Interactive Learning

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Advanced Topics in Machine Learning & Optimization — 2023-24

"Imagine that you are the leader of a colonial expedition from Earth to an extrasolar planet. Luckily, this planet is habitable and has a fair amount of vegetation suitable for feeding your group. Impor- tantly, the most abundant source of food comes from a plant whose fruits are sometimes smooth and round, but sometimes bumpy and irregular."

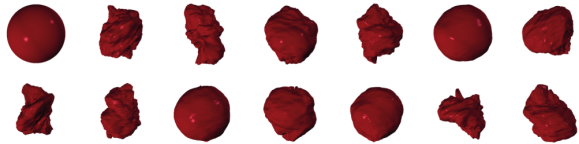

Figure 1.1: Several alien fruits, which vary in shape from round to irregular.

"The physicians assure you that the shape of a fruit is the only feature that seems related to its safety. The problem, though, is that a wide variety of fruit shapes from these plants exist: almost a continuous range from round to irregular. Since the colony has essential uses for both safe and noxious fruits, you want to be able to classify them as accurately as possible. " Credits: (Settles, 2012).

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- In other words, we want to learn a **threshold function**:

$$f_\theta(\mathbf{x}) = \begin{cases} 1 & \text{if } x_3 < \theta \\ -1 & \text{otherwise} \end{cases}$$

where $\mathbf{x}$ are measurements of fruit features and x_3 captures its shape "irregularity".

Idea: use regular **supervised learning**

- Collect a large enough training set $\mathcal{L} = \{(\mathbf{x}, y)\}$, fit threshold classifier f_θ on $\mathcal{L}$
- If maximum % errors is $\epsilon \in (0, 1)$, enough to collect $\approx \frac{1}{\epsilon}$ examples (Shalev-Shwartz and Ben-David, 2014). For instance, if max error is 1%, then we need to collect 100 examples. Considering how simple this problem is, this is a lot!

Figure 1.2: Supervised learning for the alien fruits example. Given a set of $\langle x, y \rangle$ instance-label pairs, we want to choose the threshold θ^* that classifies them most accurately.

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Figure 1.2: Supervised learning for the alien fruits example. Given a set of $\langle x, y \rangle$ instance-label pairs, we want to choose the threshold θ^* that classifies them most accurately.

- We want to find θ as quickly and as economically as possible, by requiring fewer tests.

- Can we do better?

Key features:

- Fruits $\mathbf{x}$ are plentiful and easy to harvest and measure
- Obtaining y incurs a **cost**: person that eats the fruit may get sick

So we definitely want to **minimize the number of needed labels**.

Idea: gather large set of **unlabeled** fruits $U = \{\mathbf{x}_i\}$ and arrange them by roughness.

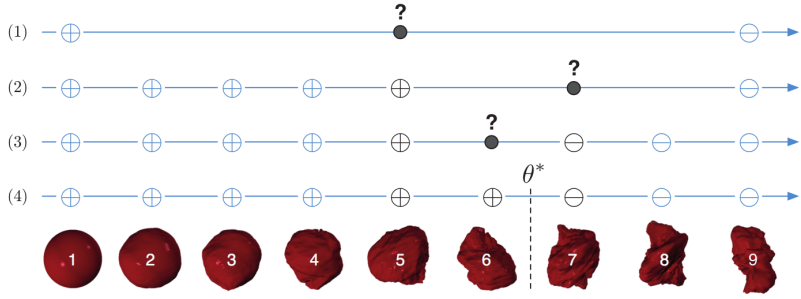

Figure 1.3: A binary search through the set of ordered, untested alien fruits. By only testing this subset of fruits, we can exponentially reduce costs while achieving the same result. The labels shown in light blue can be inferred, and therefore do not need to be tested.

Use **binary search** to find the threshold θ only takes $\approx \log_2 \frac{1}{\epsilon}$ tests! For $\epsilon = 1\%$, this amounts to ≈ 7.

Idea: gather large set of **unlabeled** fruits $U = \{\mathbf{x}_i\}$ and arrange them by roughness, then use **binary search**:

ϵ	$\frac{1}{\epsilon}$	$\log_2 \frac{1}{\epsilon}$
0.1	10	3.321
0.001	1000	9.966
0.00001	100000	16.610

- In this (cleverly designed illustrative) scenario, there is an **exponential** improvement in sample complexity

Active vs Passive

"The key hypothesis is that if the learner is allowed to choose the data from which it learns — to be active, curious, or exploratory, if you will — it can **perform better** *with* **less training**." (Settles, 2012)

Preconditions:

- Collecting unlabelled instances $\mathbf{x}$ is **cheap**
- Obtaining their labels y is **expensive**

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Example: Citizen Science

There are tons of images of celestial bodies (think sky surveys). However, in order to undestand what's in an image (is it a spiral galaxy? is it a gravitational lensing effect?) you have to ask a human expert.

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Example: Recommendation

There are millions of products on online catalogues (think Amazon), but in order to discover what are the tastes of a user, you have to actually convince them to score the items. This information is personalized, so this is the only way to obtain supervision.

Example: Scientific Discovery

- Adam, the "robot scientist" (King et al., 2009)

The Automation of Science

Ross D. King,[1]* Jem Rowland,[1] Stephen G. Oliver,[2] Michael Young,[3] Wayne Aubrey,[1] Emma Byrne,[1] Maria Liakata,[1] Magdalena Markham,[1] Pınar Pir,[2] Larisa N. Soldatova,[1] Andrew Sparkes,[1] Kenneth E. Whelan,[1] Amanda Clare[1]

The basis of science is the hypothetico-deductive method and the recording of experiments in sufficient detail to enable reproducibility. We report the development of Robot Scientist "Adam," which advances the automation of both. Adam has autonomously generated functional genomics hypotheses about the yeast *Saccharomyces cerevisiae* and experimentally tested these hypotheses by using laboratory automation. We have confirmed Adam's conclusions through manual experiments. To describe Adam's research, we have developed an ontology and logical language. The resulting formalization involves over 10,000 different research units in a nested treelike structure, 10 levels deep, that relates the 6.6 million biomass measurements to their logical description. This formalization describes how a machine contributed to scientific knowledge.

- The learner obtains labels by operating an automated testing machine.

Example: Scientific Discovery

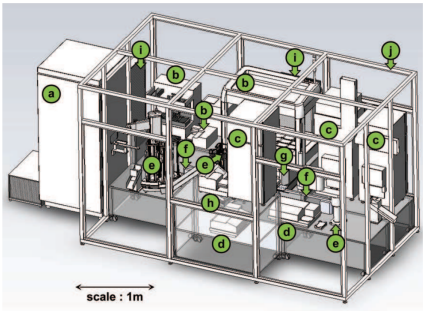

Fig. 1. The Robot Scientist Adam. The advances that distinguish Adam from other complex laboratory systems are the individual design of the experiments to test hypotheses and the utilization of complex internal cycles. Adam's basic operations are selection of specified yeast strains from a library held in a freezer, inoculation of these strains into microtiter plate wells containing rich medium, measurement of growth curves on rich medium, harvesting of a defined quantity of cells from each well, inoculation of these cells into wells containing defined media (minimal synthetic dextrose medium plus up to four added metabolites from a choice of six), and measurement of growth curves on the specified media. To achieve this functionality, Adam has the following components: a, an automated −20°C freezer; b, three liquid handlers (one of which can separately control 96 fluid channels simultaneously; c, three automated +30°C incubators; d, two automated plate readers; e, three robot arms; f, two automated plate slides; g, an automated plate centrifuge; h, an automated plate washer; i, two high-efficiency particulate air filters; and j, a rigid transparent plastic enclosure. There are also two bar code readers, seven cameras, 20 environment sensors, and four personal computers, as well as the software. Adam is capable of designing and initiating over a thousand new strain and defined-growth-medium experiments each day (from a selection of thousands of yeast strains), with each experiment lasting up to 5 days. The design enables measurement of OD_{595nm} for each experiment at least once every 30 min (more often if running at less than full capacity), allowing accurate growth curves to be recorded (typically we take over a hundred measurements a day per well), plus associated metadata. See the supporting online material for pictures and a video of Adam in action.

- Similar strategies used in chemical engineering, material engineering, *etc.*

Notation

A summary of frequently used terms:

- **Instances** $\mathbf{x} \in \mathbb{R}^d$ are *unlabelled* d-dimensional vectors of observations
- **Examples** $z = (\mathbf{x}, y)$ are instances *annotated* by a label $y \in \{0, 1\}$ or $y \in \{1, \ldots, c\}$
- A **classifier** $f : \mathbb{R}^d \to \{0, 1\}$ maps instances to labels (e.g., a neural networks, ...)
- $\mathcal{F} = \{f_\theta\}$ is a **family of classifiers** parameterized by θ (e.g., all neural networks with a specified architecture)

■ The meaning of θ depends on the model class, e.g., for neural nets with a fixed architecture, θ represents their weights; for random forests, θ represents the structure and leaves of all trees.

Assumptions

- We assume the data to be distributed according to a **ground-truth distribution** $p^*(Y, \mathbf{X})$, which combines a distribution over inputs ("how rare is this document/image?" and a distribution over labels given the input ("how likely is this document to be labeled as `funny`?")

$$p^*(Y, \mathbf{X}) \equiv p^*(Y \mid \mathbf{X}) \cdot p^*(\mathbf{X}) \tag{1}$$

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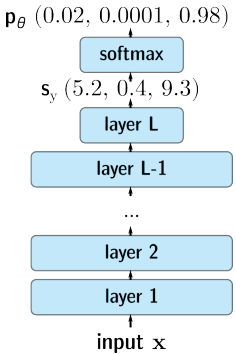
- We focus on learning a **probabilistic classifier**, written as:

$$p_\theta(Y = y \mid \mathbf{X} = \mathbf{x}) \tag{2}$$

We always predict the most likely label, that is:

$$f_\theta(\mathbf{x}) = \underset{y=1,\ldots,c}{\operatorname{argmax}} \; p_\theta(Y = y \mid \mathbf{X} = \mathbf{x}) \tag{3}$$

Possible models are anything from **logistic regression** to **neural nets** with a $\mathrm{softmax}$ activation (illustrated on the right).



Structure of your average feed-forward neural network. Notice how the output consists of per-class probabilities. Here we write the vector $\mathbf{p}$ this using the notation $p_\theta(Y \mid \mathbf{x})$.

Modeling the Annotator

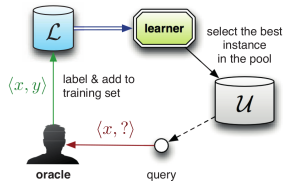
- Annotator modelled as an "**oracle**" that returns the correct label:

$$\text{annot}(\mathbf{x}) := \underset{y \in \{0,1\}}{\text{argmax}} \; p^*(Y = y \mid \mathbf{X} = \mathbf{x}) \tag{4}$$

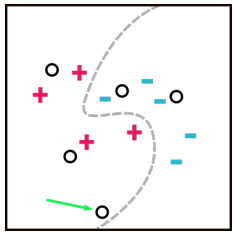
where p^* is the true (but unobserved) label distribution. In other words, we assume the annotator **always answers correctly**, i.e., they are knowledgeable and collaborative.

- Invoking the oracle comes at a **cost**, which is unknown, but usually non-negligible, instance- and class-dependent.

For simplicity, we assume the cost to be **unitary**: all questions cost the same.



