Maximum-likelihood (ML) estimation

Setting

- A training data $\mathcal{D} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ of i.i.d. examples for the target class y is available
- We assume the parameter vector $\boldsymbol{\theta}$ has a fixed but unknown value
- We estimate such value maximizing its **likelihood** with respect to the training data:

$$\boldsymbol{\theta}^* = \text{argmax}_{\boldsymbol{\theta}} p(\mathcal{D}|\boldsymbol{\theta}) = \text{argmax}_{\boldsymbol{\theta}} \prod_{j=1}^{n} p(\mathbf{x}_j|\boldsymbol{\theta})$$

- The joint probability over $\mathcal{D}$ decomposes into a product as examples are i.i.d (thus independent of each other given the distribution)

Maximum-likelihood estimation

Maximizing log-likelihood

- It is usually simpler to maximize the logarithm of the likelihood (monotonic):

$$\boldsymbol{\theta}^* = \text{argmax}_{\boldsymbol{\theta}} \ln p(\mathcal{D}|\boldsymbol{\theta}) = \text{argmax}_{\boldsymbol{\theta}} \sum_{j=1}^{n} \ln p(\mathbf{x}_j|\boldsymbol{\theta})$$

- Necessary conditions for the maximum can be obtained zeroing the gradient wrt to $\boldsymbol{\theta}$:

$$\nabla_{\boldsymbol{\theta}} \sum_{j=1}^{n} \ln p(\mathbf{x}_j|\boldsymbol{\theta}) = \mathbf{0}$$

- Points zeroing the gradient can be local or global maxima depending on the form of the distribution

Bayesian estimation

setting

- Assumes parameters $\boldsymbol{\theta}_i$ are *random variables* with some known *prior* distribution
- Predictions for new examples are obtained *integrating* over all possible values for the parameters:

$$p(\mathbf{x}|y_i, \mathcal{D}_i) = \int_{\boldsymbol{\theta}_i} p(\mathbf{x}, \boldsymbol{\theta}_i|y_i, \mathcal{D}_i) d\boldsymbol{\theta}_i$$

- probability of $\mathbf{x}$ given each class y_i is independent of the other classes y_j, for simplicity we can again write:

$$p(\mathbf{x}|y_i, \mathcal{D}_i) \to p(\mathbf{x}|\mathcal{D}) = \int_{\boldsymbol{\theta}} p(\mathbf{x}, \boldsymbol{\theta}|\mathcal{D}) d\boldsymbol{\theta}$$

- where $\mathcal{D}$ is a dataset for a certain class y and $\boldsymbol{\theta}$ the parameters of the distribution

Bayesian estimation

setting

$$p(\mathbf{x}|\mathcal{D}) = \int_{\boldsymbol{\theta}} p(\mathbf{x}, \boldsymbol{\theta}|\mathcal{D}) d\boldsymbol{\theta} = \int p(\mathbf{x}|\boldsymbol{\theta}) p(\boldsymbol{\theta}|\mathcal{D}) d\boldsymbol{\theta}$$

- $p(\mathbf{x}|\boldsymbol{\theta})$ can be easily computed (we have both form and parameters of distribution, e.g. Gaussian)
- need to estimate the parameter posterior density given the training set:

$$p(\boldsymbol{\theta}|\mathcal{D}) = \frac{p(\mathcal{D}|\boldsymbol{\theta}) p(\boldsymbol{\theta})}{p(\mathcal{D})}$$

Bayesian estimation

denominator

$$p(\boldsymbol{\theta}|\mathcal{D}) = \frac{p(\mathcal{D}|\boldsymbol{\theta}) p(\boldsymbol{\theta})}{p(\mathcal{D})}$$

- $p(\mathcal{D})$ is a constant independent of $\boldsymbol{\theta}$ (i.e. it will no influence final Bayesian decision)
- if final *probability* (not only decision) is needed we can compute:

$$p(\mathcal{D}) = \int_{\boldsymbol{\theta}} p(\mathcal{D}|\boldsymbol{\theta}) p(\boldsymbol{\theta}) d\boldsymbol{\theta}$$

Sufficient statistics
Definition

- Any function on a set of samples $\mathcal{D}$ is a *statistic*
- A statistic $\mathbf{s} = \phi(\mathcal{D})$ is *sufficient* for some parameters $\boldsymbol{\theta}$ if:

$$P(\mathcal{D}|\mathbf{s}, \boldsymbol{\theta}) = P(\mathcal{D}|\mathbf{s})$$

- If $\boldsymbol{\theta}$ is a random variable, a sufficient statistic contains all relevant information $\mathcal{D}$ has for estimating it:

$$p(\boldsymbol{\theta}|\mathcal{D}, \mathbf{s}) = \frac{p(\mathcal{D}|\boldsymbol{\theta}, \mathbf{s}) p(\boldsymbol{\theta}|\mathbf{s})}{p(\mathcal{D}|\mathbf{s})} = p(\boldsymbol{\theta}|\mathbf{s})$$

Use

- A sufficient statistic allows to compress a sample $\mathcal{D}$ into (possibly few) values
- Sample mean and covariance are sufficient statistics for true mean and covariance of the Gaussian distribution

