Analyzing the Core of Categorial Grammar

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$\square$ 1. Introduction

Explanatory Aim: To show that the $\diamond$ and $\square^{\downarrow}$ proposed in [Moortgat 95] are indeed the unary counterpart of the $\backslash, \bullet, /$ operators [Lambek 58, 61].

Innovative Aim: (i) To extend the Residuated Calculus with unary Galois operators; (ii) To give an idea on how these new operators could be applied in linguistic analysis.

More generally, this work brings to a

Methodological Aim: To give a recipe for encoding (some) algebraic principles into Gentzen sequent proof system.

□ 2. Explanatory Part

We start introducing the algebraic principle at the heart of Categorial Grammar, *residuation.*

We show how in *Display Logic* [Belnap 82, Goré 98] the residuation law is directly specified starting from structural rules and projecting them into the logical operators.

We then obtain the logical rules of NL in sequent Gentzen format, *compiling* those structural rules.

We apply this mechanism to *unary residuated* operators obtaning the sequents presentation of $\mathsf{NL}(\Diamond)$.

□ 3. Residuation (I)

Let $\mathcal{A}_i = (A_i, \sqsubseteq_{A_i})$ be partially ordered sets. A pair of functions (f, g) such that $f : A_1 \to A_2$ and $g : A_2 \to A_1$ forms a *residuated pair* if

$$[RES_1] \quad \forall x \in A_1, y \in A_2 \begin{pmatrix} f(x) \sqsubseteq_{A_2} y \\ \text{iff} \\ x \sqsubseteq_{A_1} g(y) \end{pmatrix}$$

Similarly, a triple of functions (f, g, h) such that $f : A_1 \times A_2 \to A_3$, $g : A_1 \times A_3 \to A_2$, $h : A_3 \times A_2 \to A_1$ forms a *residuated triple* if

$$[RES_2] \quad \forall x \in A_1, y \in A_2, z \in A_3 \begin{pmatrix} x \sqsubseteq_{A_1} h(z, y) \\ \text{iff} \\ f(x, y) \sqsubseteq_{A_3} z \\ \text{iff} \\ y \sqsubseteq_{A_2} g(x, z) \end{pmatrix}$$

□ 4. Residuation (II)

Saying that (f, g) is a residuated pair is equivalent to the conditions i) and ii),

i. f and g are $[\uparrow]$-functions.

ii. $$\forall y \in A_2, x \in A_1 \begin{pmatrix} f(g(y)) \sqsubseteq_{A_2} y \\ \text{and} \\ x \sqsubseteq_{A_1} g(f(x)) \end{pmatrix}$$

Similarly, saying that (f, g, h) is a residuated triple is equivalent to requiring

i. f is a $[\uparrow, \uparrow]$-function, g is an $[\downarrow, \uparrow]$-function and h is an $[\uparrow, \downarrow]$-function.

ii. $$\forall x \in A_1, y \in A_2, z \in A_3 \begin{pmatrix} f(x, g(x, z)) \sqsubseteq_{A_3} z \\ \text{and} \\ y \sqsubseteq_{A_2} g(x, f(x, y)) \\ \text{and} \\ f(h(z, y), y) \sqsubseteq_{A_3} z \\ \text{and} \\ x \sqsubseteq_{A_1} h(f(x, y), y) \end{pmatrix}$$

□ 5. Display Logic

[Belnap 1982, Goré 1998] DL generalizes Gentzen's notion of structures using multiple, complex, structural connectives.

It can *display* any particular constituent of a sequent at the right or left side.

DL's enjoy *cut-elimination* theorem, whenever the rules obey certain, easily checked, conditions.

In DL's the behavior of the logical operators is defined in terms of *structural properties* – sequent rules involving only structural operators.