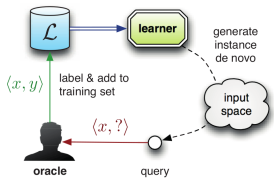
(b) pool-based sampling



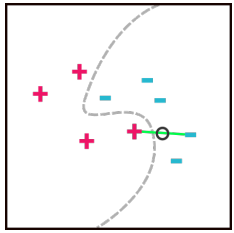
Active Learning (Pool-based). Given:

- a family of classifiers $\mathcal{F}$,
- a set of unlabelled instances $U = \{\mathbf{x}_1, \ldots, \mathbf{x}_m\} \subseteq \mathbb{R}^d$ sampled i.i.d. from $p^*(\mathbf{X})$,
- a (costly) labeling oracle $\mathrm{label} : \mathbb{R}^d \to \{0, 1\}$,

Find a classifier $\widehat{f} \in \mathcal{F}$ that achieves low risk on $p^*(\mathbf{X}, Y)$ while keeping annot. cost low



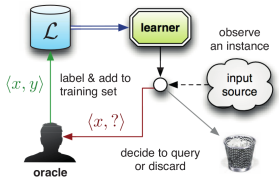
(a) query synthesis



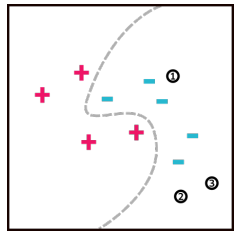
Active Learning (Query Synthesis). Given:

- a family of classifiers $\mathcal{F}$,
- a generator of instances $\mathrm{synthesize}(\text{region}) \mapsto \mathbf{x}$,
- a (costly) labeling oracle $\mathrm{label} : \mathbb{R}^d \to \{0, 1\}$,

Find a classifier $\widehat{f} \in \mathcal{F}$ that achieves low risk on $p^*(\mathbf{X}, Y)$ while keeping annot. cost low



(a) selective sampling

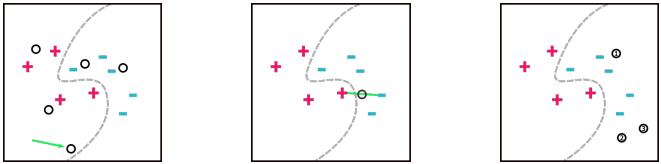


Active Learning (Selective Sampling). Given:

- a family of classifiers $\mathcal{F}$,
- a sequence of unlabelled instances $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots,$
- a (costly) labeling oracle $\mathrm{label} : \mathbb{R}^d \to \{0, 1\}$

Find a classifier $\widehat{f}_t \in \mathcal{F}$ that achieves low risk on future data $\mathbf{x}_{t+1}, \mathbf{x}_{t+2}, \ldots$ while keeping annot. cost low

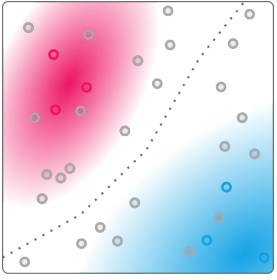
Query Sampling vs. Query Synthesis



- Left to right:
  - **Pool-based**: moderate control over queries, requires memory to store $\mathcal{U}$
  - **Query synthesis**: maximum control over queries, can generate uninterpretable queries (Baum and Lang, 1992), although deep generative models can help somehow (Nguyen et al., 2016).
  - **Selective sampling**: little control over the distribution of queries, often solved under tight memory constraints (online learning)
- We will focus on **pool-based AL**.

Strategies

Quiz Time!



- Out of the many unlabeled points (in gray), which ones would you pick for a human annotator to label?

Template

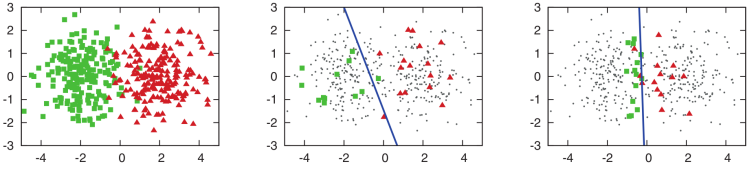
Input: models $\mathcal{F}$, examples $\mathcal{L}$, pool $\mathcal{U}$, query budget $T \geq 1$
Output: selected model $f \in \mathcal{F}$

$f \leftarrow \mathrm{fit}(\mathcal{F}, \mathcal{L})$ ▷ initialize the model
for $t = 1, 2, \ldots, T$ **do** ▷ until the budget is exhausted
  $\mathbf{x} \leftarrow \mathrm{argmax}_{x \in \mathcal{U}} \mathrm{acq}(f, \mathbf{x})$ ▷ select a query instance
  obtain label y of $\mathbf{x}$ from annotator
  $\mathcal{U} \leftarrow \mathcal{U} \setminus \{\mathbf{x}\}$ ▷ remove unlabeled instance from pool
  $\mathcal{L} \leftarrow \mathcal{L} \cup \{(\mathbf{x}, y)\}$ ▷ update training set
  $f \leftarrow \mathrm{fit}(\mathcal{F}, \mathcal{L})$ ▷ update the model
return f

- fit performs training (e.g., trains for a fixed # of epochs)
- acq scores instances based on their "informativeness"
- What instance $\mathbf{x} \in \mathcal{U}$ should be selected so to convey **as much information as possible** to f?

Uncertainty Sampling

What's the point of asking the label of instances on which the classifier is already certain?[1]



(a) a 2D toy data set (b) random sampling (c) uncertainty sampling

- **Left**: two Gaussians (40 points each)
- **Middle**: picking points completely **at random** (ignoring the class label!)
- **Right**: picking points based on **uncertainty**

[1] There is a point to doing so, as we will see later.

Uncertainty Sampling

- **Idea**: pick $\mathbf{x} \in \mathcal{U}$ on which the classifier is **most uncertain**.

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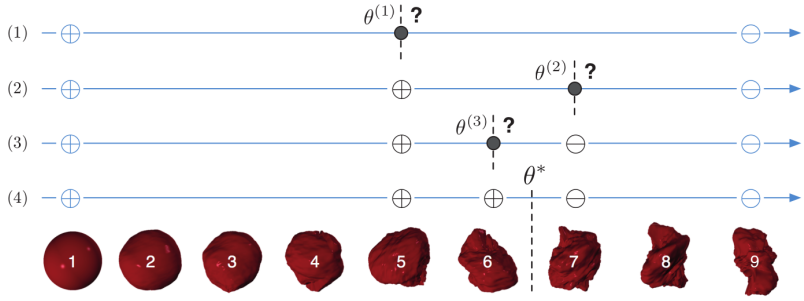


Figure 2.1: The binary search from Figure 1.3, re-interpreted as an uncertainty sampling approach. The best instance to query is deemed to be the one closest to the threshold θ.

Uncertainty Sampling

- **Idea**: pick $\mathbf{x} \in \mathcal{U}$ on which the classifier is **most uncertain**.

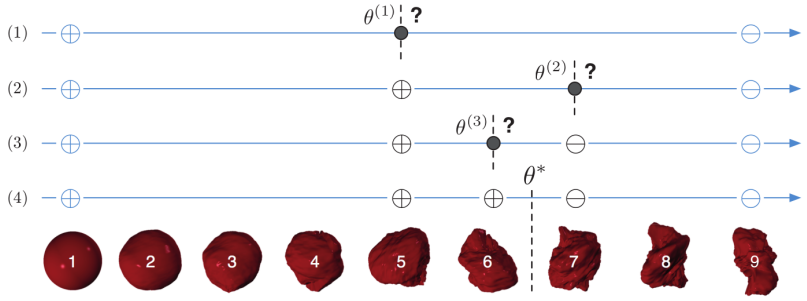


Figure 2.1: The binary search from Figure 1.3, re-interpreted as an uncertainty sampling approach. The best instance to query is deemed to be the one closest to the threshold θ.

- How should uncertainty be defined?

Uncertainty Sampling

- Define uncertainty using the **confidence**, i.e., *distance from certainty*:

$$\mathrm{acq}(\theta, \mathbf{x}) := \; 1 - p_\theta(\hat{y} \mid \mathbf{x}) \tag{5}$$

where $\hat{y}$ is the predicted label:

$$\hat{y} := f_\theta(\mathbf{x}) = \underset{y}{\mathrm{argmax}} \; p_\theta(y \mid \mathbf{x}) \tag{6}$$

Uncertainty Sampling

- Define uncertainty using the **margin**, i.e., difference in (conditional) likelihood:

$$\mathrm{acq}(\theta, \mathbf{x}) := \; p_\theta(\hat{y}' \mid \mathbf{x}) - p_\theta(\hat{y} \mid \mathbf{x}) \tag{7}$$

where $\hat{y}$ is the predicted label and $\hat{y}'$ is the **2nd best** label:

$$\hat{y} = \operatorname*{argmax}_y \; p_\theta(y \mid \mathbf{x}) \tag{8}$$

$$\hat{y}' := \operatorname*{argmax}_{y \neq \hat{y}} \; p_\theta(y \mid \mathbf{x}) \tag{9}$$

Uncertainty Sampling

- Define uncertainty using the Shannon **entropy** of the label:

$$\mathrm{acq}(\theta, \mathbf{x}) := \; H_\theta(Y \mid \mathbf{X} = \mathbf{x}) \tag{10}$$

where H_θ is defined as:

$$H_\theta(Y \mid \mathbf{X} = \mathbf{x}) := -\sum_{y \in [c]} p_\theta(y \mid \mathbf{x}) \log_2 p_\theta(y \mid \mathbf{x}) \tag{11}$$

Remark: conventionally, $0 \times \log_2 0 = 0$.

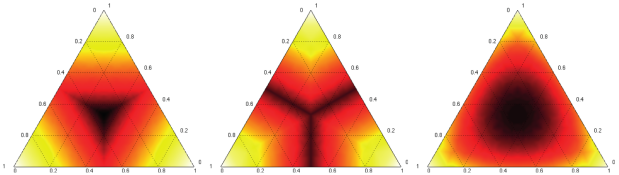
- It achieves a **minimum** on dead certain distributions:

$$p_\theta(Y \mid \mathbf{x}) = (0, 1, 0, \ldots, 0)$$

- and a **maximum** on the uniform distribution:

$$p_\theta(Y \mid \mathbf{x}) = (\frac{1}{c}, \cdots, \frac{1}{c})$$

Confidence vs. Margin vs. Entropy

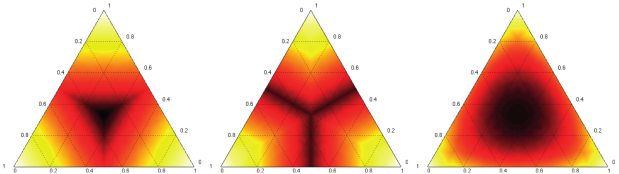


(d) least confident – ternary (e) margin – ternary (f) entropy – ternary

- **Left**: confidence considers prob. of top class only
- **Middle**: margin considers prob. of top & runner up classes
- **Right**: entropy considers prob. of all classes

If $c = 2$, they are equivalent. If $c > 2$, no obvious best choice, it really depends on the task and loss (e.g., cross-entropy vs.accuracy)

Confidence vs. Margin vs. Entropy



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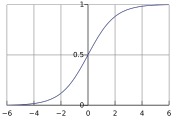
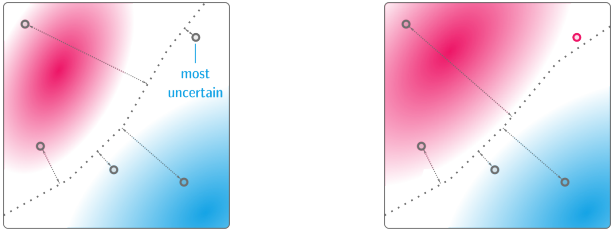
Example: for classifiers with a sigmoid top layer:



uncertainty depends on distance from separating hyperplane of predicted vs. top two vs. all classes

Example: Uncertainty Sampling

In a binary classification task (**red** vs. **blue**), when paired with a sigmoid-based classifier, uncertainty is inversely proportional to the distance from the separator between classes:



- **Left**: gray points indicate unlabelled points, and their distance from the separation surface is indicated by an arrow. Uncertainty sampling picks the closest unlabelled point.
- **Right**: that label of that point happens to be **red**, and the classifier is updated accordingly. Naturally, the distance of all other points from the separator (and hence their uncertainty) changes too.

Uncertainty Sampling

```python
def select(self, labeled_mask):
    unlabeled = np.where(labeled_mask == 0)[0]

    p = model.predict(X_train[unlabeled])
    entropy = -np.sum(p * np.log(p), axis=-1)
    most_uncerain = np.argmax(entropy)

    return unlabeled[most_uncertain]
```

- Uncertainty sampling is very **easy to implement**.
- Margin & Confidence can be defined even in terms of unnormalized scores.
- Usually performs reasonably well (though not optimally) in practice: a useful baseline/starting point.

Example: Structured Output

Consider an LSTM that takes a sequence of MNIST images $X = [\mathbf{x}_1, \ldots, \mathbf{x}_n]$ that composes a word and outputs the word itself $\mathbf{y} = (y_1, \ldots, y_n)$.

