Mathematical Logic

Introduction on Modal Logics

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TestBooks and Readings

- *Hughes, G. E., and M.J. Cresswell (1996) A New Introduction to Modal Logic. Routledge.*
  Introductory textbook. Provides an historic perspective and a lot of explanations.
- *Blackburn, Patrick, Maarten de Rijke, and Yde Venema (2001) Modal Logic. Cambridge Univ. Press*
  More modern approach. It focuses on the formalisation of frames and structures.
- *Chellas, B. F. (1980) Modal Logic: An Introduction. Cambridge Univ. Press*
  The focus is on the axiomatization of the modal operators $\Box$ and $\Diamond$.

Origins of modal logics

- (Modern modal logic) Developed in the early twentieth century,
- **Clarence Irving Lewis**, thought that Russell's description of the truth-functional conditional operator as material implication (i.e, $A \supset B$ is true if either A is false or B is true) was misleading.
  He suggested to define a new form of implication called strict implication which literally can be seen like this

$$\underline{\text{it is not possible that}}\ A \text{ is true and } B \text{ is false} \tag{1}$$

- He proposed to formalise (1) as

$$\neg\Diamond(A \wedge \neg B) \tag{2}$$

Origins of modal logics - ctn'd

The novelties in $\neg\Diamond(A \wedge \neg B)$ are:

- A modal operator $\Diamond$ for representing the fact that a statement is *possibly true* (*impossible*, *necessary*, ...)
- The fact that the truth value of $\neg\Diamond(A \wedge \neg B)$ is not a function of the truth values of A and B as it refers to a set of *possible situations* (lately called possible worlds) in which you have to consider the truth of A and B.

What is Modality?

- A modality is an expression that is used to *qualify* the truth of a judgement (or, in other words, an operator that expresses a "mode" in which a proposition is true)
- It can be seen as an operator that takes a proposition and returns a more complex proposition.

Example

Proposition	Modal Expression
John drives a Ferrari	John *is able to* drive a Ferrari
Everybody pays taxes	It is *obligatory* that everybody pays taxes

- Modalities are expressed in natural language through **modal verbs** such as *can/could, may/might, must, will/would*, and *shall/should*.

What is Modality?

- In logic modalities are formalized using an operator such as $\Box$ ($\Diamond$) that can be applied to a formula ϕ to obtain another formula $\Box\phi$ ($\Diamond\phi$).
- The truth value of $\Box\phi$ is not a function of the truth value of ϕ.

Example

- The fact that John is able to drive a Ferrari may be true independently from the fact that John is actually driving a Ferrari.
- The fact that it is *obligatory* that everybody pays taxes is typically true, and this is independent from the fact that everybody actually pays taxes.

Note: $\neg$ is not a modal operator since the truth value of $\neg\phi$ is a function of the truth value of ϕ.

Modalities

- A modality is an expression that is used to *qualify* the truth of a judgement.
- Historically, the first modalities formalized with modal logic were the so called alethic modalities i.e.,
  1. it is possible that a certain proposition holds, usually denoted with $\Diamond\phi$
  2. it is necessary that a certain proposition holds, usually denoted with $\Box\phi$
- Afterwards a number of modal logics for different "*qualifications*" have been studied. The most common are...

Modalities

Modality	Symbol	Expression Symbolised
Alethic	$\Box\phi$	it is *necessary* that ϕ
	$\Diamond\phi$	it is *possible* that ϕ
Deontic	$\mathrm{O}\phi$	it is *obligatory* that ϕ
	$\mathrm{P}\phi$	it is *permitted* that ϕ
	$\mathrm{F}\phi$	it is *forbidden* that ϕ
Temporal	$\mathrm{G}\phi$	it will *always* be the case that ϕ
	$\mathrm{F}\phi$	it will *eventually* be the case that ϕ
Epistemic	$\mathrm{B}_a\phi$	agent *a believes* that ϕ
	$\mathrm{K}_a\phi$	agent *a knows* that ϕ
Contextual	$\mathrm{ist}(c, \phi)$	ϕ is *true in the context c*
Dynamic	$[\alpha]\phi$	ϕ must be true after the execution of program α
	$\langle\alpha\rangle\phi$	ϕ can be true after the execution of program α
Computational	$\mathrm{AX}\phi$	ϕ is true for every immediate successor state
	$\mathrm{AG}\phi$	ϕ is true for every successor state
	$\mathrm{AF}\phi$	ϕ will eventually be true in all the possible evolutions
	$\mathrm{A}\phi\mathrm{U}\theta$	ϕ is true until θ becomes true
	$\mathrm{EX}\phi$	ϕ is true in at least one immediate successor state

Modal logics & relational structures

- Historically, modal logics were developed in order to formalise the different modalities that qualify the truth of a formula;
- Modern modal logics have a different goal. They are motivated by the study of relational structures.

