Analyzing the Core of Categorial Grammar

Raffaella Bernardi

UiL-OTS, University of Utrecht

`Raffaella.Bernardi@let.uu.nl`

Work in collaboration with

Carlos Areces

ILLC, University of Amsterdam

`carlos@science.uva.nl`

□ 1. Introduction

Explanatory Aim: To show that the $\Diamond, \Box^{\downarrow}$ proposed in [Moortgat 95] are the unary counterpart of the $\backslash, \bullet, /$ operators [Lambek 58, 61].

How: We will (i) introduce the algebraic principle of residuation at the heart of NL; (ii) show how to encode it proof theoretically using Display Calculus; (iii) translate the obtained proof system into Gentzen sequents format; and (iv) apply the same mechanism to the $\Diamond, \Box^{\downarrow}$.

Innovative Aim: To extend the Residuated Calculus with unary Galois operators and give an idea on how these new operators could be applied in linguistic analysis.

How: We will (i) follow the same method used for the residuation principle; and (ii) discuss type assignments involving Galois unary operators for Generalized Quantifiers.

Methodological Aim: To give a recipe for encoding (some) algebraic principles into Gentzen sequent proof system.

□ 2. Residuation (I)

Let $\mathcal{A}_i = (A_i, \sqsubseteq_{A_i})$ be partially ordered sets. A pair of functions (f, g) such that $f : A_1 \to A_2$ and $g : A_2 \to A_1$ forms a *residuated pair* if

$$[RES_1] \quad \forall x \in A_1, y \in A_2 \begin{pmatrix} f(x) \sqsubseteq_{A_2} y \\ \text{iff} \\ x \sqsubseteq_{A_1} g(y) \end{pmatrix}$$

Similarly, a triple of functions (f, g, h) such that $f : A_1 \times A_2 \to A_3$, $g : A_1 \times A_3 \to A_2$, $h : A_3 \times A_2 \to A_1$ forms a *residuated triple* if

$$[RES_2] \quad \forall x \in A_1, y \in A_2, z \in A_3 \begin{pmatrix} x \sqsubseteq_{A_1} h(z, y) \\ \text{iff} \\ f(x, y) \sqsubseteq_{A_3} z \\ \text{iff} \\ y \sqsubseteq_{A_2} g(x, z) \end{pmatrix}$$

□ 3. Residuation (II)

Saying that (f, g) is a residuated pair is equivalent to the conditions *i*) and *ii*),

i. f and g are $[\uparrow]$-functions.

ii. $$\forall y \in A_2, x \in A_1 \begin{pmatrix} f(g(y)) \sqsubseteq_{A_2} y \\ \text{and} \\ x \sqsubseteq_{A_1} g(f(x)) \end{pmatrix}$$

Similarly, saying that (f, g, h) is a residuated triple is equivalent to requiring

i. f is a $[\uparrow, \uparrow]$-function, g is an $[\downarrow, \uparrow]$-function and h is an $[\uparrow, \downarrow]$-function.

ii. $$\forall x \in A_1, y \in A_2, z \in A_3 \begin{pmatrix} f(x, g(x, z)) \sqsubseteq_{A_3} z \\ \text{and} \\ y \sqsubseteq_{A_2} g(x, f(x, y)) \\ \text{and} \\ f(h(z, y), y) \sqsubseteq_{A_3} z \\ \text{and} \\ x \sqsubseteq_{A_1} h(f(x, y), y) \end{pmatrix}$$

□ 4. Display Calculus

[Belnap 1982, Goré 1998] DC generalizes Gentzen's notion of structures using multiple, complex, structural connectives.

It can *display* any particular constituent of a sequent at the right or left side.

DC's enjoy *cut-elimination* theorem, whenever the rules obey certain, easily checked, conditions.

In DC's the behavior of the logical operators is defined in terms of *structural properties* – sequent rules involving only structural operators.

