Scientific Programming

Lecture A06 – Recursion

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Acknowledgments: Alberto Montresor

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Recursion

"Of all ideas I have introduced to children, recursion stands out as the one idea that is particularly able to evoke an excited response."
— Seymour Papert, Mindstorms

Goal of this lecture

- To learn how to formulate programs recursively
- To understand and apply the three laws of recursion
- To understand how recursion is handled by a computer system
- To understand that complex problems that may otherwise be difficult to solve, may be solved by splitting them in sub-problems
- To understand that sometimes, a recursive approach may lead to more efficient algorithms

Recursion

Definition

Recursion is the process a function goes through when one of the steps of the function involves invoking the function itself, on a smaller input. A function that goes through recursion is said to be recursive.

Recursion involves a function that calls itself

- Several mathematical functions are defined recursively
- Several problems can be defined recursively
- Some problems may be solved more efficiently with a recursive approach

Example: Factorial numbers

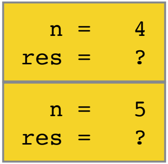
$$n! = \begin{cases} 1 & n = 1 \\ n \cdot (n-1)! & n > 1 \end{cases}$$

```
def fact(n):
    if n <= 1:
        res = 1
    else:
        res = n * fact(n-1)
    return res
```

Example: Factorial numbers

```
  n =   5
res =   ?
```

Example: Factorial numbers



Example: Factorial numbers

```
  n =   3
res =   ?
---------
  n =   4
res =   ?
---------
  n =   5
res =   ?
```

Example: Factorial numbers

n =	2
res =	?
n =	3
res =	?
n =	4
res =	?
n =	5
res =	?

Example: Factorial numbers

```
  n =   1
res =   1
---------
  n =   2
res =   ?
---------
  n =   3
res =   ?
---------
  n =   4
res =   ?
---------
  n =   5
res =   ?
```

Example: Factorial numbers

n	=	2
res	=	2
n	=	3
res	=	?
n	=	4
res	=	?
n	=	5
res	=	?

Example: Factorial numbers

```
  n =   3
res =   6
---------
  n =   4
res =   ?
---------
  n =   5
res =   ?
```

Example: Factorial numbers

```
  n =  4
res = 24

  n =  5
res =  ?
```

Example: Factorial numbers

```
  n =   5
res = 120
```

Example: Factorial numbers

```
def fact(n):
    print("Start", n)
    if n <= 1:
        res = 1
    else:
        res = n*fact(n-1)
    print("End", n, res)
    return res

print(fact(5))
```

Example: Factorial numbers

```
def fact(n):
    print("Start", n)
    if n <= 1:
        res = 1
    else:
        res = n*fact(n-1)
    print("End", n, res)
    return res

print(fact(5))
```

```
Start 5
Start 4
Start 3
Start 2
Start 1
End 1 1
End 2 2
End 3 6
End 4 24
End 5 120
```

The three laws of recursion

All recursive algorithms must obey three important laws

- A recursive algorithm must have a base case
- A recursive algorithm must call itself, recursively
- A recursive algorithm must move toward the base case

What happens with this code?

```
def fact(n):
    return n*fact(n-1)
```

Divide-et-impera

Three phases

- Divide: Break the problem in smaller and independent sub-problems
- Impera: Solve the sub-problems recursively
- Combine: "merge" the solutions of subproblems

There is not a unique recipe for divide-et-impera

- A creative effort is required

Minimum – Recursive version 1

```
def minrec(A, i):
    if i==0:
        res = A[0]
    else:
        res = min(minrec(A, i-1), A[i])
    return res

def mymin(A):
    return minrec(A, len(A)-1)
```

- `minrec()` returns the minimum of the elements between 0 and i, both included.
- `mymin()` is a wrapper to hide the recursion from the caller

Minimum – Recursive version 2

```
def minrec(A, i, j):
    if i==j:
        res = A[i]
    else:
        m = (i+j) // 2
        res = min(minrec(A, i, m), minrec(A,m+1,j))
    return res

def mymin(A):
    return minrec(A, 0, len(A)-1)
```

- `minrec()` returns the minimum of the elements between i and j, both included.
- `mymin()` is a wrapper to hide the recursion from the caller