- Computing the most likely output $\hat{\mathbf{y}}$ can be done efficiently.
- Computing the entropy amounts to:

$$H_\theta(Y \mid \mathbf{X} = \mathbf{x}) := - \sum_{\mathbf{y} \in \{1,\ldots,26\}^n} p_\theta(\mathbf{y} \mid X) \log_2 p_\theta(\mathbf{y} \mid X) \tag{12}$$

This involves summing over 26^n possible outputs, which takes time **exponential** in n.

■ Computing the most likely output can be NP-hard. For instance, if $\mathbf{y}$ is molecular structure that mast satisfy specific hard constraints (chemical validity), then finding the best structure amounts to solving a hard combinatorial problem.

Hence, the **confidence** and **margin** can also be very hard.

Uncertainty in Regression

- When considering regression models with $Y \in \mathbb{R}$, uncertainty at $\mathbf{x}$ can be implemented as **differential entropy**:

$$H_\theta(Y \mid \mathbf{X} = \mathbf{x}) := \mathbb{E}[-\log_2 p_\theta(y \mid \mathbf{x}) \mid \mathbf{x}] \tag{13}$$

$$= -\int_{\mathbb{R}} p_\theta(y \mid \mathbf{x}) \log_2 p_\theta(y \mid \mathbf{x}) \tag{14}$$

Uncertainty in Regression

- When considering regression models with $Y \in \mathbb{R}$, uncertainty at $\mathbf{x}$ can be implemented as **differential entropy**:

$$H_\theta(Y \mid \mathbf{X} = \mathbf{x}) := \mathbb{E}[-\log_2 p_\theta(y \mid \mathbf{x}) \mid \mathbf{x}] \tag{13}$$

$$= -\int_{\mathbb{R}} p_\theta(y \mid \mathbf{x}) \log_2 p_\theta(y \mid \mathbf{x}) \tag{14}$$

- As an alternative heuristic, use the **variance**:

$$\mathrm{Var}_\theta(Y \mid \mathbf{x}) := \mathbb{E}[(Y - \underbrace{\mathbb{E}[Y \mid \mathbf{x}]}_{\mu_\theta(Y \mid \mathbf{x}) :=})^2 \mid \mathbf{x}] \tag{15}$$

$$= \int_{\mathbb{R}} (y - \mu_\theta(Y \mid \mathbf{x}))^2 p_\theta(y \mid \mathbf{x}) dy \tag{16}$$

$$\mu_\theta(Y \mid \mathbf{x}) = \int_{\mathbb{R}} y \; p_\theta(y \mid \mathbf{x}) dy \tag{17}$$

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- How to compute them?

Uncertainty in Regression

- Differential entropy and variance:

$$H_\theta(Y \mid \mathbf{X} = \mathbf{x}) = -\int_{\mathbb{R}} p_\theta(y \mid \mathbf{x}) \log_2 p_\theta(y \mid \mathbf{x}) \qquad \mathrm{Var}_\theta(Y \mid \mathbf{x}) = \int_{\mathbb{R}} (p_\theta(y \mid \mathbf{x}) - \mu_\theta(Y \mid \mathbf{x}))dy \tag{18}$$

[2] See https://en.wikipedia.org/wiki/Normal_distribution.

Uncertainty in Regression

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- Both expensive to compute for general models, can **approximate** via quadrature or sampling, but **closed-form solutions** exist for some models (e.g., Gaussian Processes and NNs with a Gaussian output)

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Example: 1-dimensional Gaussian Output

Consider one-dimensional output $y \in \mathbb{R}$ and a neural net:

$$nn : \mathbf{x} \mapsto (\mu, \sigma), \qquad y \sim \mathcal{N}(\mu, \sigma) \tag{19}$$

In this case, it is well known[2] that:

$$\mathrm{Var}_\theta(Y \mid \mathbf{x}) = \sigma^2, \qquad H_\theta(Y \mid \mathbf{x}) = \frac{1}{2}\log(2\pi\sigma^2) + \frac{1}{2} \tag{20}$$

Notice that $\mathrm{Var}_\theta(Y \mid \mathbf{x}) \propto \exp H_\theta(Y \mid \mathbf{x})$, so they change monotonically.

[2] See https://en.wikipedia.org/wiki/Normal_distribution.

Uncertainty in Regression

- Differential entropy and variance:

$$H_\theta(Y \mid \mathbf{X} = \mathbf{x}) = -\int_{\mathbb{R}} p_\theta(y \mid \mathbf{x}) \log_2 p_\theta(y \mid \mathbf{x}) \qquad \mathrm{Var}_\theta(Y \mid \mathbf{x}) = \int_{\mathbb{R}} (p_\theta(y \mid \mathbf{x}) - \mu_\theta(Y \mid \mathbf{x}))dy \tag{21}$$

- Both expensive to compute for general models, can **approximate** via quadrature or sampling, but **closed-form solutions** exist for some models (e.g., Gaussian Processes and NNs with a Gaussian output)

Example: k-dimensional Gaussian Output

Consider one-dimensional output $y \in \mathbb{R}^k$ and a neural net:

$$nn : \mathbf{x} \mapsto (\boldsymbol{\mu}, S), \qquad \Sigma \leftarrow SS^T, \qquad \mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma) \tag{22}$$

with Σ PSD by construction. In this case, it is well known[3] that:

$$\mathrm{Var}_\theta(Y \mid \mathbf{x}) \propto \mathrm{tr}\Sigma \qquad H_\theta(Y \mid \mathbf{x}) \propto \log\det\Sigma \tag{23}$$

where the trace is cheap to compute but the determinant is more challenging.

[3] See https://en.wikipedia.org/wiki/Multivariate_normal_distribution.

Illustration

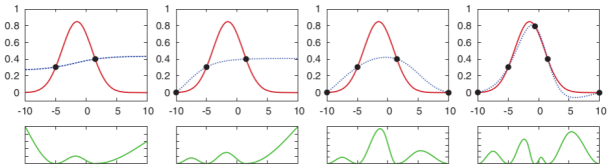
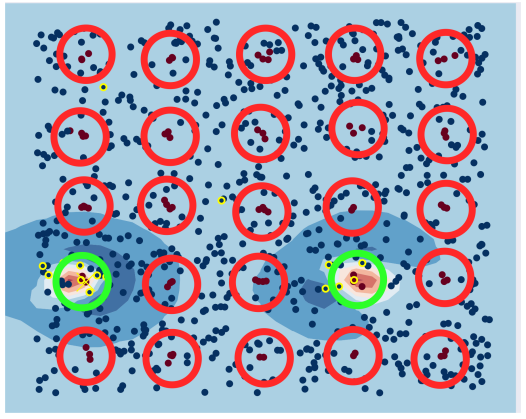
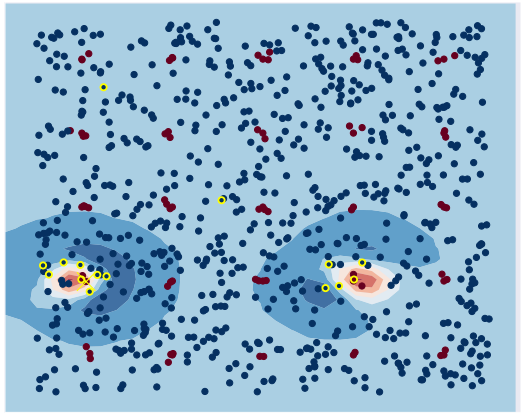


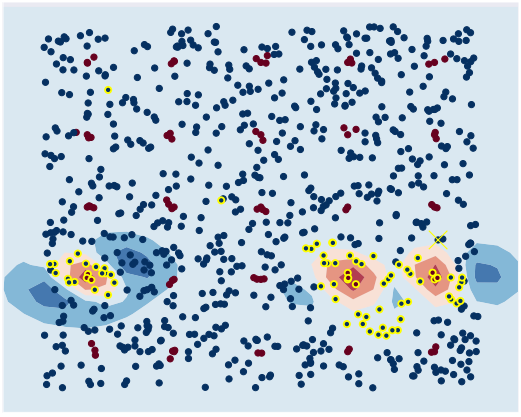
Figure 2.6: Variance-based uncertainty sampling for a toy 1D regression task. Each column represents an iteration of active learning. In the top row, solid lines show the target function to be learned, while dashed lines show a neural network approximation based on available training data (black dots). The bottom row plots the network's output variance across the input range, which is used to select the query for the next iteration.



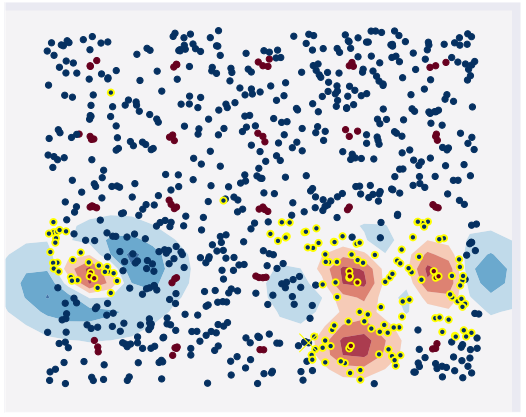
- Synthetic dataset: 25 clusters of **red** points arranged in a 5×5 grid, surrounded by a sea of **blue** points



- After **10** iterations of uncertainty sampling.



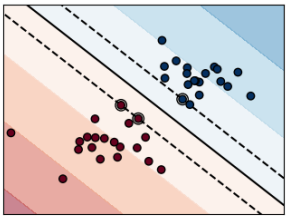
- After **70** iterations of uncertainty sampling.



- After **140** iterations of uncertainty sampling. Not nice!

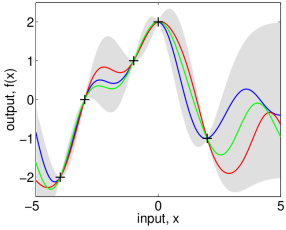
Over-confidence

■ **Discriminative** models are **over-confident**:



Uncertainty does **not** decrease with distance from the training set.

■ **Bayesian generative models** not so much:



(a), posterior

Uncertainty **does** decrease with distance from the training set.

Aleatoric vs Epistemic (Hüllermeier and Waegeman, 2021)

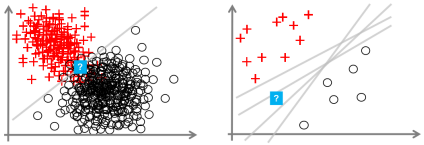


Figure 5: Left: Even with precise knowledge about the optimal hypothesis, the prediction at the query point (indicated by a question mark) is aleatorically uncertain, because the two classes are overlapping in that region. Right: A case of epistemic uncertainty due to a lack of knowledge about the right hypothesis, which is in turn caused by a lack of data.

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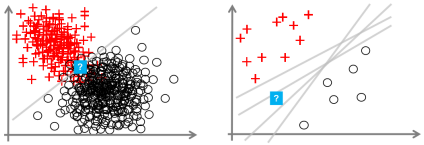

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- **Aleatoric** uncertainty ("random") captures how much we can trust the supervision itself. It cannot be decreased. (**left**)

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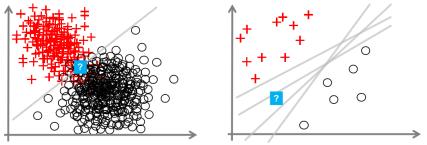

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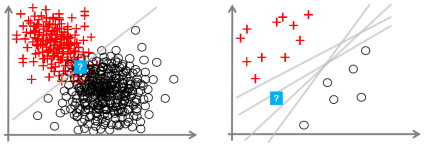

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- **Aleatoric** uncertainty ("random") captures how much we can trust the supervision itself. It cannot be decreased. (**left**)
- **Epistemic** uncertainty ("relating to knowledge") captures how little we know about the world. This reflects on uncertainty on the choice of θ. It decreases by acquiring **more data**. (**right**)
- There isn't much point in trying to reduce **aleatoric** uncertainty in AL (Sharma and Bilgic, 2017)

Uncertainty Sampling for Streaming Data

Input: models $\mathcal{F}$, bootstrap training set $\mathcal{L}$, threshold τ
Output: selected model $f \in \mathcal{F}$

$f \leftarrow \text{fit}(\mathcal{F}, \mathcal{L})$ ▷ initialize the model
for $t = 1, 2, 3, \ldots,$ **do**
  receive instance $\mathbf{x}$
  if $\text{unc}(f, \mathbf{x}) > \tau$ **then** ▷ if f is uncertain about $\mathbf{x}$
    obtain label y of $\mathbf{x}$ from annotator
    $\mathcal{L} \leftarrow \mathcal{L} \cup \{(\mathbf{x}, y)\}$ ▷ update training set
  $f \leftarrow \text{fit}(\mathcal{F}, \mathcal{L})$ ▷ update the model
return f

- The tricky bit is setting τ. Many algorithms update it dynamically by, e.g, starting from a large τ and lowering it as new data is received and the model improves

(a) target function (b) random (c) uncertainty-based selective sampling over time

Figure 2.7: Stream-based uncertainty sampling for a simple toy classification task. (a) Positive instances lie inside the black box in 2D. (b) After 100 random samples, the function learned by a neural network is still somewhat amorphous. (c) Uncertainty-based selective sampling at 20, 60, and 100 queries. The highlighted areas represent the region of uncertainty, which gradually shrinks and becomes more focused as the network grows more confident. The output of the resulting network after 100 queries is much more square-like than (b).