DL language

$$\begin{aligned}\mathsf{FORM} ::= \quad & \mathsf{ATOM} \mid \mathsf{FORM}; \mathsf{FORM} \mid \\ & \mathsf{FORM} < \mathsf{FORM} \mid \mathsf{FORM} > \mathsf{FORM} \mid \\ & \mathsf{FORM} \otimes \mathsf{FORM} \mid \mathsf{FORM} \leftarrow \mathsf{FORM} \mid \\ & \mathsf{FORM} \rightarrow \mathsf{FORM}.\end{aligned}$$

Residuation [rp]

$$\dfrac{\dfrac{Y \vdash X > Z}{X; Y \vdash Z}}{X \vdash Z < Y}$$

□ 6. Residuation in Display Logic

We can obtain $[\otimes\vdash]$, $[\vdash\leftarrow]$ and $[\vdash\rightarrow]$ directly from [rp] by projection (as *rewrite rules*).

$$\frac{Z \vdash A < B}{Z \vdash A \leftarrow B}\ [\vdash\leftarrow]$$

To obtain the still missing rules we need to use the monotonicity property of the residuated operators:

Let s be a structural operator and l its corresponding logical counterpart.

$$\frac{V \vdash X \quad W \vdash Y}{[l,s](X,Y) \vdash [s,l](V,W)} \quad \text{if } s \text{ is } [\downarrow,\downarrow]$$

$$\frac{X \vdash V \quad Y \vdash W}{[l,s](X,Y) \vdash [s,l](V,W)} \quad \text{if } s \text{ is } [\uparrow,\uparrow]$$

$$\frac{X \vdash V \quad W \vdash Y}{[l,s](X,Y) \vdash [s,l](V,W)} \quad \text{if } s \text{ is } [\uparrow,\downarrow]$$

$$\frac{V \vdash X \quad Y \vdash W}{[l,s](X,Y) \vdash [s,l](V,W)} \quad \text{if } s \text{ is } [\downarrow,\uparrow]$$

□ 7. The final system (**NL**) in **DL**

$$\frac{Z \vdash A < B}{Z \vdash A \leftarrow B}\ [\vdash\leftarrow] \qquad \frac{A \vdash X \quad Y \vdash B}{A \leftarrow B \vdash X < Y}\ [\leftarrow\vdash]$$

$$\frac{A; B \vdash Z}{A \otimes B \vdash Z}\ [\otimes \vdash] \qquad \frac{Y \vdash B \quad X \vdash A}{Y ; X \vdash B \otimes A}\ [\vdash \otimes]$$

$$\frac{Z \vdash A > B}{Z \vdash A \rightarrow B}\ [\vdash\rightarrow] \qquad \frac{X \vdash A \quad B \vdash Y}{A \rightarrow B \vdash X > Y}\ [\rightarrow\vdash]$$

$$\begin{array}{c} Y \vdash X > Z \\ \hline\hline X ; Y \vdash Z \\ \hline\hline X \vdash Z < Y \end{array} \qquad \frac{\Gamma \vdash A \quad A \vdash \Delta}{\Gamma \vdash \Delta}\ [\text{Cut}]$$

□ 8. Compiling Residuation

We want to get rid of [rp]. We should "compile" it into the logical rules.

Start with

$$\frac{X \vdash A \quad B \vdash C}{A \to B \vdash X > C}\;[\to\vdash]$$

By [rp]

$$\cfrac{\cfrac{X \vdash A \quad B \vdash C}{A \to B \vdash X > C}\;[\to\vdash]}{X; A \to B \vdash C}\;[\mathrm{rp}]$$

By compilation

$$\frac{X \vdash A \quad B \vdash C}{X; A \to B \vdash C}$$

Finally, by replacing ; by , and $\to$ by $\backslash$ and adding structural contexts we obtain [\L]

$$\frac{X \vdash A \quad \Gamma[B] \vdash C}{\Gamma[(X, A\backslash B)] \vdash C}\;[\backslash \mathrm{L}]$$

In this way we have obtained Gentzen sequents of NL [Lambek 61].

□ 9. Unary Residuated Operators in **DL**

FORM ::= ATOM | $\bullet$FORM | $\circ$FORM | $\blacklozenge$FORM | $\Box$FORM.

$$\frac{X \vdash \circ A}{X \vdash \Box A}\ [\vdash \Box] \qquad \frac{A \vdash X}{\Box A \vdash \circ X}\ [\Box \vdash]$$

$$\frac{\bullet X \vdash A}{\blacklozenge X \vdash A}\ [\blacklozenge \vdash] \qquad \frac{X \vdash A}{\bullet X \vdash \blacklozenge A}\ [\vdash \blacklozenge]$$

$$\frac{\bullet X \vdash Y}{X \vdash \circ Y}\ [\mathrm{rp}] \quad \text{(double line)}$$