Minimum – Recursive version 2 - Debug

```python
def minrec(A, i, j):
    print("Start", i, j)
    if i==j:
        res = A[i]
    else:
        m = (i+j) // 2
        res = min(minrec(A, i, m),
            minrec(A, m+1, j))
        print("COMPARE")
    print("End", i, j)
    return res

def mymin(A):
    return minrec(A, 0, len(A)-1)

L = [2, 4, 1, 3, 6, 8, 9, 12]
m = mymin(L)
```

```
Start 0 7
  Start 0 3
    Start 0 1
      Start 0 0
      End 0 0
      Start 1 1
      End 1 1
    COMPARE
    End 0 1
    Start 2 3
      Start 2 2
      End 2 2
      Start 3 3
      End 3 3
    COMPARE
    End 2 3
  COMPARE
  End 0 3
  Start 4 7
    Start 4 5
      Start 4 4
      End 4 4
      Start 5 5
      End 5 5
    COMPARE
    End 4 5
    Start 6 7
      Start 6 6
      End 6 6
      Start 7 7
      End 7 7
    COMPARE
    End 6 7
  COMPARE
  End 4 7
COMPARE
End 0 7
```

So far, you probably don't see why recursion is good...

Factorial can be defined recursively, but also "iteratively":

$$n! = \prod_{i=1}^{n} i$$

```
def fact(n):
    res = 1
    for k in range(1, n + 1):
        res = res * k
    return res
```

So far, you probably don't see why recursion is good...

Minimum can be defined recursively, but also "iteratively"

```
def mymin(A):
    res =  A[0]
    for x in A:
        res = min(res,x)
    return res
```

So far, you probably don't see why recursion is good...

Sometimes there is no advantage in performance for the recursive versions

- Both versions of `fact` require $n-1$ products,
- Both versions of `mymin` require $n-1$ comparisons, where n is the number of items in the input
- Note: Executing the recursive invocations is more costly than executing the iterations.

Version	Time (ms)
Recursive V2	645.60
Iterative	195.03
`min()` Python	17.88

Cost of `min`(10^6 integers)

So far, you probably don't see why recursion is good...

Recursion may even be dangerous

```
def minrec(L):
    if len(L) == 1:
        return L[0]
    else:
        return min(L[0], minrec(L[1:]))
```

```
RecursionError: maximum recursion depth exceeded in comparison
```

Why recursion?

- Recursion may help in solving problems that are very difficult to attack otherwise
- Recursion may lead to much more efficient algorithms, at least for very large input size

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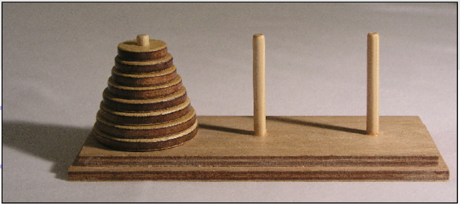
Hanoi's tower



Mathematical game

- Three pins
- n disks with different sizes
- Initially, all the disks are stacked in decreasing size order (from bottom to top) on the left pin

Goal of the game

- Stack all the disks on the right pin in decreasing size order (from bottom to top)
- Never put a larger disk on top of a smaller disk
- You can move one disk at each step
- You can use the middle pin to as support

Hanoi's tower

```
def hanoi(n, src, dst, mid):
    if n == 1:
        print(src, "-->", dst)
    else:
        hanoi(n-1, src, mid, dst)
        print(src, "-->", dst)
        hanoi(n-1, mid, dst, src)
```

Divide-et-impera

- $n-1$ disks from *src* to *middle*
- 1 disks from *src* to *dest*
- $n-1$ disks from *middle* to *dest*

https://it.wikipedia.org/wiki/Torre_di_Hanoi#/media/File:Tower_of_Hanoi_4.gif

Hanoi's tower

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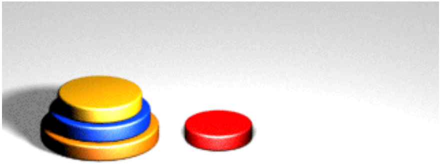

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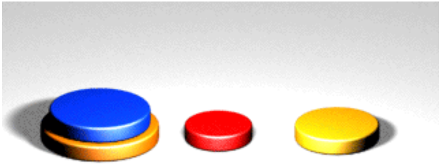