- For some problems, US **converges** to the right thing – because it is uncertain enough

(a) target function (b) initial sample (c) uncertainty-based selective sampling over time

Figure 2.8: An example of uncertainty sampling failure. (a) Positive instances lie inside the two black triangles. (b) An initial random sample fails to draw many training instances from the negative space in between the triangles. (c) The trained network becomes overly confident that instances in the center are positive. As a result, it avoids that region and begins to learn a different, more square-like shape.

- If you are unluckly, US becomes **over-confident**: in this example, the model becomes confident that the regions inside the black blob cannot be white, so it does not sample them and converges to the wrong shape.

The Story So Far

- In **active learning** the machine is allowed to ask **questions** to an oracle – and it should do so **intelligently**, so as to minimize the # of questions for obtaining a good model.
- The most common query selection strategy is **uncertainty sampling**: the machine asks the oracle to label those (unlabelled) instances on which it is most unsure.
- **Issues**:
  1. If the machine doesn't know that it doesn't know – i.e., its self-assessed uncertainty is poorly calibrated – then uncertainty sampling can ignore informative instances altogether.
  2. Common uncertainty measures (e.g., entropy) mix together epistemic and aleatoric uncertainty, only the former of which we can reduce by acquiring more labels.

Input: models $\mathcal{F}$, examples $\mathcal{L}$, pool $\mathcal{U}$, query budget $T \geq 1$
Output: selected model $f \in \mathcal{F}$

$f \leftarrow \mathrm{fit}(\mathcal{F}, \mathcal{L})$
for $t = 1, 2, \ldots, T$ **do**
  $\mathbf{x} \leftarrow \mathrm{argmax}_{\mathbf{x} \in \mathcal{U}} \mathrm{acq}(f, \mathbf{x})$
  obtain label y of $\mathbf{x}$ from annotator
  $\mathcal{U} \leftarrow \mathcal{U} \setminus \{\mathbf{x}\}$
  $\mathcal{L} \leftarrow \mathcal{L} \cup \{(\mathbf{x}, y)\}$
  $f \leftarrow \mathrm{fit}(\mathcal{F}, \mathcal{L})$
return f

- Uncertainty sampling is quite heuristic. Are there more **principled** approaches?

Version Space

- Consider a **hypothesis space** $\mathcal{F} = \{f_\theta : \mathbf{x} \mapsto y\}$ and a data set $\mathcal{L} = \{(\mathbf{x}_i, y_i)\}$

Consistency

A hypothesis $f \in \mathcal{F}$ is **consistent** with $\mathcal{L}$, written $f \models \mathcal{L}$, iff it makes *zero mistakes* on it, that is:

$$(f \models \mathcal{L}) \iff \left(\sum_{(\mathbf{x},y)\in\mathcal{L}} \mathbb{1}(f(\mathbf{x}) \neq y)\right) = 0 \tag{24}$$

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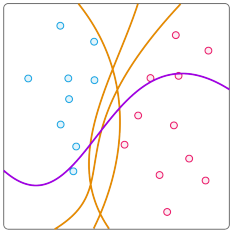
Version Space

The **version space** $VS(\mathcal{L})$ of $\mathcal{F}$ given $\mathcal{L}$ is the set of hypotheses $f \in \mathcal{F}$ that are *consistent* with $\mathcal{L}$, that is:

$$VS(\mathcal{L}) = \{f \in \mathcal{F} \ : \ f \models \mathcal{L}\} \tag{25}$$



$VS(\mathcal{L})$ contains those classifiers that are not ruled out by the examples $\mathcal{L}$ (in orange). It does not include the purple classifier though!

- If $\mathcal{L}$ is not **separable** w.r.t. $\mathcal{F}$, i.e., if there is no hypothesis $f \in \mathcal{F}$ that is constent with it, then the version space is **empty**.

This can happen in practice because:

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This can happen in practice because:

- $\mathcal{F}$ is **not expressive enough**.
  Example: *neural networks in $\mathcal{F}$ have too few layers/neurons, none of them is expressive enough to correctly label all data.*

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  Example: *$\mathcal{L}$ contains the same instance twice but annotated with different labels – e.g., $(\mathbf{x}, 1)$ and $(\mathbf{x}, 3)$ – so no $f \in \mathcal{F}$ can classify both correctly.*

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The Realizable Case

We assume the **realizable case**: $\exists f^* \in \mathcal{F}$ s.t. $y = f^*(\mathbf{x})$ for all $\mathbf{x}$ and no noise.

This implies that $f^* \in VS(\mathcal{L})$ for all choices of labeled examples $\mathcal{L}$, because the supervision $(\mathbf{x}, y)$ is always consistent with f^*. Hence, the version space is never empty, regardless of what data we see!

Version Space $\leftrightarrow$ Disagreement Region

Disagreement Region

Given $\mathcal{F}$ and $\mathcal{L}$, the **disagreement region** is the set of points $\mathbf{x} \in \mathbb{R}^d$ such that there exist two classifiers f, f' in the version space $VS(\mathcal{L})$ that produce different predictions for them:

$$DIS(\mathcal{L}) = \{\mathbf{x} \in \mathbb{R}^d \ : \ \exists f, f' \in VS(\mathcal{L}) \ . \ f(\mathbf{x}) \neq f'(\mathbf{x})\} \tag{26}$$

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- If $\mathbf{x} \notin DIS(\mathcal{L})$, then all candidate classifiers f in the version space classify it the same: acquiring its label is **pointless**.

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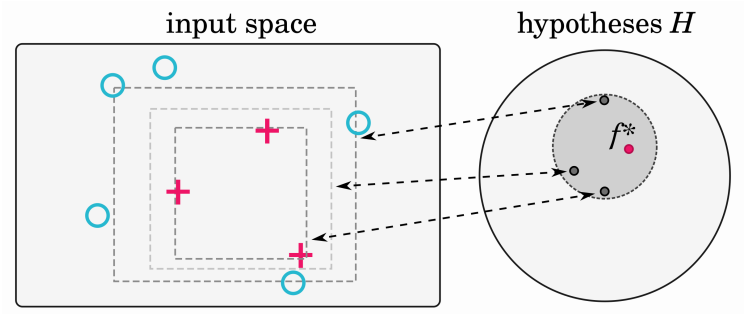
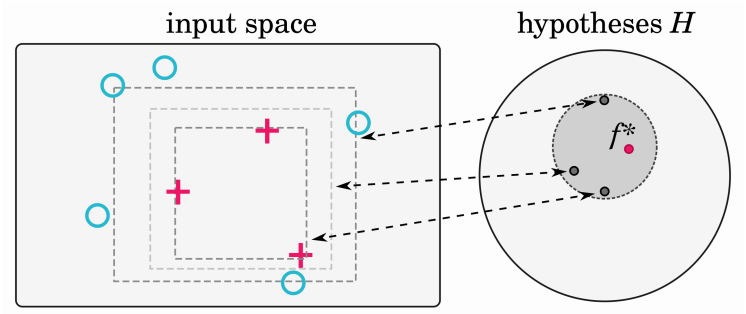
- If $\mathbf{x} \notin DIS(\mathcal{L})$, then all candidate classifiers f in the version space classify it the same: acquiring its label is **pointless**.
- If $\mathbf{x} \in DIS(\mathcal{L})$, then at least one f in the version space classifies it differently: acquiring its label is **useful**.



Left: **input space** $\mathbb{R}^d$, data set $\mathcal{L}$ of red crosses vs blue circles. **Right**: **hypothesis space** $\mathcal{F}$, each f is a point; the ground-truth f^* is in red.



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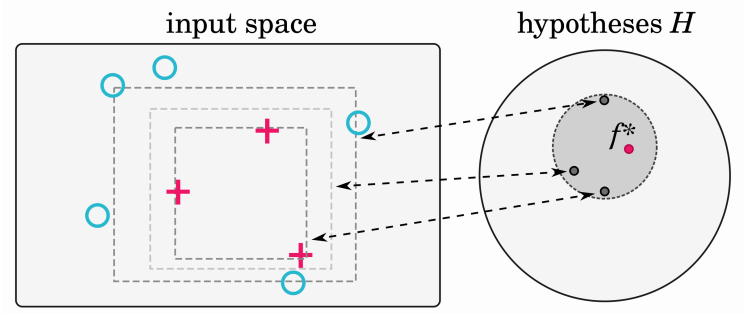
$\mathcal{F}$ is the set of $2D$ rectangles. Rectangles in instance space (left) are points in hypothesis space (right), as shown by the arrows.



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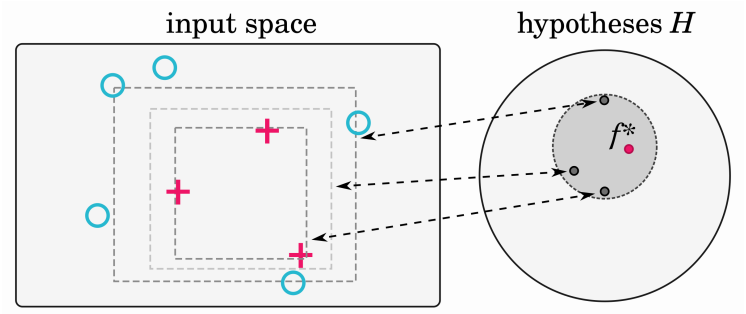
The **version space** $VS(\mathcal{L})$ contains all the rectangles (pale gray) between inner & outer rectangles (darker gray)



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The **disagreement region** $DIS(\mathcal{L})$ is the space enclosed between these two rectangles.

Version Space for Streaming AL

Input: models $\mathcal{F}$
Output: selected model $f \in \mathcal{F}$

$\mathcal{L} \leftarrow \varnothing$
$\mathcal{V} \leftarrow \mathcal{F}$ ▷ implements the version space $VS(\mathcal{L})$
for $t = 1, 2, 3, \ldots,$ **do**
    receive instance $\mathbf{x}$
    if $\mathbf{x} \in DIS(\mathcal{V})$ **then** ▷ if $\mathbf{x}$ falls in the disagreement region
        obtain label y of $\mathbf{x}$
        update $\mathcal{V} \leftarrow \{f \in \mathcal{V} \ : \ f(\mathbf{x}) = y\}$ ▷ update version space
return any $f \in \mathcal{V}$

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2: $\mathcal{V} \leftarrow \mathcal{F}$ ▷ implements the version space $VS(\mathcal{L})$
3: **for** $t = 1, 2, 3, \ldots,$ **do**
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5: **if** $\mathbf{x} \in DIS(\mathcal{V})$ **then** ▷ if $\mathbf{x}$ falls in the disagreement region
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Input: models $\mathcal{F}$
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1: $\mathcal{L} \leftarrow \varnothing$
2: $\mathcal{V} \leftarrow \mathcal{F}$ ▷ implements the version space $VS(\mathcal{L})$
3: **for** $t = 1, 2, 3, \ldots,$ **do**
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- Recall that f^* is always compatible with examples $(\mathbf{x}, y)$, so it is always in $VS(\mathcal{L}) \rightarrow$ algorithm *zooms* into it!

- This algorithm makes **no useless queries**!