Definition (Relational structure)

A relational structure is a tuple

$$\langle W, R_{a_1}, \ldots, R_{a_n} \rangle$$

where $R_{a_i} \subseteq W \times \ldots \times W$

- each $w \in W$ is called, point (world, state, time instant, situation, ...)
- each R_{a_i} is called accessibility relation (or simply relation)

Alternative notation $\langle W, R_a \rangle_{a \in A}$

The importance of relational structures

- In Computer Science, Artificial Intelligence and Knowledge Representation there are many examples of relational structures:
  - Graphs and labelled graphs;
  - Ontologies;
  - Finite state machines;
  - Computation paths; ...
- Modal logics allow us to predicate on properties of relational structures.
  - Loop detection;
  - Reachability of a (set of) node(s);
  - Properties of a relation such as Transitivity, Reflexivity, ...

Examples of Relational structures

- Strict partial order (SPO)

  $\langle W, < \rangle \quad <$ is transitive and irreflexive[1]

- Strict linear order

  $\langle W, < \rangle \quad$ (SPO) + for each $v \neq w \in W$, $v < w$ or $w < v$

- Partial order (PO)

  $\langle W, \leq \rangle \quad \leq$ is transitive, reflexive, and antisymmetric

- Linear order

  $\langle W, \leq \rangle \quad$ (PO) + for each $v, w \in W$, $v \leq w$ or $w \leq v$

- Labeled transition system (LTS)

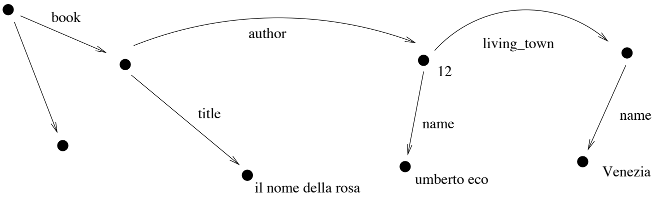
  $\langle W, R_a \rangle_{a \in A}$ and $R_a \subseteq W \times W$

- XML document

  $\langle W, R_l \rangle_{l \in L}$, W contains the components of an XML document and L is the set of labels that appear in the document

[1] Antisymmetry follows.

XML document as a relational stucture



Relational structures in FOL

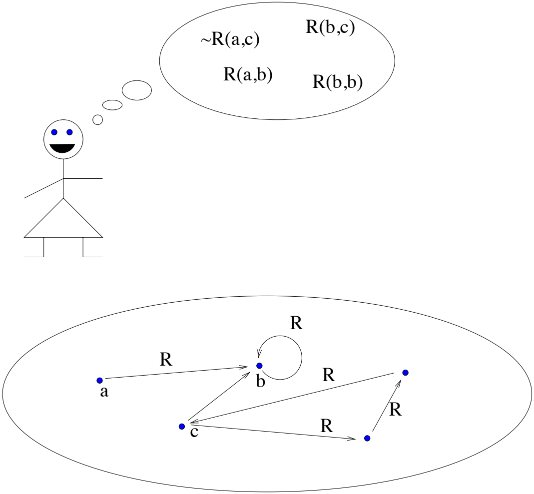
- Relational structures can be investigated in FOL;
- The language must contain at least a binary relation R, and we can formalise the properties of a relational structure using formulae such as
  - $\forall x R(x,x)$ (R is reflexive)
  - $\forall x \exists y R(x,y)$ (R is serial)
  - $\forall xy(R(x,y) \supset R(y,x))$ (R is symmetric)
  - $\ldots$
- So, why do we need modal logics?