$$\frac{A \vdash B}{\Box A \vdash \circ B}\ [\Box \vdash] \text{ by } [\mathrm{rp}] \qquad \cfrac{\cfrac{A \vdash B}{\Box A \vdash \circ B}\ [\Box \vdash]}{\bullet \Box A \vdash B}\ [\mathrm{rp}] \text{ by co} \qquad \frac{\Gamma[A] \vdash B}{\Gamma[\langle \Box^{\downarrow} A \rangle] \vdash B}\ [\Box^{\downarrow}{}_{\mathrm{L}}]$$

$$\frac{X \vdash \circ A}{X \vdash \Box A}\ [\Box \vdash] \text{ by } [\mathrm{rp}] \qquad \cfrac{\cfrac{\bullet X \vdash A}{X \vdash \circ A}\ [\mathrm{rp}]}{X \vdash \Box A}\ [\Box \vdash] \text{ by co} \qquad \frac{\langle X \rangle \vdash A}{X \vdash \Box^{\downarrow} A}\ [\Box^{\downarrow}{}_{\mathrm{R}}]$$

So we arrive to Gentzen Sequent presentation of NL($\Diamond$) [Moortgat 95].

□ 10. Innovative Part: Galois Operators.

Let $\mathcal{A}_i = (A_i, \sqsubseteq_{A_i})$ be a partially ordered set. Consider a pair of functions $f : A_1 \to A_2$ and $g : A_2 \to A_1$. The pair (f, g) is called a *Galois connection* if

$$[\text{GC}] \quad \forall x \in A_1, y \in A_2 \begin{pmatrix} y \sqsubseteq_{A_2} f(x) \\ \text{iff} \\ x \sqsubseteq_{A_1} g(y) \end{pmatrix}$$

The equivalent formulation of this property in terms of the monotonicity behavior and a composition rule is given by

i. f and g are $[\downarrow]$-functions;

ii. $\forall x \begin{pmatrix} x \sqsubseteq f(g(x)) \\ \text{and} \\ x \sqsubseteq g(f(x)) \end{pmatrix}$

$\mathsf{NL}(\diamond)$ is the pure calculus of residuation. But, note that $\backslash$ and $/$ form a *Galois* connection when their positive argument is fixed (i.e. $\backslash B, B/$).

$$[\text{GC}] \ \forall A \begin{pmatrix} A \vdash B/(A\backslash B) \\ \text{and} \\ A \vdash (B/A)\backslash B \end{pmatrix}$$

This fact is used, by the *lifting theorem*, which has found interesting linguistic applications.

□ 11. Galois operators in **DL**

We explicitly write the Galois principle for a pair of structural operators $(\sharp, \flat)$ and we project their behavior into the logical operators.

$$\frac{Z \vdash \flat A}{Z \vdash {}^0(A)}\ [\vdash {}^0(\cdot)] \qquad\qquad \frac{Z \vdash A}{{}^0(A) \vdash \flat Z}\ [{}^0(\cdot) \vdash]$$

$$\frac{Z \vdash \sharp A}{Z \vdash (A)^0}\ [\vdash (\cdot)^0] \qquad\qquad \frac{Z \vdash A}{(A)^0 \vdash \sharp Z}\ [(\cdot)^0 \vdash]$$

$$\frac{Y \vdash \flat X}{\overline{X \vdash \sharp Y}}\ [\mathrm{gc}]$$

To move closer to standard sequent presentations, we need to compile [gc] into the logical rules.

$$\frac{Z \vdash \flat A}{Z \vdash {}^0(A)}\ [\vdash^0 \cdot]\ \text{by [gc]} \qquad \frac{\dfrac{A \vdash \sharp Z}{Z \vdash \flat A}\ [\mathrm{gc}]}{Z \vdash {}^0(A)}\ [\vdash^0 \cdot]\ \text{by co} \qquad \frac{A \vdash \sharp Z}{Z \vdash {}^0(A)}\ [{}^0\cdot_{\mathrm{R}}]$$

$$\frac{Z \vdash \sharp A}{Z \vdash (A)^0}\ [\vdash \cdot^0]\ \text{by [gc]} \qquad \frac{\dfrac{A \vdash \flat Z}{Z \vdash \sharp A}}{Z \vdash (A)^0}\ [\vdash \cdot^0]\ \text{by co} \qquad \frac{A \vdash \flat Z}{Z \vdash (A)^0}\ [\cdot^0{}_{\mathrm{R}}]$$