Question

Figure 2.8: An example of uncertainty sampling failure. (a) Positive instances lie inside the two black triangles. (b) An initial random sample fails to draw many training instances from the negative space in between the triangles. (c) The trained network becomes overly confident that instances in the center are positive. As a result, it avoids that region and begins to learn a different, more square-like shape.

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- **Can we do better if we can choose $\mathbf{x}$?**

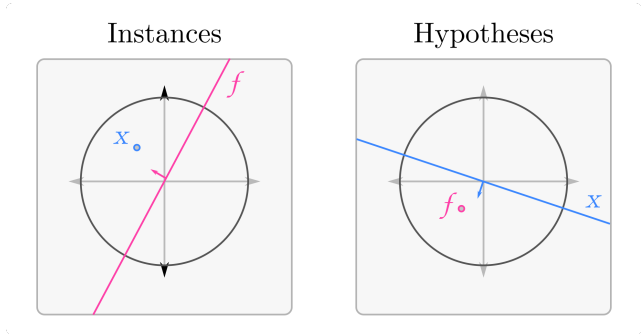
Version Space for Pool-based AL

Input: models $\mathcal{F}$
Output: selected model $f \in \mathcal{F}$

1: $\mathcal{L} \leftarrow \varnothing$
2: $\mathcal{V} \leftarrow \mathcal{F}$ ▷ implements the version space $VS(\mathcal{L})$
3: **for** $t = 1, 2, \ldots, T$ **do**
4: $\quad \mathbf{x} \leftarrow \text{argmax}_{\mathbf{x} \in \mathcal{U}} \ \text{acq}_{VS}(\mathcal{V}, \mathcal{F}, \mathbf{x})$
5: $\quad$ obtain label y of $\mathbf{x}$
6: $\quad$ update $\mathcal{V} \leftarrow \{f \in \mathcal{V} \ : \ f(\mathbf{x}) = y\}$ ▷ update version space
7: **return** any $f \in \mathcal{V}$

- We can always ensure that there is a point on which the classifiers in the version space disagree, unless the version space is empty or includes a single classifier. In this case we can simply terminate.

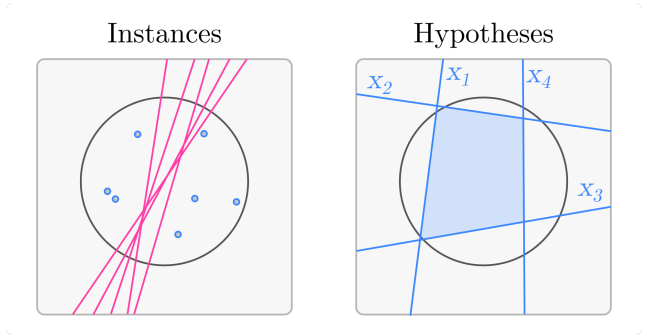
Problem: how do we define the acquisition function?



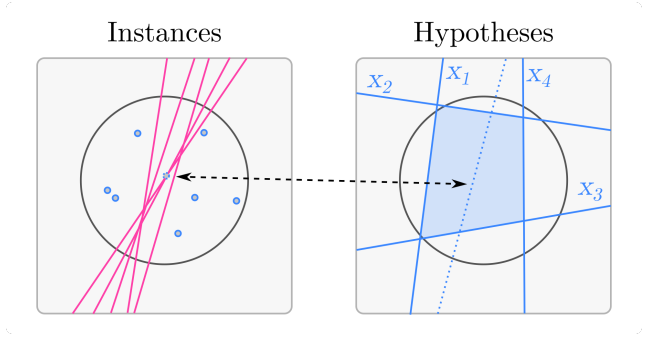
Consider the *linear classifiers*, i.e., $\mathcal{F}$ is:

$$\left\{ f_\theta(\mathbf{x}) = \mathbb{1}\left(\boldsymbol{\theta}^\top \mathbf{x} > 0\right) \;:\; \boldsymbol{\theta} \in \mathbb{R}^d,\ \underbrace{\|\boldsymbol{\theta}\|_2}_{\text{length}} = 1 \right\} \tag{27}$$

The version space of $\mathcal{L}$ is essentially the set of direction vectors $\boldsymbol{\theta}$ that classify all points correctly.



■ Classifiers are hyperplanes in instance space and instances are hyperplanes in hypothesis space. In some sense, the two spaces are "dual" of one another.



■ **Idea**: pick the point $\mathbf{x} \in \mathcal{U}$ that (greedily) restricts the version space as much as possible. In this special case, $\mathbf{x}$ passes close to the **center** of $VS(\mathcal{L})$.

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- The best we can do is to compute the **average volume** based on the probability of the **predicted labels** given by the model:

$$\underset{\mathbf{x} \in \mathcal{U}}{\text{argmin}} \ \frac{1}{c} \sum_{y=1}^{c} p_\theta(y \mid \mathbf{x}) \cdot \text{Vol}(VS(\mathcal{L} \cup \{(\mathbf{x}, y)\})) \tag{29}$$

This tells us what the volume of the VS **would be** if we were to add $\mathbf{x}$ – with an unknown label y – to the data.

Question: how to **encode** the version space?

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  - If $\mathcal{F}$ **is "simple" and/or** $\mathcal{L}$ **is small**, volume can be approximated cheaply using Monte Carlo techniques. For instance with *rejection sampling*, let $B \subseteq VS(\mathcal{L})$ of known volume:

$$\{\widetilde{\boldsymbol{\theta}}_i \sim \mathrm{Uniform}(B) \ : \ i = 1, \ldots, s\}, \qquad \mathrm{Vol}\left(VS(\mathcal{L}')\right) \approx \frac{1}{\mathrm{Vol}\,(B)} \cdot \frac{1}{s} \sum_{i=1}^{s} \mathbb{1}\left(\widetilde{\boldsymbol{\theta}}_i \in VS(\mathcal{L}')\right) \tag{31}$$

    To check, $\mathbb{1}(\boldsymbol{\theta}_i \in VS(\mathcal{L}'))$, check that $f_{\boldsymbol{\theta}}$ classifies all examples in $\mathcal{L}$ correctly.

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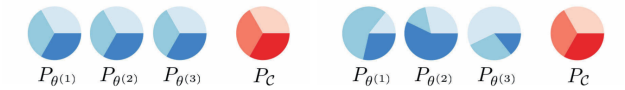
  - Otherwise (think CNN on ImageNet), can be **extremely challenging** – we cannot use VS!

Not All Classifiers in VS Think Exactly The Same



(a) uncertain but in agreement

(b) uncertain and in disagreement

Figure 3.5: Examples of committee and consensus distributions. $P_{\theta(i)}$ refers the output distribution of the ith hypothesis, and $P_{\mathcal{C}}$ represents the consensus across all committee members.

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- Let's **relax** both of them $\rightarrow$ **speed-up**!
- Moreover, version space is only non-empty in the realizable case. How do we deal with this?

Query By Committee (QBC)

- **Idea**: replace VS witha **committee** $\mathcal{C}$:
  - Select k representatives $\mathcal{C} = \{c_1, \ldots, c_k\}$ from $VS(\mathcal{L})$, with $k > 100$.
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In all cases, we end up having a set of classifiers that fit the data – assuming their accuracy is 100% – so they are all in the VS. In practice, **less than perfect accuracy is allowed**: members are "almost" in VS.

Measuring Disagreement of $\mathcal{C}$ on $\mathbf{x} \in \mathcal{U}$

- **"Hard" Voting + Entropy**:

$$\underset{\mathbf{x}\in\mathcal{U}}{\operatorname{argmax}} \; -\sum_{y} \frac{n(y,\mathbf{x})}{k} \log \frac{n(y,\mathbf{x})}{k}, \qquad n(y,\mathbf{x}) := \sum_{c\in\mathcal{C}} \mathbb{1}\left(c(\mathbf{x}) = y\right) \tag{32}$$

Each classifier votes either 0 or 1.

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- Average Kullback-Liebler divergence:

$$\underset{\mathbf{x} \in \mathcal{U}}{\operatorname{argmax}} \; \frac{1}{k} \sum_{c \in \mathcal{C}} \mathbb{KL}(p_c(Y \mid \mathbf{x}) \| p_{\mathcal{C}}(Y \mid \mathbf{x})) \tag{34}$$

$$\mathbb{KL}(p(Y \mid \mathbf{x}) \| q(Y \mid \mathbf{x})) := \sum_{y} p(y \mid \mathbf{x}) \log \frac{p(y \mid \mathbf{x})}{q(y \mid \mathbf{x})} \tag{35}$$

Very expressive, measures difference between **whole distributions**, i.e., **prob. of all possible labels**.

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Useful Concepts

The **loss** of p_θ on example $z = (\mathbf{x}, y)$ is denoted $\ell(\theta, z)$. For instance, cross-entropy loss:

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The **empirical risk** $\widehat{\mathcal{L}}_S$ of θ w.r.t. data set $S = \{z_1, \ldots, z_m\}$ sampled i.i.d. from p^* is:

$$\widehat{\mathcal{L}}_S(\theta) := \frac{1}{|S|} \sum_{z \in S} \ell(\theta, z) \tag{38}$$

It **estimates** the quality of the model from a sample S, optimized during training.

Let $\widehat{\theta}$ be the parameters obtained by training on S and $\widehat{\theta}^{+z}$ those obtained by training on $S \cup \{z\}$, i.e.,

$$\widehat{\theta} := \operatorname*{argmin}_{\theta} \ \widehat{\mathcal{L}}_S(\theta) \qquad \widehat{\theta}^{+z} := \operatorname*{argmin}_{\theta} \ \widehat{\mathcal{L}}_{S \cup \{z\}}(\theta) \tag{39}$$

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Model Improvement

The **model improvement** (MI) given by a new example $\mathbf{z} \notin S$ is the decrease in true risk:

$$\mathrm{acq}(\mathbf{x}) := \mathcal{L}^*(\widehat{\theta}) - \mathcal{L}^*(\widehat{\theta}^{+\mathbf{z}}) \tag{40}$$

The higher, the better $\longrightarrow$ pick the $\mathbf{x} \in \mathcal{U}$ that **maximizes** the improvement.

Model Improvement as Greedy Optimization

- MI amounts to solving:

$$\underset{\mathbf{x}\in\mathcal{U}}{\operatorname{argmax}} \; \mathcal{L}^*(\widehat{\theta}) - \mathcal{L}^*(\widehat{\theta}^{+\mathbf{z}}) \quad = \quad \underset{\mathbf{x}\in\mathcal{U}}{\operatorname{argmin}} \; \mathcal{L}^*(\widehat{\theta}^{+\mathbf{z}}) \tag{41}$$

It is *guaranteed* to find the best next candidate!

[4] **Note**: MI is *greedy*, *not* optimal! Non-greedy alternatives are conceptually better, but they also computationally infeasible and for this reason they are ignored in the AL literature.

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- MI is essentially a **greedy strategy** for solving:[4]

$$\underset{S\subseteq\mathcal{U}}{\operatorname{argmin}} \quad \mathcal{L}^*(\widehat{\theta}) \tag{42}$$

$$\text{s.t.} \quad |S| \leq \text{query budget} \tag{43}$$

In this view, **AL is a subset optimization problem**, and MI solves it directly.

[4] **Note**: MI is *greedy*, *not* optimal! Non-greedy alternatives are conceptually better, but they also computationally infeasible and for this reason they are ignored in the AL literature.

Model Improvement as Greedy Optimization

- MI amounts to solving:

$$\underset{\mathbf{x}\in\mathcal{U}}{\operatorname{argmax}}\ \mathcal{L}^*(\widehat{\theta}) - \mathcal{L}^*(\widehat{\theta}^{+\mathbf{z}}) \quad = \quad \underset{\mathbf{x}\in\mathcal{U}}{\operatorname{argmin}}\ \mathcal{L}^*(\widehat{\theta}^{+\mathbf{z}}) \tag{41}$$

It is *guaranteed* to find the best next candidate!

- MI is essentially a **greedy strategy** for solving:[4]

$$\underset{S\subseteq\mathcal{U}}{\operatorname{argmin}}\quad \mathcal{L}^*(\widehat{\theta}) \tag{42}$$

$$\text{s.t.}\quad |S| \leq \text{query budget} \tag{43}$$

In this view, **AL is a subset optimization problem**, and MI solves it directly.