Relational structures in first order and modal logic

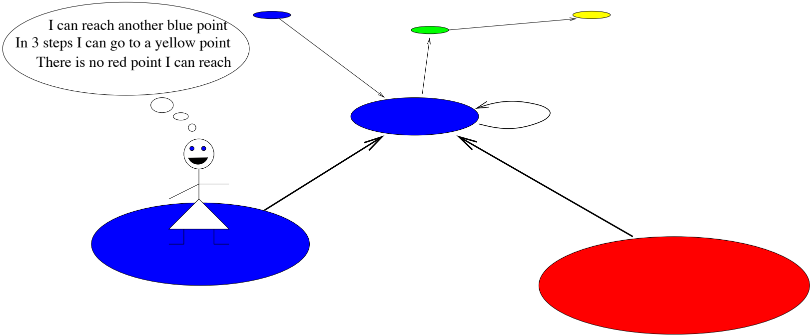
- In First Order Logic we describes a relational structure from an external point of view, (and our description is not relative to a particular point).
- Modal logics describe relational structures from an internal point of view, rather than from the top perspective
- A formula has a meaning in a point $w \in W$ of a structure

Relational structures in first order and modal logic

In first order logic, relational structures are described from the top point of view. each point of W and the relation R can be named.



In modal logics, relational structures are described from an internal perspective there is no way to mention points of W and the relation R.



An example: seriality

Let us assume to have a strict linear serial order.

- In first order logic I can observe an infinite sequence of points;
- in modal logic I know that I can always move to the next point (that is, from the point where I am I can always see (and move to) a successor point).

The Language of a basic modal logic

If $\mathcal{P}$ is a set of primitive proposition, the set of formulas of the basic modal logic is defined as follows:

- each $p \in \mathcal{P}$ is a formula (atomic formula);
- if A and B are formulas then $\neg A$, $A \wedge B$, $A \vee B$, $A \supset B$ and $A \equiv B$ are formulas
- if A is a formula $\Box A$ and $\Diamond A$ are formulas.

Intuitive interpretation of the basic modal logic

The formula $\Box\phi$ can be intuitively interpreted in many ways

- ϕ is necessarily true (classical modal logic)
- ϕ is known/believed to be true (epistemic logic)
- ϕ is provable in a theory (provability logic)
- ϕ will be always true (temporal logic)
- ...

In all these cases $\Diamond\phi$ is interpreted as $\neg\Box\neg\phi$.

In other words, $\Diamond\phi$, stands for $\neg\phi$ is not necessarily true, that is, ϕ is possibly true.

Semantics for the basic modal logic

A basic frame (or simply a frame) is an algebraic structure

$$\mathcal{F} = \langle W, R \rangle$$

where $R \subseteq W \times W$.

An interpretation $\mathcal{I}$ (or assignment) of a modal language in a frame $\mathcal{F}$, is a function

$$\mathcal{I} : P \to 2^W$$

Intuitively $w \in \mathcal{I}(p)$ means that p is true in w, or that w is of type p.

A model $\mathcal{M}$ is a pair $\langle \mathit{frame}, \mathit{interpretation} \rangle$. I.e.:

$$\mathcal{M} = \langle \mathcal{F}, \mathcal{I} \rangle$$

Satisfiability of modal formulas

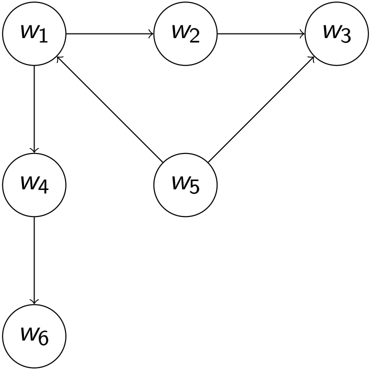
Truth is relative to a world, so we define that relation of $\models$ between a world in a model and a formula