Conjugate priors

Definition

- Given a likelihood function $p(x|\theta)$
- Given a prior distribution $p(\theta)$
- $p(\theta)$ is a *conjugate prior* for $p(x|\theta)$ if the posterior distribution $p(\theta|x)$ is in the same family as the prior $p(\theta)$

Examples

Likelihood	Parameters	Conjugate prior
Binomial	p (probability)	Beta
Multinomial	$\boldsymbol{p}$ (probability vector)	Dirichlet
Normal	μ (mean)	Normal
Multivariate normal	$\boldsymbol{\mu}_i$ (mean vector)	Normal

Probability distributions

Bernoulli distribution

- Two possible values (outcomes): 1 (success), 0 (failure).
- Parameters: p probability of success.
- Probability mass function:

$$P(x;p) = \begin{cases} p & if\ x = 1 \\ 1-p & if\ x = 0 \end{cases}$$

- $\mathrm{E}[x] = p$
- $\mathrm{Var}[x] = p(1-p)$

Example: tossing a coin

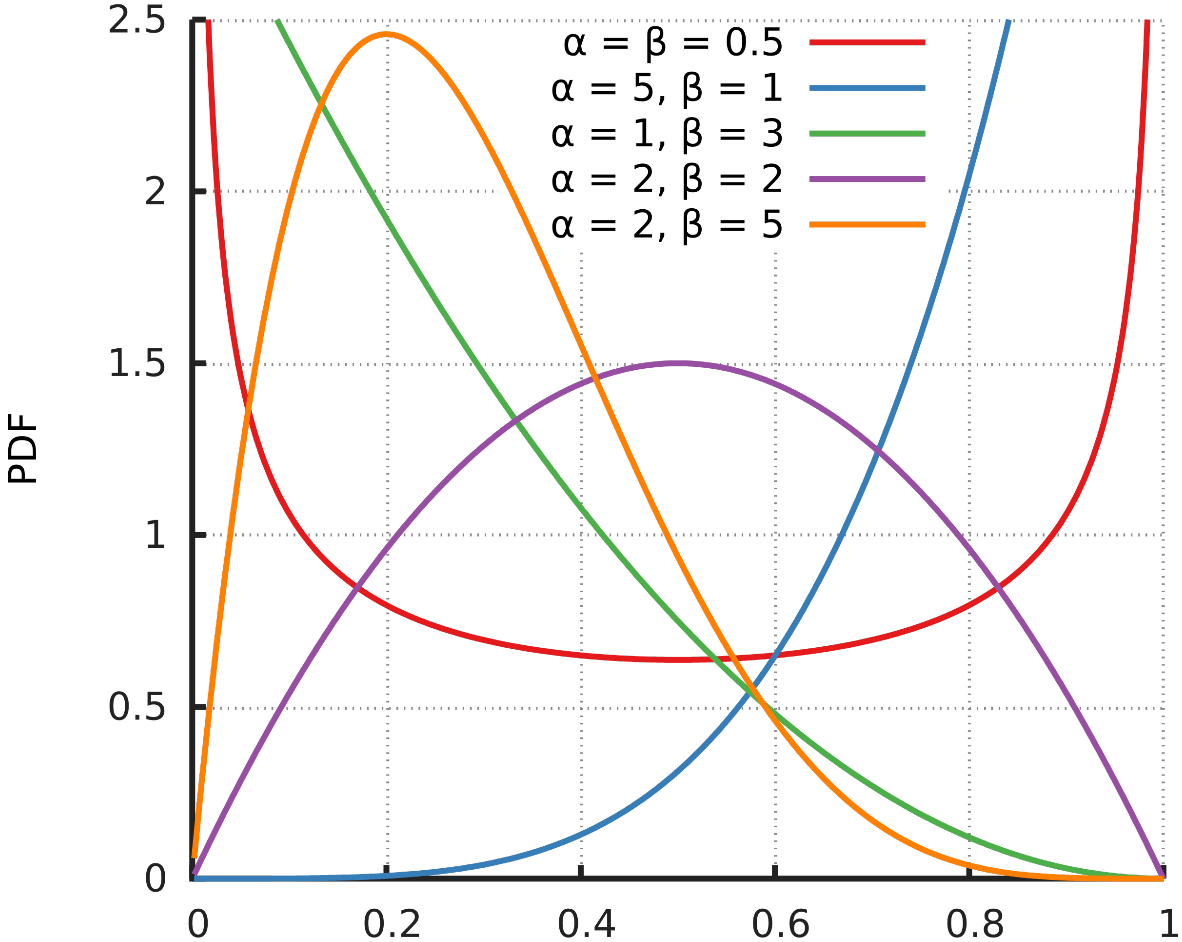
- Head (success) and tail (failure) possible outcomes
- p is probability of head

Probability distributions

Beta distribution

- Defined in the interval $[0, 1]$
- Parameters: α, β
- Probability density function:

$$p(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$



- $\mathrm{E}[x] = \frac{\alpha}{\alpha+\beta}$ $\quad\quad\quad \Gamma(x+1) = x\Gamma(x), \Gamma(1) = 1$
- $\mathrm{Var}[x] = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Note

It models the posterior distribution of parameter p of a binomial distribution after observing $\alpha - 1$ independent events with probability p and $\beta - 1$ with probability $1 - p$.