- Compare this to *uncertainty sampling*, which is not as sound

[4] **Note**: MI is *greedy*, *not* optimal! Non-greedy alternatives are conceptually better, but they also computationally infeasible and for this reason they are ignored in the AL literature.

- We want to solve:

$$\underset{\mathbf{x}\in\mathcal{U}}{\operatorname{argmin}} \; \mathcal{L}^*\big(\widehat{\theta}^{+z}\big) \tag{44}$$

[5] The unlabeled set $\mathcal{U}$ is ideally pretty large, so the approximation is reasonable.

- We want to solve:

$$\underset{\mathbf{x} \in \mathcal{U}}{\operatorname{argmin}} \ \mathcal{L}^*(\widehat{\theta}^{+z}) \tag{44}$$

Problem: $\mathcal{L}^*(\cdot)$ is an **integral over** $\mathbf{x}' \in \mathbb{R}^d$:

$$\mathcal{L}^*(\widehat{\theta}^{+z}) = \mathbb{E}_{z' \sim \rho^*}[\ell(\widehat{\theta}^{+z}, z')] = \int_{\mathbb{R}^d} \ell(\widehat{\theta}^{+z}, (\mathbf{x}', y'))d\mathbf{x}' \tag{45}$$

which is intractable

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which is intractable $\rightarrow$ **approximate using empirical average over $\mathcal{U}$**:[5]

$$\mathcal{L}^*(\widehat{\theta}^{+z}) \approx \widehat{\mathcal{L}}_{\mathcal{U}}(\widehat{\theta}^{+z}) = \frac{1}{|\mathcal{U}|}\sum_{\mathbf{x}'\in\mathcal{U}} \ell(\widehat{\theta}^{+z}, (\mathbf{x}', y')) \tag{46}$$

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Example: if ℓ is the 0–1 loss, then this amounts to:

$$\frac{1}{|\mathcal{U}|} \sum_{\mathbf{x}'\in\mathcal{U}} \mathbb{1}\left(f_{\widehat{\theta}^{+z}}(\mathbf{x}') \neq y'\right) \tag{47}$$

[5] The unlabeled set $\mathcal{U}$ is ideally pretty large, so the approximation is reasonable.

- We already decided on this approximation:

$$\mathcal{L}^*(\widehat{\theta}^{+z}) \qquad \longrightarrow \qquad \widehat{\mathcal{L}}_{\mathcal{U}}(\widehat{\theta}^{+z}) = \frac{1}{|\mathcal{U}|} \sum_{\mathbf{x}' \in \mathcal{U}} \ell(\widehat{\theta}^{+z}, (\mathbf{x}', y')) \tag{48}$$

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Problem: we don't have access to the ground-truth label $z = (\mathbf{x}, y)$ → **marginalize w.r.t.** p^*:

$$\mathbb{E}_{y \sim p^*(Y|\mathbf{x})} \left[\frac{1}{|\mathcal{U}|} \sum_{\mathbf{x}' \in \mathcal{U}} \ell(\widehat{\theta}^{+(\mathbf{x},y)}, (\mathbf{x}', y')) \right] \tag{49}$$

This averages over alternative **future models** $\widehat{\theta}^{+(\mathbf{x},y)}$ obtained after retraining on $\mathcal{L} \cup (\mathbf{x}, y)$.

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Problem: we don't have access to the ground-truth label $z = (\mathbf{x}, y)$ → **marginalize w.r.t.** p^*:

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This averages over the **unknown labels** y' of the instances in $\mathbf{x}' \in \mathcal{U}$.

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$$\mathcal{L}^*(\widehat{\theta}^{+z}) \quad \longrightarrow \quad \mathbb{E}_{y\sim p^*(Y|\mathbf{x})}\left[\frac{1}{|\mathcal{U}|}\sum_{\mathbf{x}'\in\mathcal{U}}\mathbb{E}_{y'\sim p^*(Y|\mathbf{x}')}\left[\ell(\widehat{\theta}^{+(\mathbf{x},y)},(\mathbf{x}',y'))\right]\right] \tag{51}$$

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Problem: we don't have access to p^* at all

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Problem: we don't have access to p^* at all $\rightarrow$ **estimate using model's distribution**:

$$\underbrace{\mathbb{E}_{y\sim p_{\widehat{\theta}}(Y|\mathbf{x})}\Big[\underbrace{\frac{1}{|\mathcal{U}|}\sum_{\mathbf{x}'\in\mathcal{U}}\underbrace{\mathbb{E}_{y'\sim p_{\widehat{\theta}^+}(Y|\mathbf{x}')}\left[\ell(\widehat{\theta}^+,(\mathbf{x}',y'))\right]}_{(a)}}_{(b)}\Big]}_{(c)} \tag{52}$$

where $\widehat{\theta}^+ := \widehat{\theta}^{+(\mathbf{x},y)}$. If p_θ is "good enough", then the approximation is good.

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(a) Is the **expected** loss of the updated model on $\mathbf{x}' \in \mathcal{U}$,

(b) Is the **average** expected oss of the updated model on *all* of $\mathcal{U}$,

(c) Is the above **averaged over the possible updated models** $\widehat{\theta}^{+(\mathbf{x},y)}$.

- We already decided on this approximation:

$$\mathcal{L}^*(\widehat{\theta}^{+z}) \quad \longrightarrow \quad \mathbb{E}_{y\sim p_{\widehat{\theta}}(Y|\mathbf{x})}\Big[\frac{1}{|\mathcal{U}|}\sum_{\mathbf{x}'\in\mathcal{U}} \mathbb{E}_{y'\sim p_{\widehat{\theta}^+}(Y|\mathbf{x}')}\Big[\ell(\widehat{\theta}^+,(\mathbf{x}',y'))\Big]\Big] \tag{53}$$

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Example: consider the 0–1 loss $\ell(\theta,(\mathbf{x},y)) = \mathbb{1}(f_\theta(\mathbf{x}) \neq y)$.

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$$\mathbb{E}_{y'\sim p_{\widehat{\theta}^+}(Y|\mathbf{x}')}\left[\mathbb{1}\left(f_{\widehat{\theta}^+}(\mathbf{x}') \neq y'\right)\right] = p_{\widehat{\theta}^+}(\hat{y}' \neq y' \mid \mathbf{x}'), \qquad \hat{y}' := f_{\widehat{\theta}^+}(\mathbf{x}') \tag{54}$$

$$= 1 - p_{\widehat{\theta}^+}(\hat{y}' \mid \mathbf{x}') \tag{55}$$

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Hence, the above can be rewritten as ($\frac{1}{|\mathcal{U}|}$ doesn't matter because it is independent of $\mathbf{x}$):

$$\mathbb{E}_{y\sim p_{\widehat{\theta}}(Y|\mathbf{x})}\Big[\frac{1}{|\mathcal{U}|}\sum_{\mathbf{x}'\in\mathcal{U}}\left(1 - p_{\widehat{\theta}^+}(\hat{y}' \mid \mathbf{x}')\right)\Big] \quad \propto \quad \sum_{y\in[c]} p_{\widehat{\theta}}(y \mid \mathbf{x}) \sum_{\mathbf{x}'\in\mathcal{U}}\left(1 - p_{\widehat{\theta}^+}(\hat{y}' \mid \mathbf{x}')\right) \tag{56}$$

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- We pick $\mathbf{x} \in \mathcal{U}$ that minimizes the above $\rightarrow$ **minimizes expected future confidence on $\mathcal{U}$**

- We already decided on this approximation:

$$\mathcal{L}^*(\widehat{\theta}^{+z}) \qquad \longrightarrow \qquad \mathbb{E}_{y\sim p_{\widehat{\theta}}(Y|\mathbf{x})}\Big[\frac{1}{|\mathcal{U}|}\sum_{\mathbf{x}'\in\mathcal{U}} \mathbb{E}_{y'\sim p_{\widehat{\theta}^+}(Y|\mathbf{x}')}\Big[\ell(\widehat{\theta}^+,(\mathbf{x}',y'))\Big]\Big] \tag{57}$$

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Example: consider the negative log-likelihood $\ell(\theta,(\mathbf{x},y)) = -\log p_\theta(y \mid \mathbf{x})$.

- We already decided on this approximation:

$$\mathcal{L}^*(\widehat{\theta}^{+z}) \qquad \longrightarrow \qquad \mathbb{E}_{y\sim p_{\widehat{\theta}}(Y|\mathbf{x})}\Big[\frac{1}{|\mathcal{U}|}\sum_{\mathbf{x}'\in\mathcal{U}} \mathbb{E}_{y'\sim p_{\widehat{\theta}^+}(Y|\mathbf{x}')}\Big[\ell(\widehat{\theta}^+,(\mathbf{x}',y'))\Big]\Big] \tag{57}$$

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$$\mathbb{E}_{y'\sim p_{\widehat{\theta}^+}(Y|\mathbf{x}')}\left[-\log p_{\widehat{\theta}^+}(y' \mid \mathbf{x}')\right] = -\sum_{y'\in[c]} p_{\widehat{\theta}^+}(y' \mid \mathbf{x}')\log p_{\widehat{\theta}^+}(y' \mid \mathbf{x}') \tag{58}$$

$$= H_{\widehat{\theta}^+}(Y \mid \mathbf{x}) \tag{59}$$

- We already decided on this approximation:

$$\mathcal{L}^*(\widehat{\theta}^{+z}) \quad \longrightarrow \quad \mathbb{E}_{y\sim p_{\widehat{\theta}}(Y|\mathbf{x})}\Big[\frac{1}{|\mathcal{U}|}\sum_{\mathbf{x}'\in\mathcal{U}} \mathbb{E}_{y'\sim p_{\widehat{\theta}^+}(Y|\mathbf{x}')}\Big[\ell(\widehat{\theta}^+,(\mathbf{x}',y'))\Big]\Big] \tag{57}$$

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Hence, the above can be rewritten as:

$$\mathbb{E}_{y\sim p_{\widehat{\theta}}(Y|\mathbf{x})}\Big[\frac{1}{|\mathcal{U}|}\sum_{\mathbf{x}'\in\mathcal{U}}\big(H_{\widehat{\theta}^+}(Y \mid \mathbf{x})\big)\Big] \quad \propto \quad \sum_{y\in[c]} p_{\widehat{\theta}}(y \mid \mathbf{x}) \sum_{\mathbf{x}'\in\mathcal{U}}\big(H_{\widehat{\theta}^+}(Y \mid \mathbf{x})\big) \tag{60}$$

- We already decided on this approximation:

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- We pick $\mathbf{x}\in\mathcal{U}$ that minimizes the above $\rightarrow$ **minimizes expected future entropy on $\mathcal{U}$**

- In uncertainty sampling, we pick $\mathbf{x}$ that minimizes model's estimate of current uncertainty w.r.t. itself, this is **myopic**

- In expected model improvement, we pick $\mathbf{x}$ that minimizes model's estimate of expected future uncertainty w.r.t. unlabeled set, this is **less myopic**

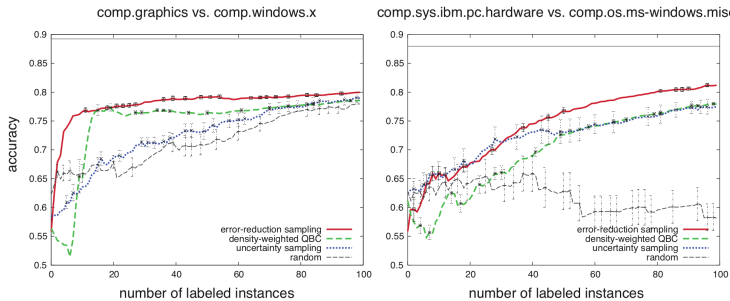

Figure 4.1: Learning curves showing that expected error reduction can outperform QBC and uncertainty sampling for two binary text classification tasks. *Source*: Adapted from Roy and McCallum (2001), with kind permission of the authors.

