **do**

  if (BinSearchMode()) **then** // Binary-search Mode

  else // Linear-search Mode

    $\langle \text{res}, \mu \rangle \leftarrow$ SMT.IncrementalSolve(φ);

Offline Schema: Mixed Linear/Binary-Search Strategy

Input: $\langle \varphi, cost, \text{lb}, \text{ub} \rangle$ // lb can be $-\infty$, ub can be $+\infty$

$\text{l} \leftarrow \text{lb}$; $\text{u} \leftarrow \text{ub}$; $\mathcal{M} \leftarrow \emptyset$; $\varphi \leftarrow \varphi \cup \{\neg(cost < \text{lb}), (cost < \text{ub})\}$;

while $(\text{l} < \text{u})$ **do**

  if (BinSearchMode()) **then** // Binary-search Mode

  else // Linear-search Mode

    $\langle \text{res}, \mu \rangle \leftarrow$ SMT.IncrementalSolve(φ);

  if (res = SAT) **then**

    $\langle \mathcal{M}, \text{u} \rangle \leftarrow \mathcal{LRA}$-Solver.Minimize($cost, \mu$);

    $\varphi \leftarrow \varphi \cup \{(cost < \text{u})\}$;

  else {res = UNSAT}

Offline Schema: Mixed Linear/Binary-Search Strategy

Input: $\langle \varphi, cost, \text{lb}, \text{ub} \rangle$ // lb can be $-\infty$, ub can be $+\infty$

$\text{l} \leftarrow \text{lb}$; $\text{u} \leftarrow \text{ub}$; $\mathcal{M} \leftarrow \emptyset$; $\varphi \leftarrow \varphi \cup \{\neg(cost < \text{lb}), (cost < \text{ub})\}$;

while ($\text{l} < \text{u}$) **do**

  if (BinSearchMode()) **then** // Binary-search Mode

  else // Linear-search Mode

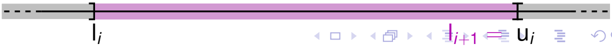
    $\langle \text{res}, \mu \rangle \leftarrow$ SMT.IncrementalSolve(φ);

  if (res = SAT) **then**

  else {res = UNSAT}

      $\text{l} \leftarrow \text{u}$;

return $\langle \mathcal{M}, \text{u} \rangle$



Offline Schema: Mixed Linear/Binary-Search Strategy

Input: $\langle \varphi, \mathit{cost}, \mathrm{lb}, \mathrm{ub} \rangle$ // lb can be $-\infty$, ub can be $+\infty$

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while ($\mathrm{l} < \mathrm{u}$) **do**

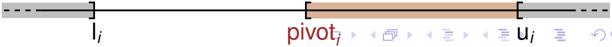
  if (BinSearchMode()) **then** // Binary-search Mode

    $\mathrm{pivot} \leftarrow \mathrm{ComputePivot}(\mathrm{l}, \mathrm{u})$;

    $\varphi \leftarrow \varphi \cup \{(\mathit{cost} < \mathrm{pivot})\}$;

    $\langle \mathrm{res}, \mu \rangle \leftarrow \mathrm{SMT.IncrementalSolve}(\varphi)$;

  else // Linear-search Mode



Offline Schema: Mixed Linear/Binary-Search Strategy

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while $(\text{l} < \text{u})$ **do**

  if (BinSearchMode()) **then** // Binary-search Mode

    pivot $\leftarrow$ ComputePivot(l, u);

    $\varphi \leftarrow \varphi \cup \{(cost < \text{pivot})\}$;

    $\langle \text{res}, \mu \rangle \leftarrow$ SMT.IncrementalSolve(φ);

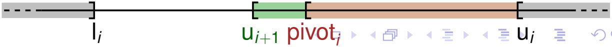
  else // Linear-search Mode

  if (res = SAT) **then**

    $\langle \mathcal{M}, \text{u} \rangle \leftarrow \mathcal{LRA}$-Solver.Minimize($cost, \mu$);

    $\varphi \leftarrow \varphi \cup \{(cost < \text{u})\}$;

  else {res = UNSAT}



Offline Schema: Mixed Linear/Binary-Search Strategy

Input: $\langle \varphi, \mathit{cost}, \text{lb}, \text{ub} \rangle$ // lb can be $-\infty$, ub can be $+\infty$

$\text{l} \leftarrow \text{lb}$; $\text{u} \leftarrow \text{ub}$; $\mathcal{M} \leftarrow \emptyset$; $\varphi \leftarrow \varphi \cup \{\neg(\mathit{cost} < \text{lb}), (\mathit{cost} < \text{ub})\}$;

while $(\text{l} < \text{u})$ **do**

  if (BinSearchMode()) **then** // Binary-search Mode

    pivot $\leftarrow$ ComputePivot(l, u);

    $\varphi \leftarrow \varphi \cup \{(\mathit{cost} < \text{pivot})\}$;