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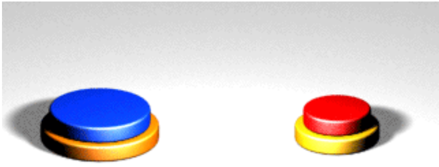

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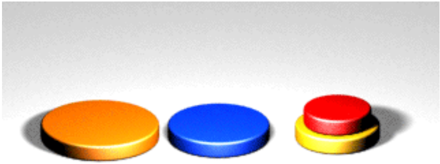

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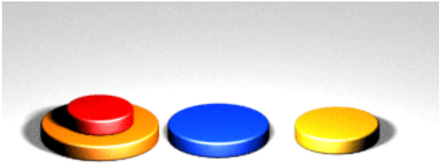

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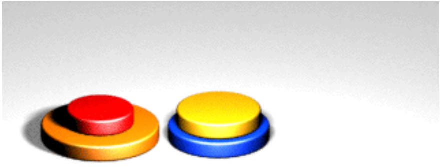

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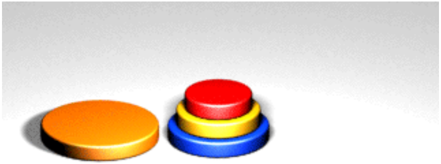

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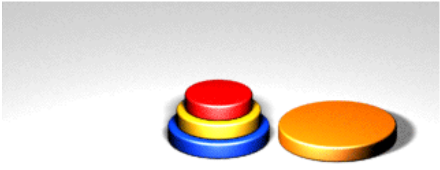

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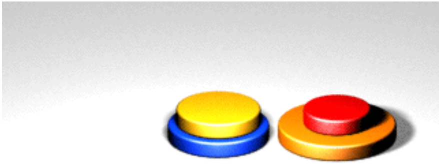

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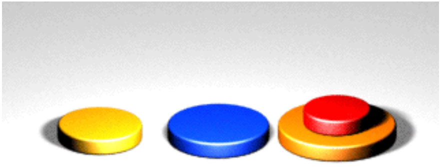

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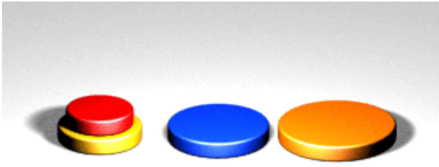

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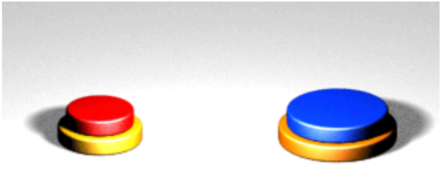

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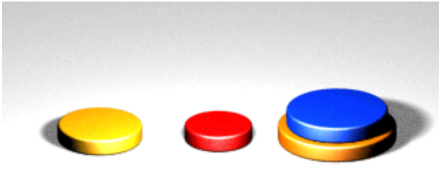

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Hanoi's tower - Bonus version

```python
def hanoi(n, src, mid, dst):
    if n == 1:
        dst.append(src.pop())
        print(rsrc, rmid, rdst)
    else:
        hanoi(n-1, src, dst, mid)
        dst.append(src.pop())
        print(rsrc, rmid, rdst)
        hanoi(n-1, mid, src, dst)

rsrc = [4,3,2,1]
rmid = []
rdst = []
print(rsrc, rmid, rdst)
hanoi(len(rsrc), rsrc, rmid, rdst)
```

```
[4, 3, 2, 1] [] []
[4, 3, 2] [1] []
[4, 3] [1] [2]
[4, 3] [] [2, 1]
[4] [3] [2, 1]
[4, 1] [3] [2]
[4, 1] [3, 2] []
[4] [3, 2, 1] []
[] [3, 2, 1] [4]
[] [3, 2] [4, 1]
[2] [3] [4, 1]
[2, 1] [3] [4]
[2, 1] [] [4, 3]
[2] [1] [4, 3]
[] [1] [4, 3, 2]
[] [] [4, 3, 2, 1]
```

Hanoi's tower - Comments

- The number of moves that are performed by this algorithm is equal to $2^n - 1$
- This number is optimal: you cannot solve this problem in a smaller number of moves
- While there exist iterative (non-recursive) solutions, none of them is as clear as the one presented here.