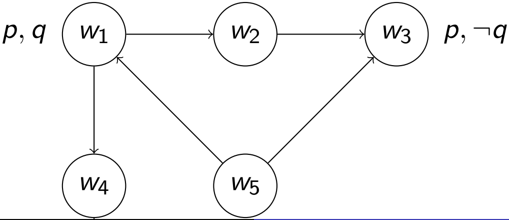
$$\mathcal{M}, w \models p \text{ iff } w \in \mathcal{I}(p)$$
$$\mathcal{M}, w \models \phi \wedge \psi \text{ iff } \mathcal{M}, w \models \phi \text{ and } \mathcal{M}, w \models \psi$$
$$\mathcal{M}, w \models \phi \vee \psi \text{ iff } \mathcal{M}, w \models \phi \text{ or } \mathcal{M}, w \models \psi$$
$$\mathcal{M}, w \models \phi \supset \psi \text{ iff } \mathcal{M}, w \models \phi \implies implies \mathcal{M}, w \models \psi$$
$$\mathcal{M}, w \models \phi \equiv \psi \text{ iff } \mathcal{M}, w \models \phi \text{ iff } \mathcal{M}, w \models \psi$$
$$\mathcal{M}, w \models \neg\phi \text{ iff not } \mathcal{M}, w \models \phi$$
$$\mathcal{M}, w \models \Box\phi \text{ iff for all } w' \text{ s.t. } wRw', \mathcal{M}, w' \models \phi$$
$$\mathcal{M}, w \models \Diamond\phi \text{ iff there is a } w' \text{ s.t. } wRw' \text{ and } \mathcal{M}, w' \models \phi$$

ϕ is globally satisfied in a model $\mathcal{M}$, in symbols, $\mathcal{M} \models \phi$ if

$$\mathcal{M}, w \models \phi \quad \text{for all } w \in W$$

Satisfiability example





Validity relation on frames

A formula ϕ is valid in a world w of a frame $\mathcal{F}$, in symbols $\mathcal{F}, w \models \phi$ iff

$$\mathcal{M}, w \models \phi \text{ for all } \mathcal{I} \text{ with } \mathcal{M} = \langle \mathcal{F}, \mathcal{I} \rangle$$

A formula ϕ is valid in a frame $\mathcal{F}$, in symbols $\mathcal{F} \models \phi$ iff

$$\mathcal{F}, w \models \phi \text{ for all } w \in W$$

If C is a class of frames, then a formula ϕ is valid in the class of frames C, in symbols $\models_{\mathsf{C}} \phi$ iff

$$\mathcal{F} \models \phi \text{ for all } \mathcal{F} \in \mathsf{C}$$

A formula ϕ is valid, in symbols $\models \phi$ iff

$$\mathcal{F} \models \phi \text{ for all models frames } \mathcal{F}$$

Logical consequence

- ϕ is a local logical consequence of Γ, in symbols $\Gamma \models \phi$, if for every model $\mathcal{M} = \langle \mathcal{F}, \mathcal{I} \rangle$ and every point $w \in W$,

$$\mathcal{M}, w \models \Gamma \text{ implies that } \mathcal{M}, w \models \phi$$

- ϕ is a local logical consequence of Γ in a class of frames C, in symbols $\Gamma \models_C \phi$ if for avery model $\mathcal{M} = \langle \mathcal{F}, \mathcal{I} \rangle$ with $\mathcal{F} \in C$ and every point $w \in W$,

$$\mathcal{M}, w \models \Gamma \text{ implies that } \mathcal{M}, w \models \phi$$

Hilbert axioms for normal modal logic

A1	$\phi \supset (\psi \supset \phi)$
A2	$(\phi \supset (\psi \supset \theta)) \supset ((\phi \supset \psi) \supset (\phi \supset \theta))$
A3	$(\neg\psi \supset \neg\phi) \supset ((\neg\psi \supset \phi) \supset \phi)$
MP	$\dfrac{\phi \quad \phi \supset \psi}{\psi}$
K	$\Box(\phi \supset \psi) \supset (\Box\phi \supset \Box\psi)$
Nec	$\dfrac{\phi}{\Box\phi}$ the necessitation rule

The above set of axioms and rules is called **K** , and every modal logic with a validity relation closed under the rules of **K** is a Normal Modal Logic.

Remark on Nec

Notice that **Nec** rule is not the same as

$$\phi \supset \Box\phi \tag{3}$$

indeed formula (3) is not valid.

Assignment Find a model in which (3) is false

Exercise

Show that each of the following formulas is not valid by constructing a frame $\mathcal{F} = (W, R)$ that contains a world that does not satisfy them.