□ 12. Genzten sequents for Galois operators

Logical Rules

$$\frac{\Gamma \vdash \Delta}{{}^0(\Delta) \vdash \flat\Gamma}\ [{}^0(\cdot)\mathrm{L}] \qquad \frac{\Delta \vdash \sharp\Gamma}{\Gamma \vdash {}^0(\Delta)}\ [{}^0(\cdot)\mathrm{R}]$$

$$\frac{\Gamma \vdash \Delta}{(\Delta)^0 \vdash \sharp\Gamma}\ [(\cdot)^0\mathrm{L}] \qquad \frac{\Delta \vdash \flat\Gamma}{\Gamma \vdash (\Delta)^0}\ [(\cdot)^0\mathrm{R}]$$

$$\frac{\Gamma \vdash \Theta \quad \Theta \vdash \Delta}{\Gamma \vdash \Delta}\ [\mathrm{Cut}]$$

The proofs below show that the $(\cdot)^0$ and ${}^0(\cdot)$ operators are indeed Galois connected,

$$\dfrac{\dfrac{A \vdash (B)^0 \quad \dfrac{B \vdash B}{(B)^0 \vdash \sharp B}\,[(\cdot)^0\mathrm{L}]}{A \vdash \sharp B}\,[\mathrm{Cut}]}{B \vdash {}^0(A)}\,[{}^0(\cdot)\mathrm{R}] \qquad \dfrac{\dfrac{A \vdash {}^0(B) \quad \dfrac{B \vdash B}{{}^0(B) \vdash \flat B}\,[{}^0(\cdot)\mathrm{L}]}{A \vdash \flat B}\,[\mathrm{Cut}]}{B \vdash (A)^0}\,[(\cdot)^0\mathrm{R}]$$

Moreover, the operators satisfy the appropriate Galois composition laws.

$$\dfrac{\dfrac{A \vdash A}{(A)^0 \vdash \sharp A}\,[(\cdot)^0\mathrm{L}]}{A \vdash {}^0((A)^0)}\,[{}^0(\cdot)\mathrm{R}] \qquad \dfrac{\dfrac{A \vdash A}{{}^0(A) \vdash \flat A}\,[{}^0(\cdot)\mathrm{L}]}{A \vdash ({}^0(A))^0}\,[(\cdot)^0\mathrm{R}]$$

From these, the fact that the operators are $[\downarrow]$-functions follows immediately.

$\Box$ 13. Galois: Linguistic Application

The addition of Galois operators to the pure logic of residuation $\mathsf{NL}(\Diamond)$ generates new derivability patterns. The one dimensional logic relation which characterizes the residuated pair $\Diamond, \Box^{\downarrow}$ is extended as shown below:

$$\begin{array}{ccccc} & & {}^0(s^0) & & \\ & & \Uparrow & & \\ \Diamond\Box^{\downarrow}s & \Rightarrow & s & \Rightarrow & \Box^{\downarrow}\Diamond s \\ & & \Downarrow & & \\ & & ({}^0s)^0 & & \end{array}$$

In [Bernardi and Moot 00] the derivability relation linking $\Diamond$ and $\Box^{\downarrow}$ is used to account for different scopal possibilities of GQs with respect to negation.

In [Moortgat 91] a GQ is assigned the type $q(np, s, s)$: It acts locally like a noun phrase but takes scope semantically at a higher sentential level. The rule of use of q is:

$$\frac{\Delta[A] \Rightarrow B \quad \Gamma[C] \Rightarrow D}{\Gamma[\Delta[q(A, B, C)]] \Rightarrow D}\ (qL)$$

□ 14. Galois: Linguistic Application Polarity Items

1. **Any student* left.
2. John didn't see *any student.*
3. *Some student* left.
4. John didn't see *some student.*