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Search: problem definition

Search over a sorted list

Let $S = s_0, s_2, \ldots, s_{n-1}$ be a list of distinct, sorted numbers, i.e. $s_0 < s_1 < \ldots < s_{n-1}$.
Searching the position of value v in S corresponds to returning the index i such that $0 \leq i < n$, if v is contained at position i, -1 otherwise.

$$index(S, v) = \begin{cases} i & \exists i \in \{0, \ldots, n-1\} : S_i = v \\ -1 & \text{otherwise} \end{cases}$$

First version – Iterative

```
def index(L, v):
    for i in range(len(L)):
        if L[i] == v:
            return i
    return -1
```

A better solution - Binary (dichotomic) search

A more efficient solution

Let's consider the median m element of the list

- If $L[m] = v$, the looked-up element has been found
- If $v < L[m]$, look in the "left part"
- If $L[m] < v$, look in the "right part"

1	5	12	15	20	23	32

21?

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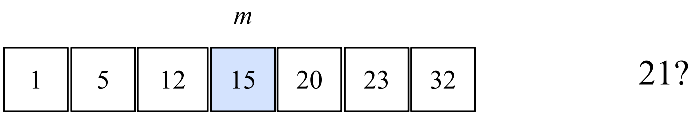

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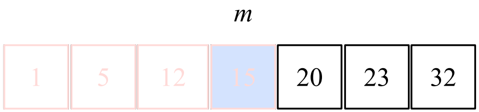

21?

A better solution - Binary (dichotomic) search

A more efficient solution

Let's consider the median m element of the list

- If $L[m] = v$, the looked-up element has been found
- If $v < L[m]$, look in the "left part"
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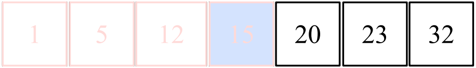

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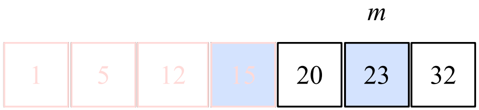

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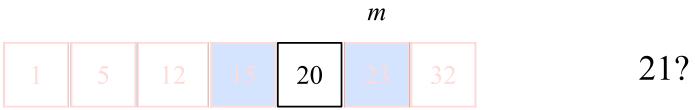

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1	5	12	15	20	23	32

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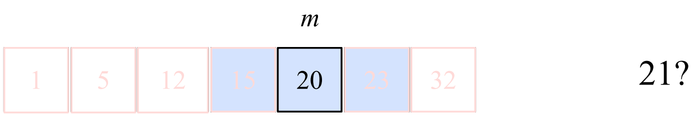


Second version – Recursive

```
def index_rec(L, i, j, v):
    print("L[",i,":",j,"]", sep="")
    if (j<i):
        return -1
    else:
        m = (i+j) // 2
        if L[m] == v:
            return m
        elif L[m] < v:
            return index_rec(L, m+1, j, v)
        else:
            return index_rec(L, i, m-1, v)

L = list(range(1000))
print(index_rec(L, 0, len(L)-1, 1000))
```

```
L[0:999]
L[500:999]
L[750:999]
L[875:999]
L[938:999]
L[969:999]
L[985:999]
L[993:999]
L[997:999]
L[999:999]
L[1000:999]
-1
```

Performance evaluation

Cost of execution of (i) `index()` method of Python lists, (ii) iterative version, (iii) recursive version over list of increasing size n. Note the different units of measures (ms versus μs).

The number of comparisons is proportional to $\log_2 n$.

n	`list.index` **(ms)**	**Iterative (ms)**	**Recursive (μs)**
10^3	0.01	0.04	2.14
10^4	0.10	0.37	2.90
10^5	0.98	3.67	3.51
10^6	9.17	36.95	4.22
10^7	91.82	364.61	5.07
10^8	920.67	3633.57	5.70

Table of contents

Gap: Problem definition

Gap

In a list L containing $n \geq 2$ integers, a gap is an index i, $0 < i < n$, such that $L[i-1] < L[i]$.

- Prove that if $n \geq 2$ and $L[0] < L[n-1]$, L contains at least one gap
- Design an algorithm that, given a list L containing $n \geq 2$ integers such that $L[0] < L[n-1]$, finds a gap in the list.