DC language

$$\begin{array}{ll} \mathsf{FORM} ::= & \mathsf{ATOM} \mid \mathsf{FORM} \otimes \mathsf{FORM} \mid \\ & \mathsf{FORM} \leftarrow \mathsf{FORM} \mid \mathsf{FORM} \rightarrow \mathsf{FORM} \\ \mathsf{STRUCT} ::= & \mathsf{FORM} \mid \mathsf{STRUCT} ; \mathsf{STRUCT} \mid \\ & \mathsf{STRUCT} < \mathsf{STRUCT} \mid \mathsf{STRUCT} > \mathsf{STRUCT}. \end{array}$$

Residuation [rp]

$$\begin{array}{c} Y \vdash X > Z \\ \hline\hline X ; Y \vdash Z \\ \hline\hline X \vdash Z < Y \end{array}$$

□ 5. Residuation in Display Calculus

We can obtain $[\otimes \vdash]$, $[\vdash\leftarrow]$ and $[\vdash\rightarrow]$ directly from [rp] by projection (as *rewrite rules*).

$$\frac{Z \vdash A < B}{Z \vdash A \leftarrow B}\ [\vdash\leftarrow]$$

To obtain the still missing rules we need to use the monotonicity property of the residuated operators:

Let s be a structural operator and l its corresponding logical counterpart.

If s is a $[\uparrow, \downarrow]$-function,

$$\frac{X \vdash V \quad W \vdash Y}{[l, s](X, Y) \vdash [s, l](V, W)}$$

E.g. $>$ is a $[\uparrow, \downarrow]$-function,

$$\frac{A \vdash X \quad Y \vdash B}{(A \leftarrow B) \vdash (X < Y)}\ [\leftarrow\vdash]$$

Similarly for the other operators, $\rightarrow, \otimes$.

□ 6. Compiling Residuation

We want to get rid of [rp]. We should "compile" it into the logical rules.

Start with

$$\frac{X \vdash A \quad B \vdash C}{A \to B \vdash X > C}\ [{\to}{\vdash}]$$

By [rp]

$$\cfrac{\cfrac{X \vdash A \quad B \vdash C}{A \to B \vdash X > C}\ [{\to}{\vdash}]}{X; A \to B \vdash C}\ [\mathrm{rp}]$$

By compilation

$$\frac{X \vdash A \quad B \vdash C}{X; A \to B \vdash C}$$

Finally, by replacing ; by , and $\to$ by $\backslash$ and adding structural contexts we obtain [\L]

$$\frac{X \vdash A \quad \Gamma[B] \vdash C}{\Gamma[(X, A\backslash B)] \vdash C}\ [\backslash \mathrm{L}]$$

In this way we have obtained Gentzen sequents of NL [Lambek 61].

□ 7. Unary Residuated Operators in DC

$$\mathsf{FORM} ::= \mathsf{ATOM} \mid \blacklozenge\mathsf{FORM} \mid \Box\mathsf{FORM}.$$

$$\mathsf{STRUCT} ::= \bullet\mathsf{FORM} \mid \circ\,\mathsf{FORM}$$

$$\frac{X \vdash \circ A}{X \vdash \Box A}\ [\vdash \Box] \qquad \frac{A \vdash X}{\Box A \vdash \circ X}\ [\Box \vdash]$$

$$\frac{\bullet A \vdash X}{\blacklozenge A \vdash X}\ [\blacklozenge \vdash] \qquad \frac{X \vdash A}{\bullet X \vdash \blacklozenge A}\ [\vdash \blacklozenge]$$

$$\frac{\bullet X \vdash Y}{\overline{X \vdash \circ Y}}\ [\mathrm{rp}]$$

$$\frac{A \vdash B}{\Box A \vdash \circ B}\ [\Box \vdash]\ \text{by [rp]} \qquad \cfrac{\cfrac{A \vdash B}{\Box A \vdash \circ B}\ [\Box \vdash]}{\bullet\Box A \vdash B}\ [\mathrm{rp}]\ \text{by co} \qquad \frac{\Gamma[A] \vdash B}{\Gamma[\langle \Box^{\downarrow} A\rangle] \vdash B}\ [\Box^{\downarrow}{}_{\mathrm{L}}]$$

$$\frac{X \vdash \circ A}{X \vdash \Box A}\ [\Box \vdash]\ \text{by [rp]} \qquad \cfrac{\cfrac{\bullet X \vdash A}{X \vdash \circ A}\ [\mathrm{rp}]}{X \vdash \Box A}\ [\Box \vdash]\ \text{by co} \qquad \frac{\langle X\rangle \vdash A}{X \vdash \Box^{\downarrow} A}\ [\Box^{\downarrow}{}_{\mathrm{R}}]$$

So we arrive to a Gentzen Sequent presentation of $\mathsf{NL}(\Diamond)$ [Moortgat 95].