    $\langle \text{res}, \mu \rangle \leftarrow$ SMT.IncrementalSolve(φ);

  else // Linear-search Mode

  if (res = SAT) **then**

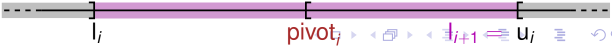
  else {res = UNSAT}

    if $((\mathit{cost} < \text{pivot}) \notin$ SMT.ExtractUnsatCore(φ)) **then**

      $\text{l} \leftarrow \text{u}$;

    else

return $\langle \mathcal{M}, \text{u} \rangle$



Offline Schema: Mixed Linear/Binary-Search Strategy

Input: $\langle \varphi, \mathit{cost}, \mathrm{lb}, \mathrm{ub} \rangle$ // lb can be $-\infty$, ub can be $+\infty$

$\mathrm{l} \leftarrow \mathrm{lb}$; $\mathrm{u} \leftarrow \mathrm{ub}$; $\mathcal{M} \leftarrow \emptyset$; $\varphi \leftarrow \varphi \cup \{\neg(\mathit{cost} < \mathrm{lb}), (\mathit{cost} < \mathrm{ub})\}$;

while $(\mathrm{l} < \mathrm{u})$ **do**

  if (BinSearchMode()) **then** // Binary-search Mode

    $\mathrm{pivot} \leftarrow \mathrm{ComputePivot(l, u)}$;

    $\varphi \leftarrow \varphi \cup \{(\mathit{cost} < \mathrm{pivot})\}$;

    $\langle \mathrm{res}, \mu \rangle \leftarrow$ SMT.IncrementalSolve(φ);

  else // Linear-search Mode

  if $(\mathrm{res} = \textsc{sat})$ **then**

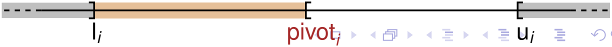
  else $\{\mathrm{res} = \textsc{unsat}\}$

    if $((\mathit{cost} < \mathrm{pivot}) \notin$ SMT.ExtractUnsatCore(φ)) **then**

    else

      $\mathrm{l} \leftarrow \mathrm{pivot}$;

      $\varphi \leftarrow (\varphi \setminus \{(\mathit{cost} < \mathrm{pivot})\} \cup \{\neg(\mathit{cost} < \mathrm{pivot})\}$;



OMT with Lexicographic Combination of Objectives [12]

The problem

Find one optimal model $\mathcal{M}$ minimizing $\underline{c} \stackrel{\text{def}}{=} cost_1, cost_2, ..., cost_k$ lexicographically.

Solution

- Intuition:

  $\{minimize\ cost_1\}$

  $when$ UNSAT

  $\{substitute\ unit\ clause\ (cost_1 < min_1)\ with\ (cost_1 = min_1)\}$

  $\{minimize\ cost_2\}$

  ...
- improvement:
  - each time UNSAT is found, add $\bigwedge_i (cost_i \leq \mathcal{M}_i(cost_i))$ to φ

Optimization problems encoded into OMT($\mathcal{LA} \cup \mathcal{T}$) I

SMT with Pseudo-Boolean Constraints & Weighted MaxSMT

$$OMT + PB: \quad \sum_j w_j \cdot A_j, \;\; w_i > 0 \;\; //(\sum_j ite(A_j, w_j, 0))$$

$$\Downarrow$$

$$\sum_j x_j, \;\; x_j \; fresh$$

$$\text{s.t.} \quad \ldots \wedge \bigwedge_j (A_j \rightarrow (x_j = w_j)) \wedge (\neg A_j \rightarrow (x_j = 0))$$

$$\wedge (x_j \geq 0) \wedge (x_j \leq w_j)$$

$$MaxSMT: \quad \langle \varphi_h, \bigwedge_j \psi_j \rangle \quad s.t. \; \psi_j \; soft, \;\; w_j = weight(\psi_j), \;\; w_i > 0$$

$$\Downarrow$$

$$minimize \; \sum_j x_j, \;\; x_j, A_j \; fresh$$

$$\varphi_h \wedge \bigwedge_j (A_j \vee \psi_j) \wedge \bigwedge_j (\neg A_j \vee (x_j = w_j)) \wedge (A_j \vee (x_j = 0))$$

$$\wedge (x_j \geq 0) \wedge (x_j \leq w_j)$$

Remark: range constraints “$(x_j \geq 0) \wedge (x_j \leq w_j)$”

$$
\begin{array}{ll}
OMT + PB: & \sum_j w_j \cdot A_j,\ \ w_i > 0 \ \ //(\sum_j ite(A_j, w_j, 0)) \\
& \Downarrow \\
& \sum_j x_j,\ \ x_j\ fresh \\
\text{s.t.} & ... \wedge \bigwedge_j (A_j \rightarrow (x_j = w_j)) \wedge (\neg A_j \rightarrow (x_j = 0)) \\
& \wedge (x_j \geq 0) \wedge (x_j \leq w_j)
\end{array}
$$

Range constraints “$(x_j \geq 0) \wedge (x_j \leq w_j)$” logically redundant, but essential for efficiency:

- Without range constraints, the SMT solver can detect the violation of a bound only after all A_i's are assigned :
  Ex: $w_1 = 4,\ w_2 = 7,\ \sum_{i=1} x_i < 10,\ A_1 = A_2 = \top,\ A_i = * \ \forall i > 2.$
- With range constraints, the SMT solver detects the violation as soon as the assigned A_i's violate a bound
  $\Longrightarrow$ drastic pruning of the search
- same for weighted MaxSMT

Remark: range constraints "$(x_j \geq 0) \wedge (x_j \leq w_j)$"

$$OMT + PB: \quad \sum_j w_j \cdot A_j,\ w_i > 0 \quad //(\sum_j ite(A_j, w_j, 0))$$

$$\Downarrow$$

$$\sum_j x_j,\ x_j \ fresh$$

$$\text{s.t.} \quad ... \wedge \bigwedge_j (A_j \rightarrow (x_j = w_j)) \wedge (\neg A_j \rightarrow (x_j = 0))$$
$$\wedge (x_j \geq 0) \wedge (x_j \leq w_j)$$

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Remark: range constraints "$(x_j \geq 0) \wedge (x_j \leq w_j)$"

$$OMT + PB: \quad \sum_j w_j \cdot A_j,\ w_i > 0 \quad //(\sum_j ite(A_j, w_j, 0))$$

$$\Downarrow$$

$$\sum_j x_j,\ x_j\ fresh$$

$$\text{s.t.} \quad ... \wedge \bigwedge_j (A_j \rightarrow (x_j = w_j)) \wedge (\neg A_j \rightarrow (x_j = 0))$$
$$\wedge (x_j \geq 0) \wedge (x_j \leq w_j)$$

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  Ex: $w_1 = 4$, $w_2 = 7$, $\sum_{i=1} x_i < 10$, $A_1 = A_2 = \top$, $A_i = * \ \forall i > 2$.
- With range constraints, the SMT solver detects the violation as soon as the assigned A_i's violate a bound
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Remark: range constraints “$(x_j \geq 0) \wedge (x_j \leq w_j)$”

$$OMT + PB: \quad \sum_j w_j \cdot A_j,\ w_i > 0 \quad //(\sum_j ite(A_j, w_j, 0))$$

$$\Downarrow$$

$$\sum_j x_j,\ x_j\ fresh$$

$$\text{s.t.} \quad \ldots \wedge \bigwedge_j (A_j \rightarrow (x_j = w_j)) \wedge (\neg A_j \rightarrow (x_j = 0))$$

$$\wedge (x_j \geq 0) \wedge (x_j \leq w_j)$$

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- Without range constraints, the SMT solver can detect the violation of a bound only after all A_i’s are assigned :
  Ex: $w_1 = 4$, $w_2 = 7$, $\sum_{i=1} x_i < 10$, $A_1 = A_2 = \top$, $A_i = * \ \forall i > 2$.
- With range constraints, the SMT solver detects the violation as soon as the assigned A_i’s violate a bound
  $\Longrightarrow$ drastic pruning of the search
- same for weighted MaxSMT

Optimization problems encoded into OMT($\mathcal{LA} \cup \mathcal{T}$) II

OMT with Min-Max [Max-Min] optimization

Given $\langle \varphi, \{cost_1, ..., cost_k\}\rangle$, find a solution which minimizes the maximum value among $\{cost_1, ..., cost_k\}$. (Max-Min dual.)

- Frequent in some applications (e.g. [66, 71])

$\Longrightarrow$ encode into OMT($\mathcal{LA} \cup \mathcal{T}$) problem $\{\varphi \wedge \bigwedge_i (cost_i \leq cost), cost\}$ s.t. *cost* fresh.

OMT with linear combinations of costs

Given $\langle \varphi, \{cost_1, ..., cost_k\}\rangle$ and a set of weights $\{w_1, ..., w_k\}$, find a solution which minimizes $\sum_i w_i \cdot cost_i$.

$\Longrightarrow$ encode into OMT($\mathcal{LA} \cup \mathcal{T}$) problem $\{\varphi \wedge (cost = \sum_i w_i \cdot cost_i), cost\}$ s.t. *cost* fresh.

These objectives can be composed with other OMT($\mathcal{LA}$) objectives.

Links I

- survey papers:
  - Roberto Sebastiani: "Lazy Satisfiability Modulo Theories". Journal on Satisfiability, Boolean Modeling and Computation, JSAT. Vol. 3, 2007. Pag 141–224, ©IOS Press.
  - Clark Barrett, Roberto Sebastiani, Sanjit Seshia, Cesare Tinelli "Satisfiability Modulo Theories". Part II, Chapter 26, The Handbook of Satisfiability. 2009. ©IOS press.
  - Leonardo de Moura and Nikolaj Bjørner. "Satisfiability modulo theories: introduction and applications". Communications of the ACM, 54 (9), 2011. ©ACM press.
- web links:
  - The SMT library SMT-LIB: `http://goedel.cs.uiowa.edu/smtlib/`
  - The SMT Competition SMT-COMP: `http://www.smtcomp.org/`
  - The SAT/SMT Schools `http://satassociation.org/sat-smt-school.html`

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