Gap – Proof by contradiction

By contradiction:

- Suppose there is no gap in the list.
- Then $L[0] \geq L[1] \geq L[2] \geq \ldots L[n-1]$, which contradicts the fact that $L[0] < L[n-1]$.

First version – Iterative

```
def gap(L):
    for i in range(1,len(L)):
        if L[i-1]<L[i]:
            return i
    return -1    # Never reached under the assumptions

L = list(range(100,0,-1))
L.append(101)
print(L)
print(gap(L))
```

First version – Iterative

```
def gap(L):
    for i in range(1,len(L)):
        if L[i-1]<L[i]:
            return i
    return -1    # Never reached under the assumptions

L = list(range(100,0,-1))
L.append(101)
print(L)
print(gap(L))
```

```
[100, 99, 98, 97, ..., 3, 2, 1, 101]
100
```

Gap – Proof by induction

Let's reformulate the property in this way:

- Let L be a list of size n
- Let i, j be two indexes, such that $0 \leq i < j < n$ and $L[i] < L[j]$

In other words, there are more than two elements in the slice $L[i : j + 1]$ and the first element $L[i]$ is smaller than the last $L[j]$.

Gap – Proof by induction

We want to prove by induction on the size n of the slice that the slice contains a gap.

- Base case: $n = j - i + 1 = 2$, i.e. $j = i + 1$:
  $L[i] < L[j]$ implies that $L[i] < L[i+1]$, which is a gap.
- Inductive ipothesis: there is a gap in any slice of L smaller than n, where the first element is smaller than the last element.
- Inductive step: let's consider any element m such that $i < m < j$. There are two cases:
  - If $L[m] \leq L[i] < L[j]$, then there is a gap between m and j, as the slice $L[m : j+1]$ is smaller than n.
  - If $L[i] < L[m]$, then there is a gap between i and m, as the slice $L[i : m+1]$ is smaller than n

Second version – Recursive

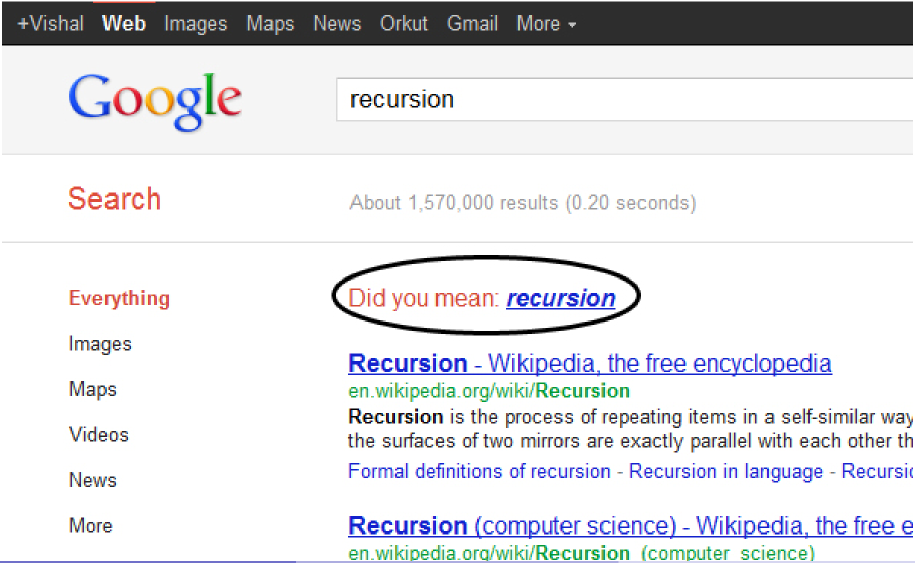
```
def gaprec(L, i, j):
    if j == i+1:
        return j
    else:
        m = (i+j) // 2
        if L[m] < L[j]:
            return gaprec(L,m,j)
        else:
            return gaprec(L,i,m)

def gap(L):
    return gaprec(L,0,len(L)-1)
```

Performance evaluation

Cost of execution of the iterative and recursive version of `gap()` over list of increasing size n. Note the different units of measures (ms versus μs).