Lexicon	
didn't	$\in ((s/(np\backslash s))\backslash s)/(np\backslash({}^{0}s)^{0})$
any N	$\in q(np, ({}^{0}s)^{0}, ({}^{0}s)^{0})$
some N	$\in q(np, \Box^{\downarrow}\Diamond s, \Box^{\downarrow}\Diamond s)$

$$\dfrac{\dfrac{np \Rightarrow np \quad s \Rightarrow ({}^{0}s)^{0}}{np \bullet np\backslash s \Rightarrow ({}^{0}s)^{0}}\ (\backslash L) \qquad \boxed{({}^{0}s)^{0} \not\Rightarrow \Box^{\downarrow}\Diamond s}}{\underbrace{q(np, ({}^{0}s)^{0}, ({}^{0}s)^{0})}_{\text{Any student}} \bullet \underbrace{np\backslash s}_{\text{left}} \Rightarrow \Box^{\downarrow}\Diamond s}\ (qL)$$

$\square$ 15. Polarity Items (I)

$$\dfrac{\dfrac{\dfrac{np \Rightarrow np \qquad \dfrac{np \Rightarrow np \quad \boxed{s \Rightarrow s_1}}{np \bullet np\backslash s \Rightarrow s_1}\,(\backslash L)}{np \bullet ((np\backslash s)/np \bullet np) \Rightarrow s_1}\,(/L) \qquad \boxed{s_2 \Rightarrow ({}^0 s)^0}}{\dfrac{np \bullet ((np\backslash s)/np \bullet \boxed{q(np, s_1, s_2)}) \Rightarrow ({}^0 s)^0}{(np\backslash s)/np \bullet q(np, s_1, s_2) \Rightarrow np\backslash({}^0 s)^0}\,(\backslash R)}\,(qL) \qquad \dfrac{np \Rightarrow s/(np\backslash s) \quad s \Rightarrow \square^{\downarrow}\Diamond s}{np \bullet (s/(np\backslash s))\backslash s \Rightarrow \square^{\downarrow}\Diamond s}\,(\backslash L)}{\underbrace{np}_{\text{subject}} \bullet (\underbrace{((s/(np\backslash s))\backslash s)/(np\backslash({}^0 s)^0)}_{\text{negation}} \bullet (\underbrace{(np\backslash s)/np}_{\text{trans. verb}} \bullet \underbrace{q(np, s_1, s_2)}_{\text{gen. quantifier}})) \Rightarrow \square^{\downarrow}\Diamond s}\,(/L)$$

Wide Scope Negation ($\neg$GQ).

□ 16. Polarity Items (II)

$$
\dfrac{
  \dfrac{
    np \Rightarrow np
    \qquad
    \dfrac{
      \dfrac{s \Rightarrow ({}^0 s)^0 \qquad np \Rightarrow np}{np\backslash s \Rightarrow np\backslash({}^0 s)^0}\,(\backslash R - L)
      \qquad
      \dfrac{np \Rightarrow s/(np\backslash s) \qquad \boxed{s \Rightarrow s_1}}{np \bullet (s/(np\backslash s))\backslash s \Rightarrow s_1}\,(\backslash L)
    }{np \bullet (((s/(np\backslash s))\backslash s)/(np\backslash({}^0 s)^0) \bullet np\backslash s) \Rightarrow s_1}\,(/L)
  }{np \bullet (((s/(np\backslash s))\backslash s)/(np\backslash({}^0 s)^0) \bullet ((np\backslash s)/np \bullet np)) \Rightarrow s_1}\,(/L)
  \qquad
  \boxed{s_2 \Rightarrow \Box^{\downarrow}\Diamond s}
}{
  \underbrace{np}_{\text{subject}} \bullet (\underbrace{((s/(np\backslash s))\backslash s)/(np\backslash({}^0 s)^0)}_{\text{negation}} \bullet (\underbrace{(np\backslash s)/np}_{\text{trans. verb}} \bullet \underbrace{\boxed{q(np, s_1, s_2)}}_{\text{gen. quantifier}})) \Rightarrow \Box^{\downarrow}\Diamond s
}\,(qL)
$$

Narrow Scope Negation (GQ¬).

$\square$ 17. Conclusion

We have shown that

1. DL provides new insight on categorial type logics.
2. It can be used to understand how algebraic principles can characterize the behavior of logical operators.
3. In particular, DL sheds lights on the residuation behavior of the logical operators of $\mathsf{NL}(\diamond)$.
4. Moreover, it helps extend $\mathsf{NL}(\diamond)$ with primitive operators that resemble a pair of split negations. (In a similar way we could introduce a second pair of negation-like operators, governed by a dual Galois connection law.)
5. Finally, we have provided some ideas on how the new system can be used in analyzing linguistic phenomena.