Undecidable two-dimensional products of modal logics with diagonal constant

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Product frames and products of modal logics

For $\mathfrak{F}_1 = \langle W_1, R_1 \rangle$ and $\mathfrak{F}_2 = \langle W_2, R_2 \rangle$,

the **product frame** is $\mathfrak{F}_1 \times \mathfrak{F}_2 = \langle W_1 \times W_2, R_h, R_v \rangle$, where

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$x R_1 x'$ and $y = y'$

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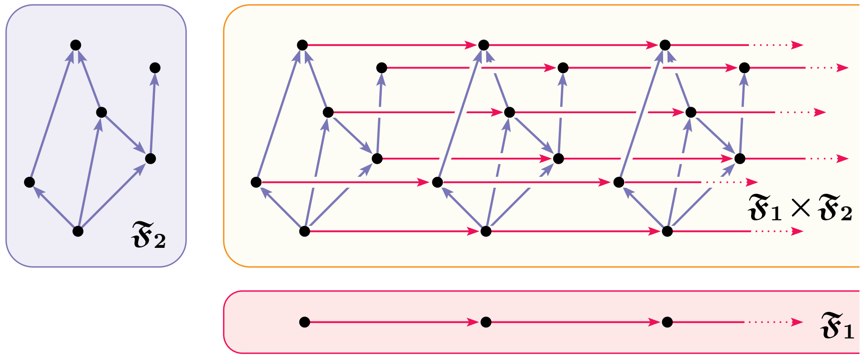

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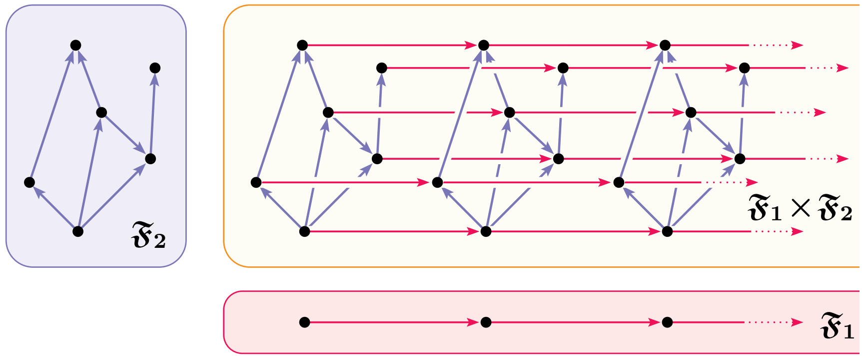

The **product** of two <u>Kripke complete</u> unimodal logics L_1, L_2 is the **bimodal logic**

$$\boxed{L_1 \times L_2 \quad = \quad \textbf{Logic\_of}\ \{\mathfrak{F}_1 \times \mathfrak{F}_2 \mid \mathfrak{F}_1 \in \textbf{Fr}\, L_1,\ \mathfrak{F}_2 \in \textbf{Fr}\, L_2\}}$$

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in the language with the modal operators $\Box_1, \Box_2, \Diamond_1, \Diamond_2$

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- one-variable fragment of modal and intuitionistic predicate logics

Finite model properties

Product logics are determined by classes of product frames, but there are **non-product frames** for product logics **!**

- A product logic $L_1 \times L_2$ has the **product fmp** if any $\varphi \notin L_1 \times L_2$ fails in a finite **product** frame for $L_1 \times L_2$.
- A product logic $L_1 \times L_2$ has the **(abstract) fmp** if any $\varphi \notin L_1 \times L_2$ fails in a finite (not necessarily product) frame for $L_1 \times L_2$.

$$\boxed{\text{product fmp} \quad \Longrightarrow \quad \text{fmp} \qquad \not\Longleftarrow}$$

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  - If both component logics are determined by
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  - But it can be even Π^1_1**-complete** (like $\mathbf{K4} \times \textbf{Logic\_of}\ (\mathbb{N})$, $\mathbf{S4} \times \mathbf{GL.3}$)

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  - $\langle W_1 \times W_2, R_h, R_v, \rangle = \mathfrak{F}_1 \times \mathfrak{F}_2$
  - $D = \{\langle u, u \rangle \mid u \in W_1 \cap W_2\}$

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- The **δ-product** of two Kripke complete unimodal logics L_1, L_2 is

  the **3-modal logic**

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$\uparrow$

in the language with $\Box_1, \Box_2, \Diamond_1, \Diamond_2, \delta$

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- $\ldots$?