Bernoulli distribution

Setting

- Boolean event: $x = 1$ for success, $x = 0$ for failure (e.g. tossing a coin)
- Parameters: θ = probability of success (e.g. head)
- Probability mass function

$$P(x|\theta) = \theta^x (1-\theta)^{1-x}$$

- Beta conjugate prior:

$$P(\theta) = \frac{\Gamma(\alpha)}{\Gamma(\alpha_h)\Gamma(\alpha_t)} \theta^{\alpha_h - 1} (1-\theta)^{\alpha_t - 1}$$

Bernoulli distribution
Maximum likelihood estimation: example

- Dataset $\mathcal{D} = \{H, H, T, T, T, H, H\}$ of N realizations (e.g. head/tail coin toss results)
- Likelihood function:

$$p(\mathcal{D}|\theta) = \theta \cdot \theta \cdot (1-\theta) \cdot (1-\theta) \cdot (1-\theta) \cdot \theta \cdot \theta = \theta^h (1-\theta)^t$$

- Maximum likelihood parameter:

$$\frac{\partial}{\partial\theta} \ln p(\mathcal{D}|\theta) = 0 \qquad \Rightarrow \qquad \begin{aligned} \frac{\partial}{\partial\theta} h \ln\theta + t \ln(1-\theta) &= 0 \\ h\frac{1}{\theta} - t\frac{1}{1-\theta} &= 0 \\ h(1-\theta) &= t\theta \\ \theta &= \frac{h}{h+t} \end{aligned}$$

- h, t are the sufficient statistics

Bernoulli distribution
Bayesian estimation: example

- Parameter posterior is proportional to:

$$P(\theta|\mathcal{D}) \propto P(\mathcal{D}|\theta)P(\theta) \propto \theta^h(1-\theta)^t \theta^{\alpha_h - 1}(1-\theta)^{\alpha_t - 1}$$

- i.e. the posterior has a beta distribution with parameters $h + \alpha_h, t + \alpha_t$:

$$P(\theta|\mathcal{D}) \propto \theta^{h+\alpha_h-1}(1-\theta)^{t+\alpha_t-1}$$

- The prediction for a new event is the expected value of the posterior beta:

$$\begin{aligned} P(x|\mathcal{D}) &= \int P(x|\theta)P(\theta|\mathcal{D})d\theta = \int \theta P(\theta|\mathcal{D})d\theta \\ &= \mathrm{E}_{P(\theta|\mathcal{D})}[\theta] = \frac{h+\alpha_h}{h+t+\alpha_h+\alpha_t} \end{aligned}$$

Bernoulli distribution

Interpreting priors

- Our prior knowledge is encoded as a number $\alpha = \alpha_h + \alpha_t$ of imaginary experiments
- we assume α_h times we observed heads
- α is called *equivalent sample size*
- $\alpha \rightarrow 0$ reduces estimation to the classical ML approach (frequentist)

Probability distributions

Multinomial distribution (one sample)

- Models the probability of a certain outcome for an event with m possible outcomes.
- Parameters: $p_1, \ldots, p_m$ probability of each outcome
- Probability mass function:

$$P(x_1, \ldots, x_m; p_1, \ldots, p_m) = \prod_{i=1}^{m} p_i^{x_i}$$

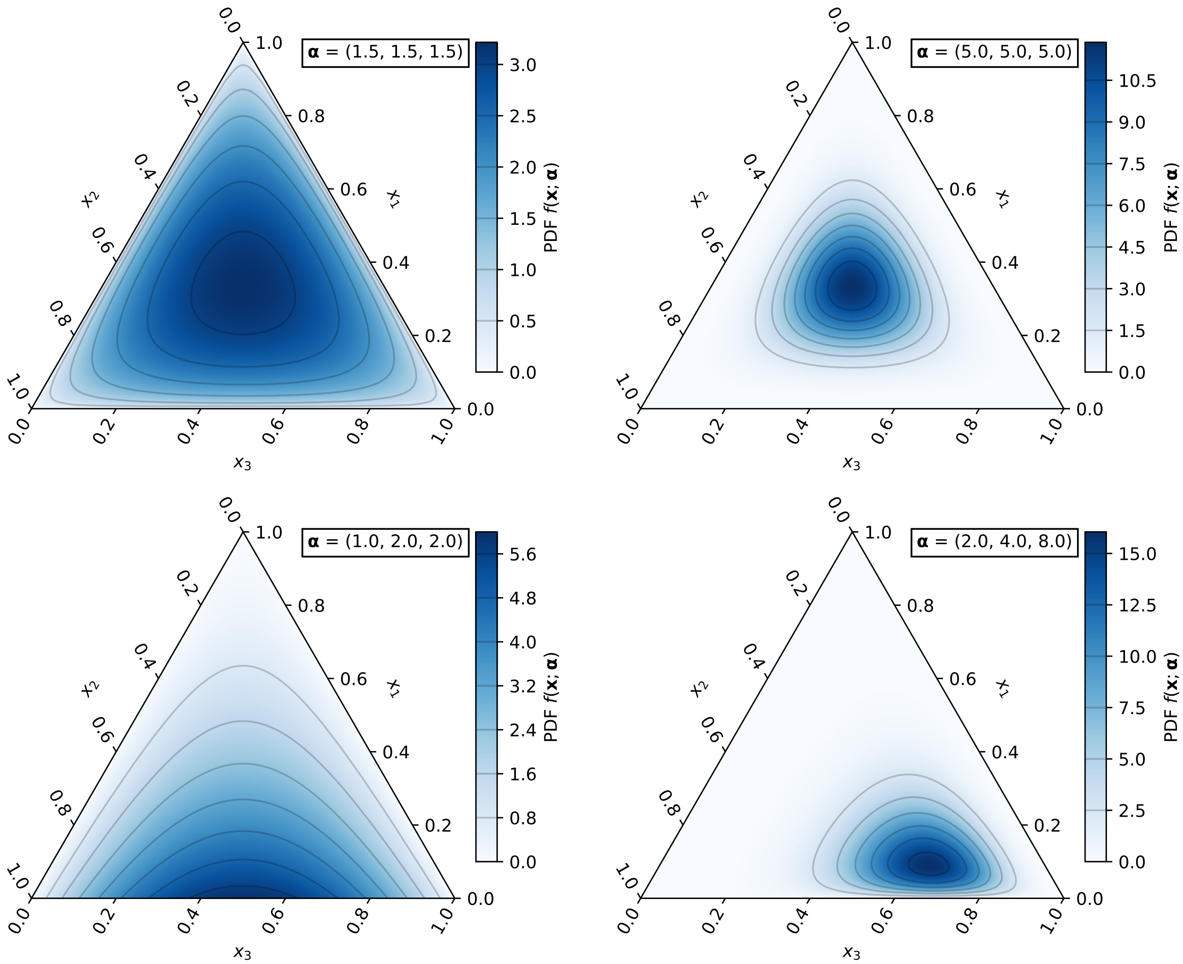
- where $x_1, \ldots, x_m$ is a vector with $x_i = 1$ for outcome i and $x_j = 0$ for all $j \neq i$.
- $\mathrm{E}[x_i] = p_i$
- $\mathrm{Var}[x_i] = p_i(1 - p_i)$
- $\mathrm{Cov}[x_i, x_j] = -p_i p_j$