1. $\Box\bot$
2. $\Diamond\phi \supset \Box\phi$
3. $\Diamond\Box\phi \supset \Box\Diamond\phi$

Multi-Modal Logics

All the definitions given for basic modal logic can be generalized in the case in which we have n $\Box$-operators $\Box_1, \ldots, \Box_n$ (and also $\Diamond_1, \ldots, \Diamond_n$), which are interpreted in the frame

$$\mathcal{F} = (W, R_1, \ldots R_n)$$

Every $\Box_i$ and $\Diamond_i$ is interpreted w.r.t. the relation R_i.

A logic with n modal operators is called Multi-Modal.

Multi-Modal logics are often used to model Multi-Agent systems where modality $\Box_i$ is used to express the fact that "agent i knows (believes) that ϕ".

Exercises

Exercise

Let $\mathcal{F} = (W, R_1, \ldots, R_n)$ be a frame for the modal language with n modal operator $\Box_1, \ldots, \Box_n$. Show that the following properties holds:

1. $\mathcal{F} \models \mathbf{K}_i$ (where $\mathbf{K}_i$ is obtained by replacing $\Box$ with $\Box_i$ in the axiom $\mathbf{K}$)
2. If $R_i \subseteq R_j$ then $\mathcal{F} \models \Diamond_i \phi \supset \Diamond_j \phi$
3. If $R_i \subseteq R_j$ then $\mathcal{F} \models \Box_j \phi \supset \Box_i \phi$
4. $\mathcal{F} \not\models \Box_i p \supset \Box_j p$ for any primitive proposition p
5. If $R_i \subseteq R_j \circ R_k$, then[a] $\mathcal{F} \models \Diamond_i \phi \supset \Diamond_j \Diamond_k \phi$

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[a]Given two binary relations R and S on the set W, $R \circ S = \{(v, u) | (v, w) \in R \text{ and } (w, u) \in S\}$

Other exercises

Exercise

Prove that the following formulae are valid:

- $\models \Box(\phi \wedge \psi) \equiv \Box\phi \wedge \Box\psi$
- $\models \Diamond(\phi \vee \psi) \equiv \Diamond\phi \vee \Diamond\psi$
- $\models \neg\Diamond\phi \equiv \Box\neg\phi$
- $\neg\Box\Diamond\Diamond\Box\Box\Diamond\Box\phi \equiv \Diamond\Box\Box\Diamond\Diamond\Box\Diamond\neg\phi$ (i.e., pushing in $\neg$ changes $\Box$ into $\Diamond$ and $\Diamond$ into $\Box$)

Suggestion: keep in mind the analogy $\Box/\forall$ and $\Diamond/\exists$.

Exercise

Consider the frame $\mathcal{F} = (W, R)$ with

- $W = \{0, 1, \ldots n - 1\}$
- $R = \{(0, 1), (1, 2), \ldots, (n - 1, 0)\}$

Show that the following formulas are valid in $\mathcal{F}$

1. $\Box\phi \equiv \Diamond\phi$
2. $\phi \equiv \underbrace{\Box \ldots \Box}_{n} \phi$

Answer also the following questions:

3. can you explain which property of the frame R is formalized by formula 1 and 2?
4. Can you imagine another frame $\mathcal{F}'$, different from $\mathcal{F}$ that satisfies formulas 1 and 2?

Expressing properties on structures

formula true at w	property of w
$\Diamond\top$	w has a successor point
$\Diamond\Diamond\top$	w has a successor point with a successor point
$\underbrace{\Diamond\ldots\Diamond}_{n}\top$	there is a path of length n starting at w
$\Box\bot$	w does not have any successor point
$\Box\Box\bot$	every successor of w does not have a successor point
$\underbrace{\Box\ldots\Box}_{n}\bot$	every path starting form w has length less then n

Expressing properties on structures

formula true at w	property of w
$\Diamond p$	w has a successor point which is p
$\Diamond\Diamond p$	w has a successor point with a successor point which is p
$\underbrace{\Diamond \ldots \Diamond}_{n} p$	there is a path of length n starting at w and ending at a point which is p
$\Box p$	every successor of w are p
$\Box\Box p$	all the successors of the successors of w are p
$\underbrace{\Box \ldots \Box}_{n} p$	all the paths of length n starting form w ends in a point which is p