n	**Iterative (ms)**	**Recursive (μs)**
10^3	0.06	2.05
10^4	0.61	2.78
10^5	6.11	3.36
10^6	62.44	4.01
10^7	621.69	4.87
10^8	6205.72	5.47

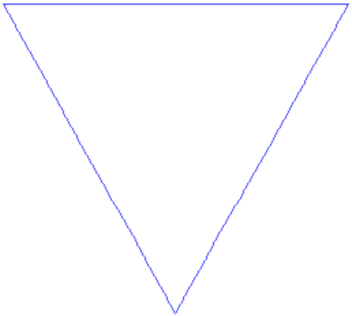
Recursion: see Recursion



Recursion: see Recursion

The Koch snowflake is a mathematical curve and one of the earliest fractal curves to have been described.

This particular curve has a finite area and a perimeter that tends to infinity.



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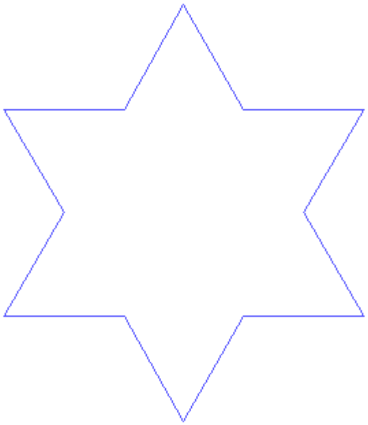

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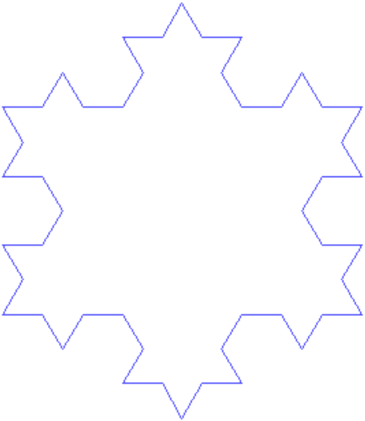

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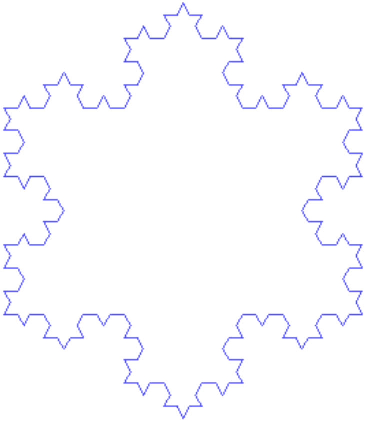

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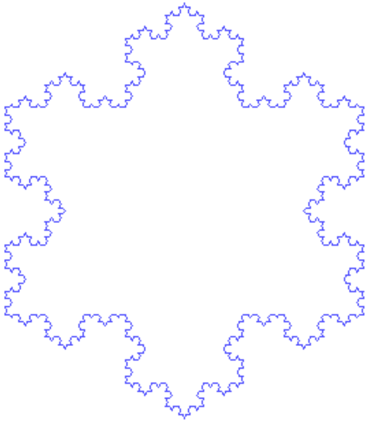

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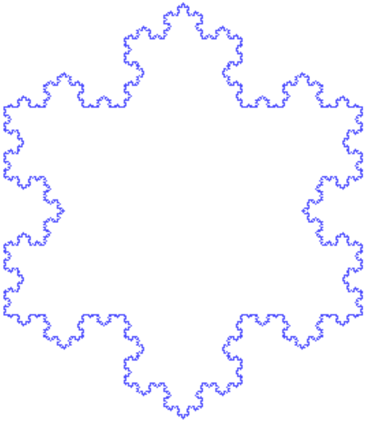

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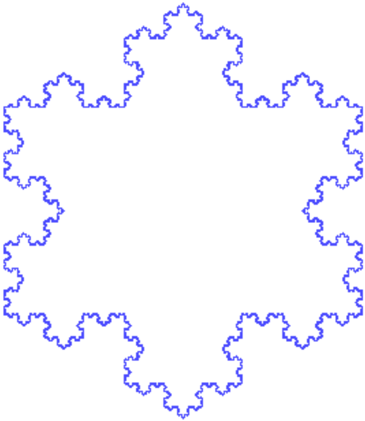

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