Probability distributions

Dirichlet distribution

- Defined: $\boldsymbol{x} \in [0, 1]^m, \sum_{i=1}^{m} x_i = 1$
- Parameters: $\boldsymbol{\alpha} = \alpha_1, \ldots, \alpha_m$
- Probability density function:

$$p(x_1, \ldots, x_m; \boldsymbol{\alpha}) = \frac{\Gamma(\alpha_0)}{\prod_{i=1}^{m} \Gamma(\alpha_i)} \prod_{i=1}^{m} x_i^{\alpha_i - 1}$$



- $\mathrm{E}[x_i] = \frac{\alpha_i}{\alpha_0}$ where $\alpha_0 = \sum_{j=1}^{m} \alpha_j$
- $\mathrm{Var}[x_i] = \frac{\alpha_i(\alpha_0 - \alpha_i)}{\alpha_0^2(\alpha_0+1)}$ $\mathrm{Cov}[x_i, x_j] = \frac{-\alpha_i \alpha_j}{\alpha_0^2(\alpha_0+1)}$

Note

It models the posterior distribution of parameters $\boldsymbol{p}$ of a multinomial distribution after observing $\alpha_i - 1$ times each mutually exclusive event

Multinomial distribution

Setting

- Categorical event with r states $x \in \{x^1, \ldots, x^r\}$ (e.g. tossing a six-faced dice)
- One-hot encoding $\mathbf{z}(x) = [z_1(x), \ldots, z_r(x)]$ with $z_k(x) = 1$ if $x = x^k$, 0 otherwise.
- Parameters: $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_r]$ probability of each state
- Probability mass function

$$P(x|\boldsymbol{\theta}) = \prod_{k=1}^{r} \theta_k^{z_k(x)}$$

- Dirichlet conjugate prior:

$$P(\boldsymbol{\theta}|\psi) = P(\boldsymbol{\theta}|\alpha_1, \ldots, \alpha_r) = \frac{\Gamma(\alpha)}{\prod_{k=1}^{r} \Gamma(\alpha_k)} \prod_{k=1}^{r} \theta_k^{\alpha_k - 1}$$

Multinomial distribution
Maximum likelihood estimation: example

- Dataset $\mathcal{D}$ of N realizations (e.g. results of tossing a dice)
- Likelihood function:

$$p(\mathcal{D}|\boldsymbol{\theta}) = \prod_{j=1}^{N} \prod_{k=1}^{r} \theta_k^{z_k(x_j)} = \prod_{k=1}^{r} \theta_k^{N_k}$$

- Maximum likelihood parameter:

$$\theta_k = \frac{N_k}{N}$$

- $N_1, \ldots, N_r$ are the sufficient statistics

Multinomial distribution
Bayesian estimation: example

- Parameter posterior is proportional to:

$$P(\boldsymbol{\theta}|\mathcal{D}) \propto P(\mathcal{D}|\boldsymbol{\theta}) P(\theta) \propto \prod_{k=1}^{r} \theta_k^{N_k} \theta_k^{\alpha_k - 1}$$

- i.e. the posterior has a Dirichlet distribution with parameters $N_k + \alpha_k, k = 1, \ldots, r$:

$$P(\boldsymbol{\theta}|\mathcal{D}) \propto \prod_{k=1}^{r} \theta_k^{N_k + \alpha_k - 1}$$