□ 8. Galois Operators

Let $\mathcal{A}_i = (A_i, \sqsubseteq_{A_i})$ be a partially ordered set. Consider a pair of functions $f : A_1 \to A_2$ and $g : A_2 \to A_1$. The pair (f, g) is called a *Galois connection* if

$$[\mathrm{GC}] \quad \forall x \in A_1, y \in A_2 \begin{pmatrix} y \sqsubseteq_{A_2} f(x) \\ \text{iff} \\ x \sqsubseteq_{A_1} g(y) \end{pmatrix}$$

The equivalent formulation of this property in terms of the monotonicity behavior and a composition rule is given by

i. f and g are $[\downarrow]$-functions;

ii. $\forall x \begin{pmatrix} x \sqsubseteq f(g(x)) \\ \text{and} \\ x \sqsubseteq g(f(x)) \end{pmatrix}$

$\mathsf{NL}(\Diamond)$ is the pure calculus of residuation. But, note that $\backslash$ and $/$ form a *Galois* connection when their positive argument is fixed (i.e. $\backslash B, B/$).

$$[\mathrm{GC}] \ \ \forall A \begin{pmatrix} A \vdash B/(A\backslash B) \\ \text{and} \\ A \vdash (B/A)\backslash B \end{pmatrix}$$

This fact is used, by the *lifting theorem*, which has found interesting linguistic applications.

□ 9. Galois operators in DC

We explicitly write the Galois principle for a pair of structural operators $(\sharp, \flat)$ and we project their behavior into the logical operators.

$$\frac{Z \vdash \flat A}{Z \vdash^0 (A)}\ [\vdash {}^0(\cdot)] \qquad \frac{Z \vdash A}{{}^0(A) \vdash \flat Z}\ [{}^0(\cdot) \vdash]$$

$$\frac{Z \vdash \sharp A}{Z \vdash (A)^0}\ [\vdash (\cdot)^0] \qquad \frac{Z \vdash A}{(A)^0 \vdash \sharp Z}\ [(\cdot)^0 \vdash]$$

$$\frac{Y \vdash \flat X}{X \vdash \sharp Y}\ [\mathrm{gc}]$$

To move closer to standard sequent presentations, we need to compile [gc] into the logical rules.

$$\frac{Z \vdash \flat A}{Z \vdash {}^0(A)}\ [\vdash^0 \cdot]\ \text{by [gc]} \qquad \frac{\dfrac{A \vdash \sharp Z}{Z \vdash \flat A}\,[\mathrm{gc}]}{Z \vdash {}^0(A)}\ [\vdash^0 \cdot]\ \text{by co} \qquad \frac{A \vdash \sharp Z}{Z \vdash {}^0(A)}\ [{}^0\cdot_{\mathrm{R}}]$$

$$\frac{Z \vdash \sharp A}{Z \vdash (A)^0}\ [\vdash \cdot^0]\ \text{by [gc]} \qquad \frac{\dfrac{A \vdash \flat Z}{Z \vdash \sharp A}}{Z \vdash (A)^0}\ [\vdash \cdot^0]\ \text{by co} \qquad \frac{A \vdash \flat Z}{Z \vdash (A)^0}\ [\cdot^0{}_{\mathrm{R}}]$$

