Mathematical Logic

Practical Class: Formalization in Propositional Logic

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Truth Tables

F	G	$\neg F$	$F \wedge G$	$F \vee G$	$F \rightarrow G$
T	T	F	T	T	T
T	F	F	F	T	F
F	T	T	F	T	T
F	F	T	F	F	T

Truth tables of some propositional logical symbols.

Truth Tables: Example

Compute the truth table of $(F \vee G) \wedge \neg(F \wedge G)$.

F	G	$F \vee G$	$F \wedge G$	$\neg(F \wedge G)$	$(F \vee G) \wedge \neg(F \wedge G)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Intuitively, what does this formula represent?

Truth Tables

Recall some definitions

- Two formulas F and G are **logically equivalent** (denoted with $F \equiv G$) if for each interpretation $\mathcal{I}$, $\mathcal{I}(F) = \mathcal{I}(G)$.
- Let F and G be formulas. G is a **logical consequence** of F (denoted with $F \models G$) if each interpretation satisfying F satisfies also G.
- Let F be a formula:
  - F is **valid** if every interpretation satisfies F
  - F is **satisfiable** if F is satisfied by some interpretation
  - F is **unsatisfiable** if there isn't any interpretation satisfying F

Truth Tables: Example (2)

Use the truth tables method to determine whether $(p \rightarrow q) \vee (p \rightarrow \neg q)$ is valid.

p	q	$p \rightarrow q$	$\neg q$	$p \rightarrow \neg q$	$(p \rightarrow q) \vee (p \rightarrow \neg q)$
T	T	T	F	F	**T**
T	F	F	T	T	**T**
F	T	T	F	T	**T**
F	F	T	T	T	**T**

The formula is valid since it is satisfied by every interpretation.

Truth Tables: Example (3)

Use the truth tables method to determine whether $(\neg p \vee q) \wedge (q \rightarrow \neg r \wedge \neg p) \wedge (p \vee r)$ (denoted with F) is satisfiable.

p	q	r	$\neg p \vee q$	$\neg r \wedge \neg p$	$q \rightarrow \neg r \wedge \neg p$	$(p \vee r)$	F
T	T	T	T	F	F	T	F
T	T	F	T	F	F	T	F
T	F	T	F	F	T	T	F
T	F	F	F	F	T	T	F
F	T	T	T	F	F	T	F
F	T	F	T	T	T	F	F
F	F	T	T	F	T	T	**T**
F	F	F	T	T	T	F	F

There exists an interpretation satisfying F, thus F is satisfiable.

Truth Tables: Example (4)

Use the truth tables method to determine whether $p \wedge \neg q \rightarrow p \wedge q$ is a logical consequence of $\neg p$.

p	q	$\neg p$	$p \wedge \neg q$	$p \wedge q$	$p \wedge \neg q \rightarrow p \wedge q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	**T**	F	F	**T**
F	F	**T**	F	F	**T**

Truth Tables: Example (5)

Use the truth tables method to determine whether $p \rightarrow (q \wedge \neg q)$ and $\neg p$ are logically equivalent.

p	q	$q \wedge \neg q$	$p \rightarrow (q \wedge \neg q)$	$\neg p$
T	T	F	**F**	**F**
T	F	F	**F**	**F**
F	T	F	**T**	**T**
F	F	F	**T**	**T**

Truth Tables: Exercises

Compute the truth tables for the following propositional formulas:

- $(p \rightarrow p) \rightarrow p$
- $p \rightarrow (p \rightarrow p)$
- $p \vee q \rightarrow p \wedge q$
- $p \vee (q \wedge r) \rightarrow (p \wedge r) \vee q$
- $p \rightarrow (q \rightarrow p)$
- $(p \wedge \neg q) \vee \neg(p \leftrightarrow q)$

Truth Tables: Exercises

Use the truth table method to verify whether the following formulas are valid, satisfiable or unsatisfiable:

- $(p \rightarrow q) \wedge \neg q \rightarrow \neg p$
- $(p \rightarrow q) \rightarrow (p \rightarrow \neg q)$
- $(p \vee q \rightarrow r) \vee p \vee q$
- $(p \vee q) \wedge (p \rightarrow r \wedge q) \wedge (q \rightarrow \neg r \wedge p)$
- $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$
- $(p \vee q) \wedge (\neg q \wedge \neg p)$
- $(\neg p \rightarrow q) \vee ((p \wedge \neg r) \leftrightarrow q)$
- $(p \rightarrow q) \wedge (p \rightarrow \neg q)$
- $(p \rightarrow (q \vee r)) \vee (r \rightarrow \neg p)$

Truth Tables: Exercises

Use the truth table method to verify whether the following logical consequences and equivalences are correct:

- $(p \rightarrow q) \models \neg p \rightarrow \neg q$
- $(p \rightarrow q) \wedge \neg q \models \neg p$
- $p \rightarrow q \wedge r \models (p \rightarrow q) \rightarrow r$
- $p \vee (\neg q \wedge r) \models q \vee \neg r \rightarrow p$
- $\neg(p \wedge q) \equiv \neg p \vee \neg q$
- $(p \vee q) \wedge (\neg p \rightarrow \neg q) \equiv q$
- $(p \wedge q) \vee r \equiv (p \rightarrow \neg q) \rightarrow r$
- $(p \vee q) \wedge (\neg p \rightarrow \neg q) \equiv p$
- $((p \rightarrow q) \rightarrow q) \rightarrow q \equiv p \rightarrow q$

Formalizing English Sentences

Exercise

Let's consider a propositional language where p means *"Paola is happy"*, q means *"Paola paints a picture"*, and r means *"Renzo is happy"*.
Formalize the following sentences:

1. *"if Paola is happy and paints a picture then Renzo isn't happy"*
   $p \wedge q \rightarrow \neg r$
2. *"if Paola is happy, then she paints a picture"*
   $p \rightarrow q$
3. *"Paola is happy only if she paints a picture"*
   $\neg(p \wedge \neg q)$ which is equivalent to $p \rightarrow q$!!!

The precision of formal languages avoid the ambiguities of natural languages.

Formalizing English Sentences

Exercise

Let $A =$**"Angelo comes to the party"**, $B =$**"Bruno comes to the party"**, $C =$**"Carlo comes to the party"**, and $D =$**"Davide comes to the party"**.
Formalize the following sentences:

1. *"If Davide comes to the party then Bruno and Carlo come too"*
2. *"Carlo comes to the party only if Angelo and Bruno do not come"*
3. *"If Davide comes to the party, then, if Carlo doesn't come then Angelo comes"*
4. *"Carlo comes to the party provided that Davide doesn't come, but, if Davide comes, then Bruno doesn't come"*
5. *"A necessary condition for Angelo coming to the party, is that, if Bruno and Carlo aren't coming, Davide comes"*
6. *"Angelo, Bruno and Carlo come to the party if and only if Davide doesn't come, but, if neither Angelo nor Bruno come, then Davide comes only if Carlo comes"*

Formalizing English Sentences

Exercise - Solution

1. *"If Davide comes to the party then Bruno and Carlo come too"*
   $D \rightarrow B \wedge C$
2. *"Carlo comes to the party only if Angelo and Bruno do not come"*
   $C \rightarrow \neg A \wedge \neg B$
3. *"If Davide comes to the party, then, if Carlo doesn't come then Angelo comes"*
   $D \rightarrow (\neg C \rightarrow A)$

Formalizing English Sentences

Exercise - Solution

1. *"Carlo comes to the party provided that Davide doesn't come, but, if Davide comes, then Bruno doesn't come"*
   $(C \rightarrow \neg D) \wedge (D \rightarrow \neg B)$
2. *"A necessary condition for Angelo coming to the party, is that, if Bruno and Carlo aren't coming, Davide comes"*
   $A \rightarrow (\neg B \wedge \neg C \rightarrow D)$
3. *"Angelo, Bruno and Carlo come to the party if and only if Davide doesn't come, but, if neither Angelo nor Bruno come, then Davide comes only if Carlo comes"*
   $(A \wedge B \wedge C \leftrightarrow \neg D) \wedge (\neg A \wedge \neg B \rightarrow (D \rightarrow C))$