- The prediction for a new event is the expected value of the posterior Dirichlet:

$$P(x_k|\mathcal{D}) = \int \theta_k P(\boldsymbol{\theta}|\mathcal{D}) d\boldsymbol{\theta} = \mathrm{E}_{P(\boldsymbol{\theta}|\mathcal{D})}[\theta_k] = \frac{N_k + \alpha_k}{N + \alpha}$$

APPENDIX

Appendix
Additional reference material

Maximum-likelihood estimation

Multivariate Gaussian case: proof (mean)

- The gradient wrt to the mean is:

$$\nabla_{\boldsymbol{\mu}} \sum_{j=1}^{n} -\frac{1}{2}(\mathbf{x}_j - \boldsymbol{\mu})^t \Sigma^{-1} (\mathbf{x}_j - \boldsymbol{\mu}) - \frac{1}{2} \ln (2\pi)^d |\Sigma| = \\ \sum_{j=1}^{n} \Sigma^{-1} (\mathbf{x}_j - \boldsymbol{\mu})$$

Note

$$\begin{aligned} \frac{\partial}{\partial \mathbf{x}} \mathbf{x}^T A \mathbf{x} &= A^T \mathbf{x} + A \mathbf{x} \\ &= 2A\mathbf{x} \quad \text{for symmetric } A \end{aligned}$$

Maximum-likelihood estimation

Multivariate Gaussian case: proof (mean)

- Setting the gradient to zero gives:

$$\sum_{j=1}^{n} \Sigma^{-1} (\mathbf{x}_j - \boldsymbol{\mu}) = \mathbf{0}$$

$$\sum_{j=1}^{n} (\mathbf{x}_j - \boldsymbol{\mu}) = \Sigma\, \mathbf{0} = \mathbf{0}$$

$$\sum_{j=1}^{n} \mathbf{x}_j = \sum_{j=1}^{n} \boldsymbol{\mu} = n\, \boldsymbol{\mu}$$

$$\boldsymbol{\mu} = \frac{1}{n} \sum_{j=1}^{n} \mathbf{x}_j$$

Maximum-likelihood estimation

Multivariate Gaussian case: proof (covariance)

- The gradient wrt to the covariance is:

$$\frac{\partial}{\partial \Sigma} \sum_{j=1}^{n} -\frac{1}{2}(\mathbf{x}_j - \boldsymbol{\mu})^t \Sigma^{-1} (\mathbf{x}_j - \boldsymbol{\mu}) - \frac{1}{2} \ln (2\pi)^d |\Sigma| = \\ -\frac{1}{2} \left(\sum_{j=1}^{n} \frac{\partial}{\partial \Sigma} (\mathbf{x}_j - \boldsymbol{\mu})^t \Sigma^{-1} (\mathbf{x}_j - \boldsymbol{\mu}) + \sum_{j=1}^{n} \frac{\partial}{\partial \Sigma} \ln (2\pi)^d |\Sigma| \right)$$

Maximum-likelihood estimation

Multivariate Gaussian case: proof (covariance)

$$\frac{\partial}{\partial\Sigma}(\mathbf{x}_j-\boldsymbol{\mu})^t\Sigma^{-1}(\mathbf{x}_j-\boldsymbol{\mu}) =$$
$$(\mathbf{x}_j-\boldsymbol{\mu})(\mathbf{x}_j-\boldsymbol{\mu})^t\frac{\partial}{\partial\Sigma}\Sigma^{-1} =$$
$$-(\mathbf{x}_j-\boldsymbol{\mu})(\mathbf{x}_j-\boldsymbol{\mu})^t\Sigma^{-2}$$

Note

Use matrix derivative rule:

$$\frac{\partial}{\partial B}tr(ABC) = CA$$

Where $A = (\mathbf{x}_j-\boldsymbol{\mu})^t$, $B = \Sigma^{-1}$, $C = (\mathbf{x}_j-\boldsymbol{\mu})$ and $tr(ABC) = ABC$ as ABC is a scalar.

Maximum-likelihood estimation

Multivariate Gaussian case: proof (covariance)

$$\frac{\partial}{\partial\Sigma}\ln(2\pi)^d|\Sigma| = \frac{1}{(2\pi)^d}|\Sigma|^{-1}\frac{\partial}{\partial\Sigma}(2\pi)^d|\Sigma| =$$
$$\frac{1}{(2\pi)^d}|\Sigma|^{-1}(2\pi)^d\frac{\partial}{\partial\Sigma}|\Sigma| = |\Sigma|^{-1}|\Sigma|\Sigma^{-1} = \Sigma^{-1}$$

Note

Use matrix derivative rule:

$$\frac{\partial}{\partial A}|A| = |A|A^{-1}$$

Maximum-likelihood estimation

Multivariate Gaussian case: proof (covariance)

- Combining and putting equal to zero:

$$-\frac{1}{2}\left(\sum_{j=1}^{n}\overbrace{-(\mathbf{x}_j-\boldsymbol{\mu})(\mathbf{x}_j-\boldsymbol{\mu})^t\Sigma^{-2}}^{\frac{\partial}{\partial\Sigma}(\mathbf{x}_j-\boldsymbol{\mu})^t\Sigma^{-1}(\mathbf{x}_j-\boldsymbol{\mu})}+\sum_{j=1}^{n}\overbrace{\Sigma^{-1}}^{\frac{\partial}{\partial\Sigma}\ln(2\pi)^d|\Sigma|}\right) = 0$$