Main results I: undecidability

Let

$$\mathfrak{F} = \langle \omega + 1, R \rangle$$

where

$$R = \{\langle \omega, n \rangle \mid \text{for all } n < \omega\} \cup \{\langle n+1, n \rangle \mid \text{for all } n < \omega\}$$

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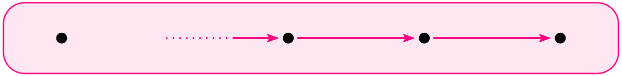

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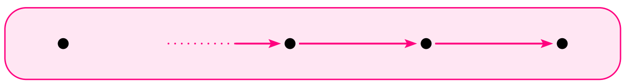

If $\mathcal{C}$ is any class of δ-product frames such that $\mathfrak{F} \times^{\delta} \mathfrak{F} \in \mathcal{C}$

then **Logic_of** $(\mathcal{C})$ is **undecidable**

Most surprising example: $\boxed{\mathbf{K} \times^{\delta} \mathbf{K}}$

Main results II: no fmp

Let

- $\mathfrak{G} = \langle \omega + 1, S \rangle$ with $S = \{\langle \omega, n \rangle \mid n < \omega\}$ (infinite fan)
- $S^{\textit{refl}} =$ reflexive closure of S
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then **Logic_of** $(\mathcal{C})$ **does not** have the (abstract) fmp

Examples: $\mathbf{K} \times^{\delta} \mathbf{K}, \quad \mathbf{K} \times^{\delta} \mathbf{K4}, \quad \mathbf{K} \times^{\delta} \mathbf{S4}, \quad \mathbf{K} \times^{\delta} \mathbf{S5}$

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- $\mathbf{K} \times \mathbf{K}$ is also decidable and has the **product fmp**
- All known **undecidable** product-like logics have some kind of 'forward going' **universal modality**:

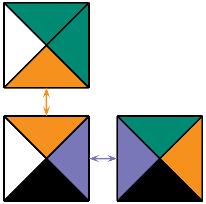
$\boxed{\mathbf{K4} \times \mathbf{K4}, \quad \mathbf{K} \times \mathbf{K} \text{ with universal modality}}$

Undecidability proof (in new paper)

By reduction of the $\mathbb{N} \times \mathbb{N}$ tiling problem:

Given a finite set T of tile types $t = (\mathit{left}(t), \mathit{right}(t), \mathit{up}(t), \mathit{down}(t))$

decide whether there exists $\tau : \mathbb{N} \times \mathbb{N} \longrightarrow T$ such that, for all $i, j \in \mathbb{N}$,



$$\mathit{up}(\tau(i,j)) = \mathit{down}(\tau(i,j+1))$$

and

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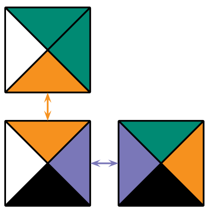

$$\mathit{up}(\boldsymbol{\tau}(\boldsymbol{i}, \boldsymbol{j})) = \mathit{down}(\boldsymbol{\tau}(\boldsymbol{i}, \boldsymbol{j} + 1))$$

and

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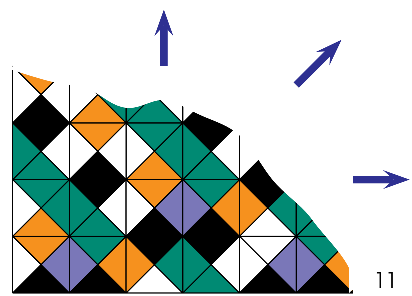
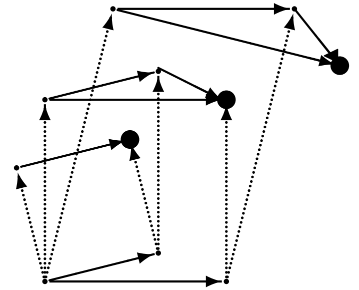


(Berger 1966): The $\mathbb{N} \times \mathbb{N}$ tiling problem is **undecidable**

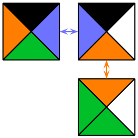
Undecidability proof: the formulas

(γ): **Generating a $\mathbb{N} \times \mathbb{N}$-like grid 'upside down' so that all points are $\Box_1\Box_2$-accessible from the root (like $\mathfrak{F} \times \mathfrak{F}$):**

$$\Diamond_2\Diamond_1\delta$$
$$\Box_2\Diamond_1\Diamond_1\delta$$
$$\Box_1\Diamond_2\delta$$
$$\Box_1\Box_2(\delta \wedge \Diamond_1\top \rightarrow \Diamond_2\top)$$
$$\Box_1\Box_2(\delta \rightarrow \Box_1\Box_2\delta)$$



(ϑ): **encoding tiling rules**



$$\Box_1\Box_2 \bigvee_{t\in T} \Big(t \wedge \bigwedge_{t'\neq t} \neg t'\Big)$$
$$\Box_1\Box_2 \bigwedge_{\textit{right}(t')\neq\textit{left}(t)} \big(t \rightarrow \Box_1\neg t'\big)$$
$$\Box_1\Box_2 \bigwedge_{\textit{up}(t')\neq\textit{down}(t)} \big(t \rightarrow \Box_2\neg t'\big)$$

Claim. $(\vartheta \wedge \gamma)$ is **satisfied** in a δ-product frame in $\mathcal{C}$ iff T **tiles** $\mathbb{N}\times\mathbb{N}$

Future work

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- Explore connections with other **undecidable extensions of products**

  say, with the universal modality ($=$ global consequence relation)