Formalizing English Sentences

Exercise

Formalize the following arguments and verify whether they are correct:

- *"If you play and you study you'll pass the exams, while if you play and don't study you won't pass. Thus, if you play, either you study and you'll pass the exams, or you don't study and you won't pass."*

Formalizing English Sentences

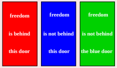
Exercise

- *"If you play and you study you'll pass the exams, while if you play and don't study you won't pass. Thus, if you play, either you study and you'll pass the exams, or you don't study and you won't pass."*
  1. $p \wedge s \rightarrow e$
  2. $p \wedge \neg s \rightarrow \neg e$
  3. $p \rightarrow (s \wedge e) \vee (\neg s \wedge \neg e)$
- We need to prove that $1. \wedge 2. \models 3.$
  Use truth tables

The 3 doors

Problem

Kyle, Neal, and Grant find themselves trapped in a dark and cold dungeon (HOW they arrived there is another story). After a quick search the boys find three doors, the first one red, the second one blue, and the third one green.
Behind one of the doors is a path to freedom. Behind the other two doors, however, is an evil fire-breathing dragon. Opening a door to the dragon means almost certain death.
On each door there is an inscription:



Given the fact that at LEAST ONE of the three statements on the three doors is true and at LEAST ONE of them is false, which door would lead the boys to safety?

The 3 doors: Solution

Language

- r: *"freedom is behind the red door"*
- b: *"freedom is behind the blue door"*
- g: *"freedom is behind the green door"*

Axioms

1. *"behind one of the door is a path to freedom, behind the other two doors is an evil dragon"*
   $(r \wedge \neg b \wedge \neg g) \vee (\neg r \wedge b \wedge \neg g) \vee (\neg r \wedge \neg b \wedge g)$
2. *"at least one of the three statements is true"*
   $r \vee \neg b$
3. *"at least one of the three statements is false"*
   $\neg r \vee b$

The 3 doors: Solution (2)

Axioms

1. $(r \wedge \neg b \wedge \neg g) \vee (\neg r \wedge b \wedge \neg g) \vee (\neg r \wedge \neg b \wedge g)$
2. $r \vee \neg b$
3. $\neg r \vee b$

Solution

r	b	g	2	3	$2 \wedge 3$
T	F	F	T	F	F
F	T	F	F	T	F
F	F	T	T	T	T

Freedom is behind the **green door!**

Traffic Light

Problem

Define a propositional language which allows to describe the state of a traffic light on different instants. With the language defined above provide a (set of) formulas which expresses the following facts:

- the traffic light is either green, or red or orange;
- the traffic light switches from green to orange, from orange to red, and from red to green;
- it can keep the same color over at most 3 successive states.

Traffic Light

Solution

- g_k ="*traffic light is green at instant k*", r_k ="*traffic light is red at instant k*" and o_k ="*traffic light is orange at instant k*".
- Let's formalize the traffic light behavior:
  1. *"the traffic light is either green, or red or orange"*
     $(g_k \leftrightarrow (\neg r_k \wedge \neg o_k)) \wedge (r_k \leftrightarrow (\neg g_k \wedge \neg o_k)) \wedge (o_k \leftrightarrow (\neg r_k \wedge \neg g_k))$
  2. *"the traffic light switches from green to orange, from orange to red, and from red to green"*
     $(g_{k-1} \rightarrow (g_k \vee o_k)) \wedge (o_{k-1} \rightarrow (o_k \vee r_k)) \wedge (r_{k-1} \rightarrow (r_k \vee g_k))$
  3. *"it can keep the same color over at most 3 successive states"*
     $(g_{k-3} \wedge g_{k-2} \wedge g_{k-1} \rightarrow \neg g_k) \wedge (r_{k-3} \wedge r_{k-2} \wedge r_{k-1} \rightarrow \neg r_k) \wedge (o_{k-3} \wedge o_{k-2} \wedge o_{k-1} \rightarrow \neg o_k)$

Graph Coloring Problem

Problem

Provide a propositional language and a set of axioms that formalize the graph coloring problem of a graph with at most n nodes, with connection degree $\leq m$, and with less then $k+1$ colors.