Maximum-likelihood estimation

Multivariate Gaussian case: proof (covariance)

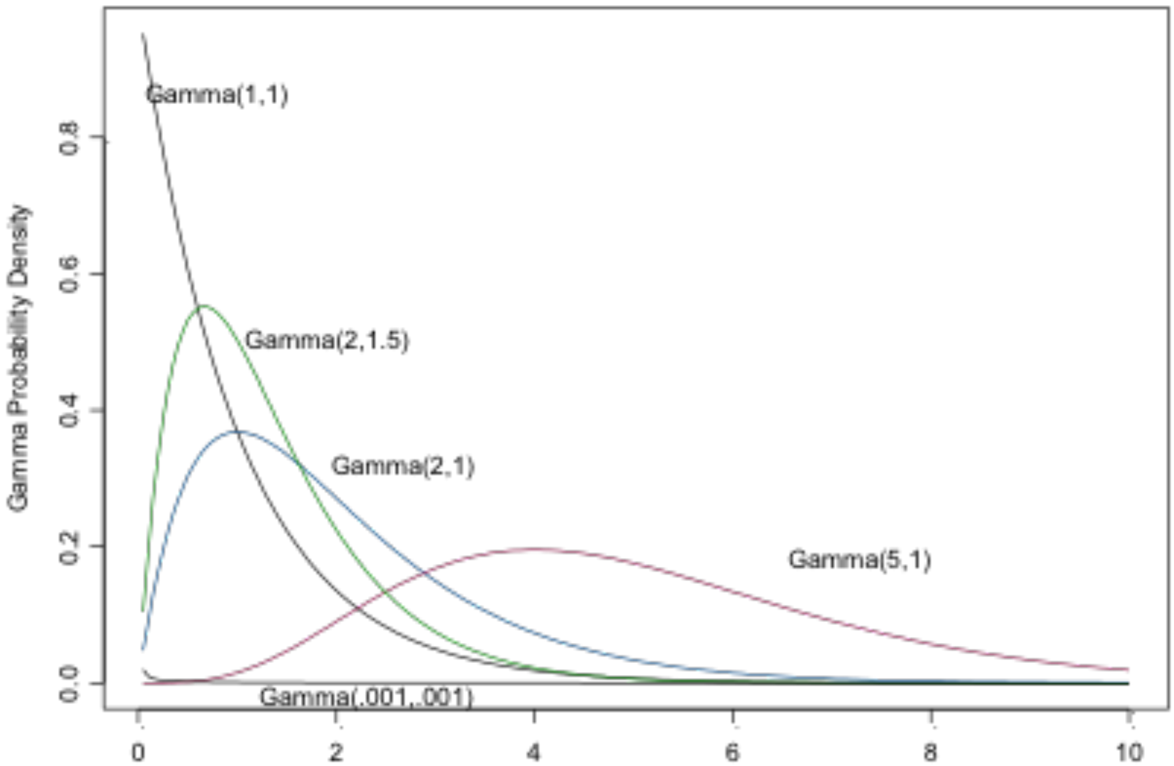
$$\sum_{j=1}^{n} \Sigma^{-1} = \sum_{j=1}^{n} (\mathbf{x}_j - \boldsymbol{\mu})(\mathbf{x}_j - \boldsymbol{\mu})^t \Sigma^{-2}$$
$$\Sigma^2 \sum_{j=1}^{n} \Sigma^{-1} = \Sigma^2 \sum_{j=1}^{n} (\mathbf{x}_j - \boldsymbol{\mu})(\mathbf{x}_j - \boldsymbol{\mu})^t \Sigma^{-2}$$
$$\sum_{j=1}^{n} \Sigma = \sum_{j=1}^{n} (\mathbf{x}_j - \boldsymbol{\mu})(\mathbf{x}_j - \boldsymbol{\mu})^t$$
$$n\Sigma = \sum_{j=1}^{n} (\mathbf{x}_j - \boldsymbol{\mu})(\mathbf{x}_j - \boldsymbol{\mu})^t$$
$$\Sigma = \frac{1}{n} \sum_{j=1}^{n} (\mathbf{x}_j - \boldsymbol{\mu})(\mathbf{x}_j - \boldsymbol{\mu})^t$$

Bayesian estimation

Gamma distribution

- Defined in the interval $[0, \infty]$
- Parameters: $\alpha > 0$ (shape) $\beta > 0$ (rate)
- Probability density function:

$$p(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$



- $\mathrm{E}[x] = \frac{\alpha}{\beta}$
- $\mathrm{Var}[x] = \frac{\alpha}{\beta^2}$

Note

Used to model the prior distribution of the *precision* (inverse variance, i.e. $\lambda = 1/\sigma^2$).