□ 10. Genzten sequents for Galois operators

$$\frac{\Gamma \vdash A}{{}^0(A) \vdash \flat\Gamma}\ [{}^0(\cdot)\mathrm{L}] \qquad \frac{A \vdash \sharp\Gamma}{\Gamma \vdash {}^0(A)}\ [{}^0(\cdot)\mathrm{R}]$$

$$\frac{\Gamma \vdash A}{(A)^0 \vdash \sharp\Gamma}\ [(\cdot)^0\mathrm{L}] \qquad \frac{A \vdash \flat\Gamma}{\Gamma \vdash (A)^0}\ [(\cdot)^0\mathrm{R}]$$

$$\frac{\Gamma \vdash \Theta \quad \Theta \vdash \Delta}{\Gamma \vdash \Delta}\ [\mathrm{Cut}]$$

The proofs below show that the $(\cdot)^0$ and ${}^0(\cdot)$ operators are indeed Galois connected,

$$\dfrac{\dfrac{A \vdash (B)^0 \quad \dfrac{B \vdash B}{(B)^0 \vdash \sharp B}\ [(\cdot)^0\mathrm{L}]}{A \vdash \sharp B}\ [\mathrm{Cut}]}{B \vdash {}^0(A)}\ [{}^0(\cdot)\mathrm{R}] \qquad \dfrac{\dfrac{A \vdash {}^0(B) \quad \dfrac{B \vdash B}{{}^0(B) \vdash \flat B}\ [{}^0(\cdot)\mathrm{L}]}{A \vdash \flat B}\ [\mathrm{Cut}]}{B \vdash (A)^0}\ [(\cdot)^0\mathrm{R}]$$

Moreover, the operators satisfy the appropriate Galois composition laws.

$$\dfrac{\dfrac{A \vdash A}{(A)^0 \vdash \sharp A}\ [(\cdot)^0\mathrm{L}]}{A \vdash {}^0((A)^0)}\ [{}^0(\cdot)\mathrm{R}] \qquad \dfrac{\dfrac{A \vdash A}{{}^0(A) \vdash \flat A}\ [{}^0(\cdot)\mathrm{L}]}{A \vdash ({}^0(A))^0}\ [(\cdot)^0\mathrm{R}]$$

From these, the fact that the operators are $[\downarrow]$-functions follows immediately.

□ 11. Galois: Linguistic Application

Generalized Quantifiers act locally like a noun phrase but take scope semantically at a higher sentential level. In [Moortgat 91] this is expressed by means of the type-assignment $q(np, s, s)$, where q follows the rule of use of below:

$$\frac{\Delta[A] \Rightarrow B \quad \Gamma[C] \Rightarrow D}{\Gamma[\Delta[q(A, B, C)]] \Rightarrow D}\ (qL)$$

The system NL +qL predicts that all GQs have the same scope possibilities contrary to linguistic data [Beghelli and Stowell 97].

In [Bernardi and Moot 00] the derivability relation $\Diamond\Box^{\downarrow}s \Rightarrow s \Rightarrow \Box^{\downarrow}\Diamond s$ is used to account for the different scopal possibilities analysed in [Beghelli and Stowell 97].

However, the linear order given by $\Diamond, \Box^{\downarrow}$, is not enough to deal with polarity items, e.g. *any student* vs. *some student*.

The addition of Galois operators generates new derivability patterns:

$$\begin{array}{ccc} s & \Rightarrow & {}^{0}(s^{0}) \\ & \searrow & \\ & & ({}^{0}s)^{0} \end{array} \qquad \text{hence} \qquad \begin{array}{ccc} s & \Rightarrow & \Box^{\downarrow}\Diamond s \\ & \searrow & \not\Updownarrow \\ & & ({}^{0}s)^{0} \end{array}$$

□ 12. Galois: Linguistic Application Polarity Items

1. **Any student* left.
2. John didn't see *any student.*
3. *Some student* left.
4. John didn't see *some student.*

Lexicon	
didn't	$\in ((s/(np\backslash s))\backslash s)/(np\backslash({}^0 s)^0)$
any N	$\in q(np, ({}^0 s)^0, ({}^0 s)^0)$
some N	$\in q(np, \Box^{\downarrow}\Diamond s, \Box^{\downarrow}\Diamond s)$

$$\dfrac{\dfrac{np \Rightarrow np \quad s \Rightarrow ({}^0 s)^0}{np \bullet np\backslash s \Rightarrow ({}^0 s)^0}\,(\backslash L) \qquad \boxed{({}^0 s)^0 \not\Rightarrow \Box^{\downarrow}\Diamond s}}{\underbrace{q(np, ({}^0 s)^0, ({}^0 s)^0)}_{\text{Any student}} \bullet \underbrace{np\backslash s}_{\text{left}} \Rightarrow \Box^{\downarrow}\Diamond s}\,(qL)$$