- We already decided on this approximation:

$$\mathcal{L}^*(\widehat{\theta}^{+z}) \qquad \longrightarrow \qquad \mathbb{E}_{y\sim p_{\widehat{\theta}}(Y|\mathbf{x})}\Big[\frac{1}{|\mathcal{U}|}\sum_{\mathbf{x}'\in\mathcal{U}}\mathbb{E}_{y'\sim p_{\widehat{\theta}^+}(Y|\mathbf{x}')}\Big[\ell(\widehat{\theta}^+,(\mathbf{x}',y'))\Big]\Big] \tag{61}$$

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Problem: computing $\widehat{\theta}^+$ requires to **fit model on** $\mathcal{L}\cup\{(\mathbf{x},y)\}$ (**slow**)

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Problem: this has to be done $|\mathcal{U}|\times[c]$ times.

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Problem: computing $\widehat{\theta}^+$ requires to **fit model on** $\mathcal{L}\cup\{(\mathbf{x},y)\}$ (**slow**)

Problem: this has to be done $|\mathcal{U}|\times[c]$ times.

Problem: this has to be done in each iteration of active learning.

- We already decided on this approximation:

$$\mathcal{L}^*(\widehat{\theta}^{+z}) \qquad \longrightarrow \qquad \mathbb{E}_{y\sim p_{\widehat{\theta}}(Y|\mathbf{x})}\Big[\frac{1}{|\mathcal{U}|}\sum_{\mathbf{x}'\in\mathcal{U}}\mathbb{E}_{y'\sim p_{\widehat{\theta}^+}(Y|\mathbf{x}')}\Big[\ell(\widehat{\theta}^+,(\mathbf{x}',y'))\Big]\Big] \tag{61}$$

Problem: computing $\widehat{\theta}^+$ requires to **fit model on** $\mathcal{L}\cup\{(\mathbf{x},y)\}$ (**slow**)

Problem: this has to be done $|\mathcal{U}|\times[c]$ times.

Problem: this has to be done in each iteration of active learning.

- Only **practical** for classes of models that support **closed-form updates** (e.g., Gaussian Processes) or stable **incremental learning** (e.g., perceptron-like learning algorithms).

Expected Model Change

- Unless a candidate $(\mathbf{x}, y)$ induces a large change in the model $\widehat{\theta}$ upon retraining, then it cannot reduce the model's risk by much: **change is a prerequisite for improvement**.

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Inituition:

$$\ell(\widehat{\theta}, \mathbf{z}') - \ell(\widehat{\theta}^{+\mathbf{z}}, \mathbf{z}') \quad \leq \quad |\ell(\widehat{\theta}, \mathbf{z}') - \ell(\widehat{\theta}^{+\mathbf{z}}, \mathbf{z}')| \quad \leq \quad c \cdot \|\widehat{\theta} - \widehat{\theta}^{+\mathbf{z}}\|, \qquad c > 0 \tag{62}$$

where $\|\cdot\|$ is, e.g., the Euclidean norm. This formally holds for all c-Lipshitz loss functions ℓ.

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where $\|\cdot\|$ is, e.g., the Euclidean norm. This formally holds for all c-Lipshitz loss functions ℓ.

- Large change also occurs when the loss *increases* – hence the absolute value in the second step of Eq. 62.

All in all, EMC looks for examples $\mathbf{x} \in \mathcal{U}$ that "make a difference" one way or the other.

But once $(\mathbf{x}, y)$ is acquired it is added to the training set $\mathcal{L}$ on which $\widehat{\theta}$ is fit, so loss is likely to go *down* rather than *up*.

Expected Model Change

- The trick is that if $\widehat{\theta}$ is obtained via **gradient descent**, the difference $\widehat{\theta} - \widehat{\theta}^{+z}$ is easy to compute:

$$\widehat{\theta} - \widehat{\theta}^{+z} = \eta \cdot \nabla_\theta \ell(\theta, z) \tag{63}$$

where η is the learning rate. This gives **expected gradient length**:

$$\mathrm{acq}_{\mathsf{EGL}}(\mathbf{x}) := \mathbb{E}_{y \sim p_\theta(Y|\mathbf{x})} \left[\|\nabla_\theta \ell(\widehat{\theta}, (\mathbf{x}, y))\|^2 \right] \tag{64}$$

The square does not change ranking of examples & avoids computing a square root.

- Quite cheap to compute using automatic differentiation packages (using Jacobian to parallelize over $\mathcal{U}$)
- Assuming η is constant across examples and GD, the computation is exact. For other optimizers, it is an approximation

Are Uncertain Points Representative?

Figure 5.1: An illustration of when uncertainty sampling can be a poor strategy. Shaded polygons represent labeled instances in $\mathcal{L}$, and circles represent unlabeled instances in $\mathcal{U}$. Since A is on the decision boundary, it would be queried as the most uncertain. However, B would probably provide more information about the input distribution as a whole.

Diversity-based Selection

Idea: pick instances $\mathbf{x} \in \mathcal{U}$ that are *both* **locally informative** and also **similar** to as many other unlabeled points as possible:

$$\underset{\mathbf{x} \in \mathcal{U}}{\operatorname{argmax}} \ \operatorname{acq}(f, \mathbf{x}) \cdot \left(\frac{1}{|\mathcal{U}|} \sum_{\mathbf{x}' \in \mathcal{U}} \operatorname{sim}(\mathbf{x}, \mathbf{x}') \right)^{\beta} \tag{65}$$

where:

- $\operatorname{acq}(f, \mathbf{x})$ is a "standard" acquisition function based on, e.g., pointwise uncertainty.
- $\operatorname{sim}(\mathbf{x}, \mathbf{x}')$ measures the **similarity** between $\mathbf{x}$ and $\mathbf{x}'$, e.g., a Gaussian kernel, Pearson's correlation coefficient, Spearman's rank correlation. **Application specific**.
- $\beta > 0$ is a hyper-parameter

Intuitively, $\mathbf{x}$'s label conveys information about the label on the other points in $\mathcal{U}$

Example

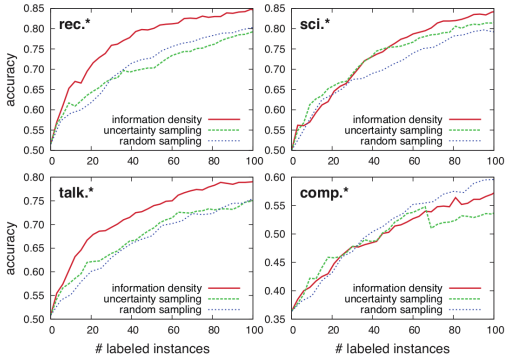

Figure 5.2: Learning curves showing that, by explicitly weighting queries by their representativeness among the input instances, information density can yield better results than the base uncertainty sampling heuristic by itself.

- We optimize:

$$\underset{\mathbf{x}\in\mathcal{U}}{\operatorname{argmax}}\ \operatorname{acq}(f,\mathbf{x})\cdot\left(\frac{1}{|\mathcal{U}|}\sum_{\mathbf{x}'\in\mathcal{U}}\operatorname{sim}(\mathbf{x},\mathbf{x}')\right)^{\beta} \tag{66}$$

Properties:

- Tends to work better than pure more "local" acquisition functions (Settles, 2012)
- Even when uncertainty sampling is worse than random, information density performs well

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- Similarity computation can be sped-up using **caching**: "simply" store similarity matrix $S_{ij} = [\operatorname{sim}(\mathbf{x}_i, \mathbf{x}_j)]$ for all $\mathbf{x}_i, \mathbf{x}_j \in \mathcal{U}$ (only needs to be done **once**)

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- Approximate using **clustering**: cluster $\mathcal{U}$ so that points within cluster are similar and points across clusters are not $\rightarrow$ block-diagonal similarity matrix, lowers storage requirement from $O(|\mathcal{U}|^2)$ to $O(\sum_i |\text{cluster}_i|^2)$

- Do we gain anything by “summarizing” the data using **clustering**?

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Idea:

- Cluster unlabeled data set $\mathcal{U} \rightarrow \{C_i \subset \mathcal{U} \ : \ i \in [k]\}$
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Problems:

- $\mathcal{U}$ may not have a good clustering structure or $\mathrm{sim}(\cdot, \cdot)$ may not be able to capture it
- How many clusters and at what granularity?
- Clusters of $\mathbf{x}$'s may not correlate well with label y.

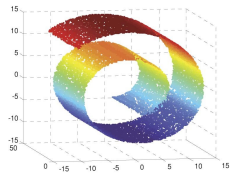

Figure: the swiss roll dataset has no obvious clustering structure.

Extensions

- Consider a neural network $f_\theta : \mathbb{R}^d \to [c]$:

$$f_\theta(\mathbf{x}) = \operatorname*{argmax}_{y \in [c]} \; p_\theta(y \mid \mathbf{x})$$

$$p_\theta(y \mid \mathbf{x}) = \mathrm{softmax}(W \phi_\omega(\mathbf{x}))_y$$

where:

- $\theta = \{W, \omega\}$ are parameters
- $\phi_\omega : \mathbb{R}^d \to \mathbb{R}^k$ is an embedding function (e.g., convolutions + pooling layers)
- $W \in \mathbb{R}^{c \times k}$ are the parameters of the top dense layer

Deep Architectures

- Deep NNs have a number of **quirks**:

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  - **Expensive** to fit on data: training a ResNet on a realistic data set can take minutes to days → hard to ensure responsivity
  - Quite **insensitive** to the addition of a single example → what's the point of querying individual instances?
  - Training is **stochastic** (i.e., not 100% stable) → changes in performance can depend on factors other than new labeled examples, high variance

Overconfidence

Problem: Deep NNs tend to be very **overconfident** even away from the training set $\rightarrow$ their *uncertainty* cannot be trusted

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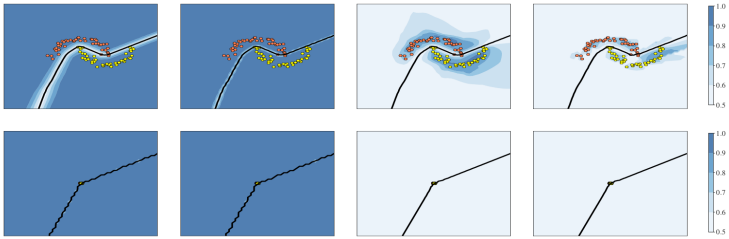

(a) MAP (b) Temp. scaling (c) Bayesian (last-layer) (d) Bayesian (all-layer)

Figure 1. Binary classification on a toy dataset using a MAP estimate, temperature scaling, and both last-layer and all-layer Gaussian approximations over the weights which are obtained via Laplace approximations. Background color and black line represent confidence and decision boundary, respectively. Bottom row shows a zoomed-out view of the top row. The Bayesian approximations—even in the last-layer case—give desirable uncertainty estimates: confident close to the training data and uncertain otherwise. MAP and temperature scaling yield overconfident predictions. The optimal temperature is picked as in Guo et al. (2017).

Credit: (Kristiadi et al., 2020).

Aleatoric vs Epistemic (Hüllermeier and Waegeman, 2021)

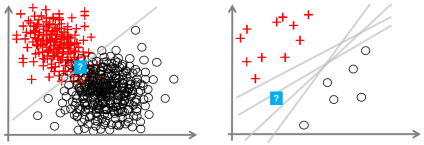

Figure 5: Left: Even with precise knowledge about the optimal hypothesis, the prediction at the query point (indicated by a question mark) is aleatorically uncertain, because the two classes are overlapping in that region. Right: A case of epistemic uncertainty due to a lack of knowledge about the right hypothesis, which is in turn caused by a lack of data.

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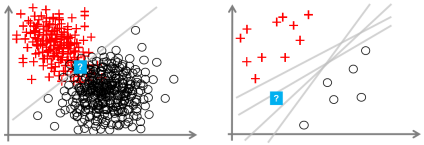

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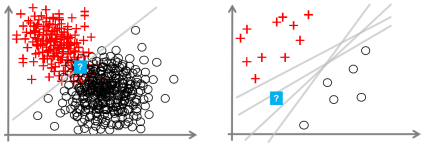

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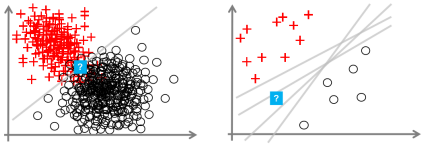

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- **Epistemic** uncertainty ("relating to knowledge") captures how little we know about the world. This reflects on uncertainty on the choice of θ. It decreases by acquiring **more data**. (**right**)
- There isn't much point in trying to reduce **aleatoric** uncertainty in AL (Sharma and Bilgic, 2017)

Bayesian NNs

- The problem with NNs is that uncertainty depends on a *single model*:
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- Compute predictions by marginalizing over θ:

$$p(y \mid \mathbf{x}) = \int \underbrace{p(y \mid \mathbf{x}, \theta)}_{\text{NN with params } \theta} \cdot \underbrace{p(\theta \mid \mathcal{L})}_{\text{posterior over params}} \, d\theta \tag{67}$$

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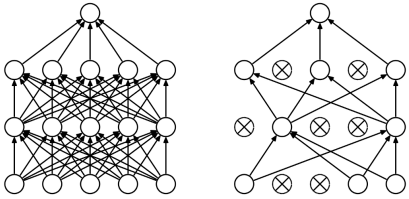
- Learn by updating distribution:

$$p(\theta \mid \mathcal{L}) \quad \rightarrow \quad p(\theta \mid \mathcal{L} \cup \{(\mathbf{x}, y)\}) \tag{68}$$

Not trivial! Is there an **efficient approximation**?