Bayesian estimation

Univariate normal case: unknown μ and $\lambda = 1/\sigma^2$

- Examples are drawn from:

$$p(x|\mu, \lambda) \sim N(\mu, 1/\lambda)$$

- The Prior of mean and precision is the NormalGamma distribution:

$$\begin{aligned} p(\mu, \lambda) &= p(\mu|\lambda)p(\lambda) = N(\mu|\mu_0, \frac{1}{\kappa_0 \lambda})Ga(\lambda|\alpha_0, \beta_0) \\ &= NG(\mu, \lambda|\mu_0, \kappa_0, \alpha_0, \beta_0) \end{aligned}$$

Univariate normal case: unknown μ and $\lambda = 1/\sigma^2$
***a posteriori* parameter density**

$$
\begin{aligned}
p(\mu,\lambda|\mathcal{D}) = \frac{1}{\mathcal{D}}\prod_{j=1}^{n}\overbrace{\frac{\lambda^{1/2}}{\sqrt{2\pi}}\exp\left[-\frac{\lambda}{2}(x_j-\mu)^2\right]}^{p(x_j|\mu,\lambda)}\overbrace{\frac{(\kappa_0\lambda)^{1/2}}{\sqrt{2\pi}}\exp\left[-\frac{\kappa_0\lambda}{2}(\mu-\mu_0)^2\right]}^{p(\mu|\lambda)} \\
\underbrace{\frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)}\lambda^{\alpha_0-1}\exp(-\beta_0\lambda)}_{p(\lambda)} \\
\propto \lambda^{\alpha_0+n/2-1}\exp(-\beta_0\lambda)\lambda^{1/2}\exp -\frac{\lambda}{2}\left[\sum_{j=1}^{n}(x_j-\mu)^2-\kappa_0(\mu-\mu_0)^2\right]
\end{aligned}
$$

***a posteriori* parameter density is still NormalGamma**

$$p(\mu,\lambda|\mathcal{D}) = NG(\mu,\lambda|\mu_n,\kappa_n,\alpha_n,\beta_n)$$

Univariate normal case: unknown μ and $\lambda = 1/\sigma^2$
***a posteriori* parameter density is still NormalGamma**

$$p(\mu,\lambda|\mathcal{D}) = NG(\mu,\lambda|\mu_n,\kappa_n,\alpha_n,\beta_n)$$

where

$$
\begin{aligned}
\mu_n &= \frac{\kappa_0\mu_0+n\hat{\mu}_n}{k_0+n} \\
\kappa_n &= k_0+n \\
\alpha_n &= \alpha_0+n/2 \\
\beta_n &= \beta_0+\frac{1}{2}\sum_{j=1}^{n}(x_j-\hat{\mu}_n)^2+\frac{\kappa_0 n(\hat{\mu}_n-\mu_0)^2}{2(\kappa_0+n)}
\end{aligned}
$$

Univariate normal case: unknown μ and $\lambda = 1/\sigma^2$
Interpreting the posterior

- Posterior mean is weighted average of prior (μ_0) and sample (μ_n) means, weighted by κ_0 and n respectively

$$\mu_n = \frac{\kappa_0\mu_0+n\hat{\mu}_n}{k_0+n}$$

- Posterior κ_n is increased by the number of samples n

$$\kappa_n = k_0+n$$

- Posterior α_n is increased by half the number of samples n

$$\alpha_n = \alpha_0+n/2$$

Univariate normal case: unknown μ and $\lambda = 1/\sigma^2$
Interpreting the posterior

- Posterior sum of squares (β_n) is sum of prior sum of squares (β_0) and sample sum of squares $\frac{1}{2}\sum_{j=1}^{n}(x_j - \hat{\mu}_n)^2$ and a term due to the discrepancy between the sample mean and the prior mean.