- node degree: number of adjacent nodes
- connection degree of a graph: max among all the degree of its nodes
- Graph coloring problem: given a non-oriented graph, associate a color to each of its nodes in such a way that no pair of adjacent nodes have the same color.

Graph Coloring: Propositional Formalization

Language

- For each $1 \leq i \leq n$ and $1 \leq c \leq k$, color_{ic} is a proposition, which intuitively means that *"the i-th node has the c color"*
- For each $1 \leq i \neq j \leq n$, edge_{ij} is a proposition, which intuitively means that *"the i-th node is connected with the j-th node"*.

Axioms

- for each $1 \leq i \leq n$, $\bigvee_{c=1}^{k} \text{color}_{ic}$
  "each node has at least one color"
- for each $1 \leq i \leq n$ and $1 \leq c, c' \leq k$, $\text{color}_{ic} \rightarrow \neg\text{color}_{ic'}$
  "every node has at most 1 color"
- for each $1 \leq i, j \leq n$ and $1 \leq c \leq k$, $\text{edge}_{ij} \rightarrow \neg(\text{color}_{ic} \wedge \text{color}_{jc})$
  "adjacent nodes do not have the same color"
- for each $1 \leq i \leq n$, and each $J \subseteq \{1..n\}$, where $|J| = m$,
  $\bigwedge_{j \in J} \text{edge}_{ij} \rightarrow \bigwedge_{j \notin J} \neg\text{edge}_{ij}$
  "every node has at most m connected nodes"

Sudoku Example

Problem

Sudoku is a placement puzzle. The aim of the puzzle is to enter a numeral from 1 through 9 in each cell of a grid, most frequently a 9×9 grid made up of 3×3 subgrids (called "regions"), starting with various numerals given in some cells (the "givens"). Each row, column and region must contain only one instance of each numeral. Its grid layout is like the one shown in the following schema

		9				7		
	4		5		9		1	
3				1				2
	1			6			7	
		2	7		1	8		
	5			4			3	
7				3				4
	8		2		4		6	
		6				5		

Provide a formalization in propositional logic of the sudoku problem, so that any truth assignment to the propositional variables that satisfy the axioms is a solution for the puzzle.

Sudoku Example: Solution

Language

For $1 \leq n, r, c \leq 9$,define the proposition

$$in(n, r, c)$$

which means that the number n has been inserted in the cross between row r and column c.

Sudoku Example: Solution

Axioms

1. *"A raw contains all numbers from 1 to 9 "*

$$\bigwedge_{r=1}^{9}\left(\bigwedge_{n=1}^{9}\left(\bigvee_{c=1}^{9} \textbf{in}(n, r, c)\right)\right)$$

2. *"A column contains all numbers from 1 to 9"*

$$\bigwedge_{c=1}^{9}\left(\bigwedge_{n=1}^{9}\left(\bigvee_{r=1}^{9} \textbf{in}(n, r, c)\right)\right)$$

3. *"A region (sub-grid) contains all numbers from 1 to 9"*

$$\textbf{for any} \quad 0 \leq k, h \leq 2 \qquad \bigwedge_{n=1}^{9}\left(\bigvee_{r=1}^{3}\left(\bigvee_{c=1}^{3} \textbf{in}(n, 3*k+r, 3*h+c)\right)\right)$$

4. *"A cell cannot contain two numbers"*

$$\textbf{for any } 1 \leq n, n', c, r \leq 9 \textbf{ and } n \neq n' \qquad \textbf{in}(n, r, c) \rightarrow \neg\textbf{in}(n', r, c)$$