Dropout

- Randomly set nodes to 0 with a fixed probability.



(a) Standard Neural Net (b) After applying dropout.

Figure 1: Dropout Neural Net Model. **Left**: A standard neural net with 2 hidden layers. **Right**: An example of a thinned net produced by applying dropout to the network on the left. Crossed units have been dropped.

- Used as a **regularization technique**: by randomly removing neurons, prevents them from relying on each other "too much"

Dropout as Bayesian Approximation

- Computing class probabilities:

$$p(y \mid \mathbf{x}, \mathcal{L}) = \int p(y \mid \mathbf{x}, \boldsymbol{\theta}) p(\boldsymbol{\theta} \mid \mathcal{L}) d\boldsymbol{\theta} \tag{69}$$

$$\approx \int p(y \mid \mathbf{x}, \boldsymbol{\theta}) p_{\text{dropout}}(\boldsymbol{\theta}) d\boldsymbol{\theta} \tag{70}$$

$$\approx \frac{1}{R} \sum_{r=1}^{R} p(y \mid \mathbf{x}, \widehat{\boldsymbol{\theta}}_r), \qquad \widehat{\boldsymbol{\theta}}_r \sim p_{\text{dropout}}(\boldsymbol{\theta}) \tag{71}$$

In other words, run NN R times with **dropout enabled** (during inference!) then average the R vectors of class probabilities.

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- Immediately leads to **more calibrated output probabilities**!

BALD (Gal et al., 2017)

Question: does dropout help with query selection too?

BALD (Gal et al., 2017)

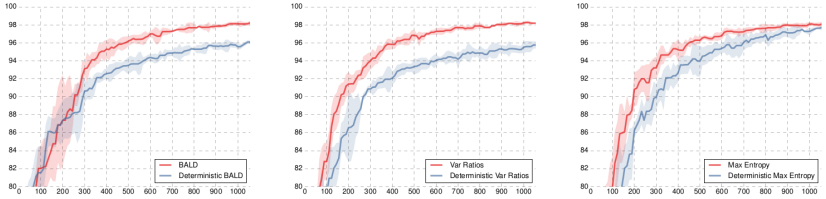
Question: does dropout help with query selection too? **Yes.**

- Uncertainty sampling:

$$\mathrm{acq}_{UNC}(\mathbf{x}) = -\sum_{y \in [c]} p(Y = y \mid \mathbf{x}, \mathcal{L}) \log p(Y = y \mid \mathbf{x}, \mathcal{L}) \tag{72}$$

Simply run the NN multiple times on your input $\mathbf{x}$ with different (random) dropout masks, then average the resulting probabilities.

Illustration



(a) BALD (b) Var Ratios (c) Max Entropy

Figure 2. Test accuracy as a function of number of acquired images for various acquisition functions, using both a **Bayesian CNN (red) and a deterministic CNN (blue)**.

- For all choices of acquisition function, the dropout-based uncertainty helps!

- Let us look at **batch-based** active learning.

Batch Selection

Given $\mathcal{L}$, $\mathcal{U}$ and a classifier $f \in \mathcal{F}$ trained on $\mathcal{L}$, find a batch $B \subseteq \mathcal{U}$ of $b \gg 1$ unlabeled instances that brings maximal information to the model:

$$\underset{B \subseteq \mathcal{U}}{\operatorname{argmax}} \ \operatorname{acq}_{BALD}(f, B) \tag{73}$$

$$\text{s.t. } |B| = b \tag{74}$$

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$$\text{s.t. } |B| = b \tag{74}$$

Advantages:

- Only retrain the model after ever b examples, meaning that supervision has an effect.
- Retraining is less frequent, leading to faster overall execution (at the expense of possibly instance selection, because b examples depend on a fixed f).
- Supports parallel annotation for, e.g., crowd-sourcing scenarios.

Question: can regular acquisition function (like BALD) be extended to this setting?

- Natural generalization of instance-level strategies:

$$\mathrm{acq}(f, B) = \sum_{\mathbf{x} \in B} \mathrm{acq}(f, \mathbf{x}) \tag{75}$$

How well does this work?

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$$\mathrm{acq}(f, B) = \sum_{\mathbf{x} \in B} \mathrm{acq}(f, \mathbf{x}) \tag{75}$$

How well does this work?

- This ignores *correlation* between instances in $\mathbf{x}$:
  - Even if all of them are informative, they may **carry the same information**
  - We want B to be informative **as a whole**!

Illustration

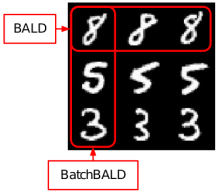

Figure 1: *Idealised acquisitions of BALD and BatchBALD.* If a dataset were to contain many (near) replicas for each data point, then BALD would select all replicas of a single informative data point at the expense of other informative data points, wasting data efficiency.

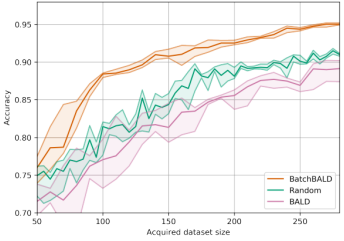

Figure 2: *Performance on* Repeated MNIST *with acquisition size 10.* See section 4.1 for further details. BatchBALD outperforms BALD while BALD performs worse than random acquisition due to the replications in the dataset.

(Credit: (Kirsch et al., 2019).)

- The problem with the "natural generalization":

$$\operatorname{acq}(f, B) = \sum_{\mathbf{x} \in B} \operatorname{acq}(f, \mathbf{x}) \tag{76}$$

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Idea: don't break the acquisition function into a sum! For BALD, this means replacing:

$$\sum_{\mathbf{x} \in B} \Big\{ \underbrace{H(Y \mid \mathbf{x}, \mathcal{L}) - \mathbb{E}_{\boldsymbol{\theta} \sim p(\boldsymbol{\theta} \mid \mathcal{L})}[H(Y \mid \mathbf{x}, \boldsymbol{\theta})]}_{MI(Y, \Theta \mid \mathbf{x}, \mathcal{L})} \Big\} \tag{77}$$

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- In other words, **don't assume independence** between the elements of B!

Illustration

(a) BALD **(b) BatchBALD**

Figure 3: *Intuition behind* BALD *and* BatchBALD *using I-diagrams [30]. BALD* overestimates the joint mutual information. *BatchBALD*, however, takes the overlap between variables into account and will strive to acquire a better cover of $\boldsymbol{\omega}$. Areas contributing to the respective score are shown in grey, and areas that are double-counted in dark grey.

(Credit: (Kirsch et al., 2019).)

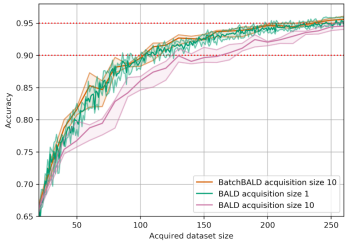

Figure 5: *Performance on* MNIST. BatchBALD outperforms BALD with acquisition size 10 and performs close to the optimum of acquisition size 1.

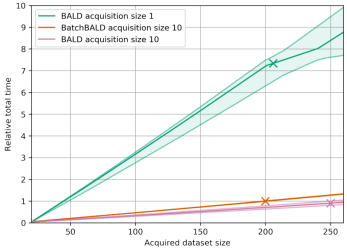

Figure 6: *Relative total time on* MNIST. Normalized to training BatchBALD with acquisition size 10 to 95% accuracy. The stars mark when 95% accuracy is reached for each method.

(Credit: (Kirsch et al., 2019).)

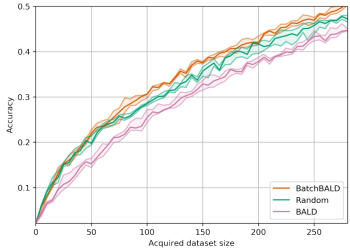

Figure 7: *Performance on* EMNIST. BatchBALD consistently outperforms both random acquisition and BALD while BALD is unable to beat random acquisition.

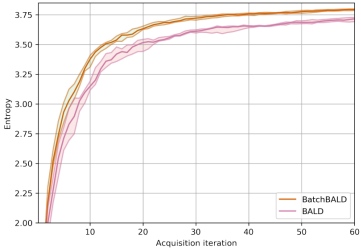

Figure 8: *Entropy of acquired class labels over acquisition steps on* EMNIST. BatchBALD steadily acquires a more diverse set of data points.

(Credit: (Kirsch et al., 2019).)

Conclusion and Further Reading

Take-away

- AL useful when supervision is expensive high $\rightarrow$ choose it wisely
- Many variants: pool-based, streaming, and query synthesis
- Many practical approaches: **uncertainty**-based (uncertainty sampling, QBC, expected gradient length), **diversity**-based (information density).

Some can be derived from *version spaces* and *model improvement*.

- Deep variants select entire **batches** and often rely on Bayesian techniques
- Critique & realistic annotators, costs, *etc.*: (Herde et al., 2021) (Settles, 2011)
- Plenty of room for new research ;-)

References

Baum, E. B. and Lang, K. (1992). Query learning can work poorly when a human oracle is used. In *International joint conference on neural networks*, volume 8, page 8.

Gal, Y. and Ghahramani, Z. (2016). Dropout as a bayesian approximation: Representing model uncertainty in deep learning. In *international conference on machine learning*, pages 1050–1059. PMLR.

Gal, Y., Islam, R., and Ghahramani, Z. (2017). Deep bayesian active learning with image data. In *International Conference on Machine Learning*, pages 1183–1192. PMLR.

Herde, M., Huseljic, D., Sick, B., and Calma, A. (2021). A survey on cost types, interaction schemes, and annotator performance models in selection algorithms for active learning in classification. *arXiv preprint arXiv:2109.11301*.

Hüllermeier, E. and Waegeman, W. (2021). Aleatoric and epistemic uncertainty in machine learning: An introduction to concepts and methods. *Machine Learning*, 110(3):457–506.

King, R. D., Rowland, J., Oliver, S. G., Young, M., Aubrey, W., Byrne, E., Liakata, M., Markham, M., Pir, P., Soldatova, L. N., et al. (2009). The automation of science. *Science*, 324(5923):85–89.

Kirsch, A., Van Amersfoort, J., and Gal, Y. (2019). Batchbald: Efficient and diverse batch acquisition for deep bayesian active learning. *Advances in neural information processing systems*, 32:7026–7037.

Kristiadi, A., Hein, M., and Hennig, P. (2020). Being bayesian, even just a bit, fixes overconfidence in relu networks. In *International Conference on Machine Learning*, pages 5436–5446. PMLR.

Nguyen, A., Dosovitskiy, A., Yosinski, J., Brox, T., and Clune, J. (2016). Synthesizing the preferred inputs for neurons in neural networks via deep generator networks. *Advances in neural information processing systems*, 29:3387–3395.

Settles, B. (2011). From theories to queries: Active learning in practice. In *Active Learning and Experimental Design workshop In conjunction with AISTATS 2010*, pages 1–18. JMLR Workshop and Conference Proceedings.

Settles, B. (2012). *Active learning*.

Shalev-Shwartz, S. and Ben-David, S. (2014). *Understanding machine learning: From theory to algorithms*. Cambridge university press.

Sharma, M. and Bilgic, M. (2017). Evidence-based uncertainty sampling for active learning. *Data Mining and Knowledge Discovery*, 31(1):164–202.