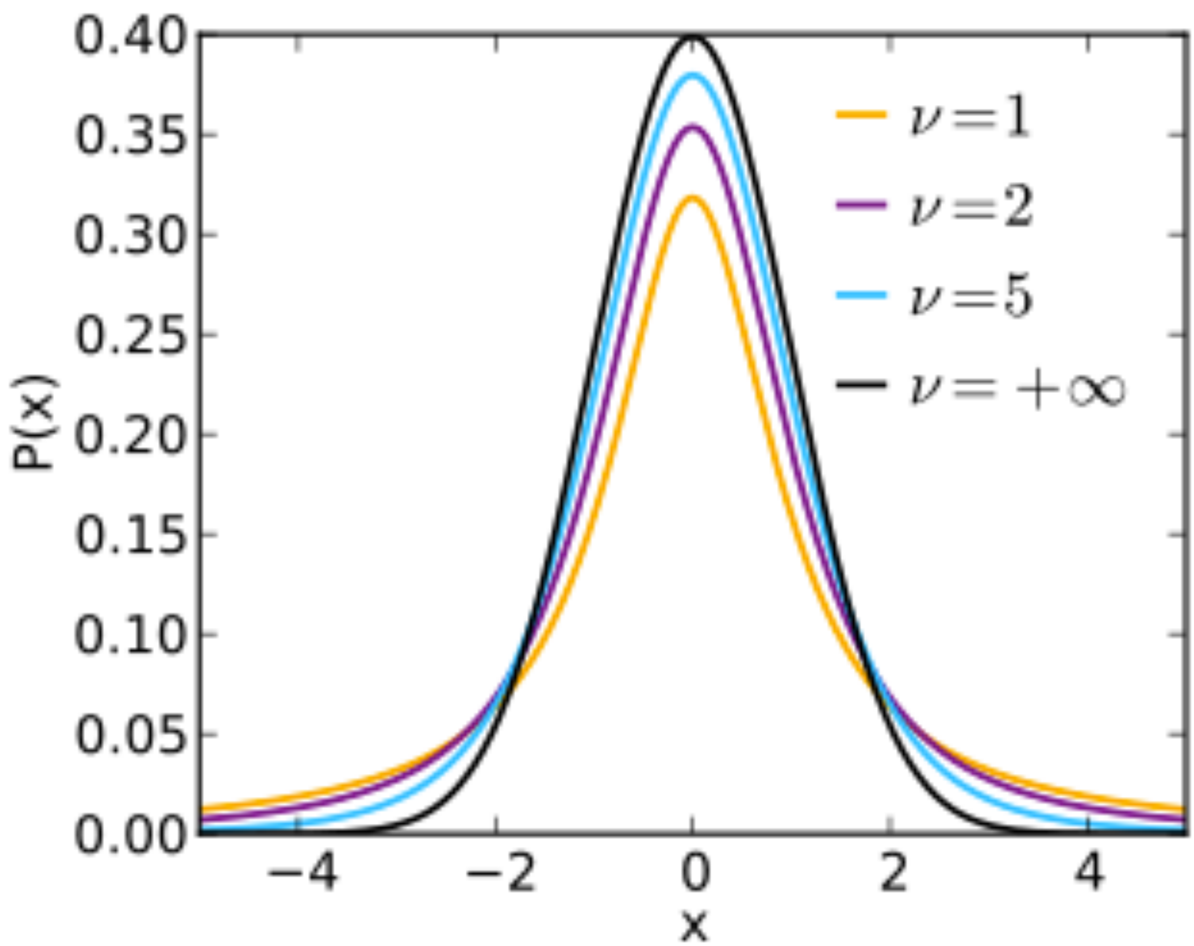
$$\beta_n = \beta_0 + \frac{1}{2}\sum_{j=1}^{n}(x_j - \hat{\mu}_n)^2 + \frac{\kappa_0 n(\hat{\mu}_n - \mu_0)^2}{2(\kappa_0 + n)}$$

Univariate normal case: unknown μ and $\lambda = 1/\sigma^2$

Computing the posterior predictive

$$\begin{aligned} p(x|\mathcal{D}) &= \int_\mu \int_\lambda p(x|\mu,\lambda)p(\mu,\lambda|\mathcal{D})d\mu d\lambda \\ &= \frac{P(x,\mathcal{D})}{P(\mathcal{D})} = t_{2\alpha_n}\left(x|\mu_n, \frac{\beta_n(\kappa_n+1)}{\alpha_n\kappa_n}\right) \end{aligned}$$

- It is a T-distribution with mean μ_n and precision $\frac{\beta_n(\kappa_n+1)}{\alpha_n\kappa_n}$ (proof omitted)



Bayesian estimation

Wishart distribution

- Defined over $d \times d$ positive semi-definite matrix
- Parameters: $\nu > d-1$ (degree of freedom) $T > 0$ ($d \times d$ scale matrix)
- Probability density function:
$$p(X; \nu, T) = \frac{1}{2^{\nu d/2}|T|^{\nu/2}\Gamma_d(\nu/2)}|X|^{\frac{\nu-d-1}{2}} \exp -\frac{1}{2}tr(T^{-1}X)$$
- $\mathrm{E}[X] = \nu T$
- $\mathrm{Var}[X_{ij}] = \nu(T_i i^2 + T_{ii}T_{jj})$

Note

Used to model the prior distribution of the *precision* matrix (inverse covariance matrix, i.e. $\Lambda = \Sigma^{-1}$). T is the prior covariance

Bayesian estimation

Multivariate normal case: unknown $\boldsymbol{\mu}$ and Σ

- Examples are drawn from:
$$p(x|\boldsymbol{\mu}, \Lambda) \sim N(\boldsymbol{\mu}, \Lambda^{-1})$$
- The Prior of mean and precision is the NormalWishart distribution:
$$p(\boldsymbol{\mu}, \Lambda) = p(\boldsymbol{\mu}|\Lambda)p(\Lambda) = N(\boldsymbol{\mu}|\boldsymbol{\mu}_0, (\kappa_0\Lambda)^{-1})Wi(\Lambda|\nu, T)$$

Multivariate normal case: unknown $\boldsymbol{\mu}$ and Σ

***a posteriori* parameter density**

$$p(\boldsymbol{\mu}, \Lambda|\mathcal{D}) = N(\mu|\mu_n(\kappa_n\Lambda)^{-1})Wi(\Lambda|\nu_n, T_n)$$

where

$$\begin{aligned}
\mu_n &= \frac{\kappa_0\mu_0 + n\hat{\mu}_n}{k_0 + n} \\
T_n &= T + \sum_{i=1}^{n}(x_i - \hat{\mu}_n)(x_i - \hat{\mu}_n)^T + \frac{\kappa n}{\kappa + n}(\mu_0 - \hat{\mu}_n)(\mu_0 - \hat{\mu}_n)^T \\
\nu_n &= \nu + n \qquad \kappa_n = \kappa + n
\end{aligned}$$

Computing the posterior predictive

$$p(x|\mathcal{D}) = t_{\nu_n - d + 1}\left(x|\mu_n, \frac{T_n(\kappa_n + 1)}{\kappa_n(\nu_n - d + 1)}\right)$$