$\square$ 13. Polarity Items (I)

$$
\dfrac{
\dfrac{
\dfrac{
np \Rightarrow np \qquad
\dfrac{np \Rightarrow np \quad \boxed{s \Rightarrow s_1}}{np \bullet np\backslash s \Rightarrow s_1}\,(\backslash L)
}{np \bullet ((np\backslash s)/np \bullet np) \Rightarrow s_1}\,(/L)
\qquad \boxed{s_2 \Rightarrow ({}^0 s)^0}
}{np \bullet ((np\backslash s)/np \bullet \boxed{q(np, s_1, s_2)}) \Rightarrow ({}^0 s)^0}\,(qL)
}{(np\backslash s)/np \bullet q(np, s_1, s_2) \Rightarrow np\backslash({}^0 s)^0}\,(\backslash R)
\qquad
\dfrac{np \Rightarrow s/(np\backslash s) \quad s \Rightarrow \square^{\downarrow}\Diamond s}{np \bullet (s/(np\backslash s))\backslash s \Rightarrow \square^{\downarrow}\Diamond s}\,(\backslash L)
}{
\underbrace{np}_{\text{subject}} \bullet (\underbrace{((s/(np\backslash s))\backslash s)/(np\backslash({}^0 s)^0)}_{\text{negation}} \bullet (\underbrace{(np\backslash s)/np}_{\text{trans. verb}} \bullet \underbrace{q(np, s_1, s_2)}_{\text{gen. quantifier}})) \Rightarrow \square^{\downarrow}\Diamond s
}\,(/L)
$$

Wide Scope Negation (¬GQ).

□ 14. Polarity Items (II)

$$
\dfrac{
\dfrac{
np \Rightarrow np
\qquad
\dfrac{
\dfrac{s \Rightarrow ({}^{0}s)^{0} \quad np \Rightarrow np}{np\backslash s \Rightarrow np\backslash({}^{0}s)^{0}}\,(\backslash R-L)
\quad
\dfrac{np \Rightarrow s/(np\backslash s) \quad \boxed{s \Rightarrow s_1}}{np \bullet (s/(np\backslash s))\backslash s \Rightarrow s_1}\,(\backslash L)
}{np \bullet (((s/(np\backslash s))\backslash s)/(np\backslash({}^{0}s)^{0}) \bullet np\backslash s) \Rightarrow s_1}\,(/L)
}{np \bullet (((s/(np\backslash s))\backslash s)/(np\backslash({}^{0}s)^{0}) \bullet ((np\backslash s)/np \bullet np)) \Rightarrow s_1}\,(/L)
\qquad
\boxed{s_2 \Rightarrow \Box^{\downarrow}\Diamond s}
}{
\underbrace{np}_{\text{subject}} \bullet (\underbrace{((s/(np\backslash s))\backslash s)/(np\backslash({}^{0}s)^{0})}_{\text{negation}} \bullet (\underbrace{(np\backslash s)/np}_{\text{trans. verb}} \bullet \underbrace{\boxed{q(np, s_1, s_2)}}_{\text{gen. quantifier}})) \Rightarrow \Box^{\downarrow}\Diamond s
}\,(qL)
$$

Narrow Scope Negation (GQ¬).

□ 15. Conclusion

We have shown that

1. DC provides new insight on categorial type logics.
2. It can be used to understand how algebraic principles can characterize the behavior of logical operators.
3. In particular, DC sheds lights on the residuation behavior of the logical operators of $\mathsf{NL}(\diamond)$.
4. Moreover, it helps extend $\mathsf{NL}(\diamond)$ with primitive operators that resemble a pair of split negations.
5. Finally, we have provided some ideas on how the new system can be used in analyzing linguistic phenomena.