Scope Phenomena and Adverb Order in Type Logical Grammar

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$\square$ 1. Introduction

Applications of the residuation calculus: We will (i) expalain the logical behavior of the unary operators $\Diamond$ $\Box^{\downarrow}$ introduced in [Moortgat 95], showing their relation with the $\backslash, \bullet, /$ operators of NL [Lambek 58, 61]; (ii) present their application to scope phenomena and adverb order.

The other algebraic principle of NL : We will show (i) that the $\backslash, /$ are (similarly) related to another pair of unary operators, the Galois operators; (ii) how these new operators can help improve on the linguistic analysis considered in the first part.

Future Research: We will give some insight on (i) how $\Diamond, \Box^{\downarrow}$ can be connected to the operators $\uparrow, \downarrow$ of Dynamic Montague Grammar [Groenendijk & Stokhof 1990]; (ii) how the Galois operators allow to introduce the Argument Raising Law presented in [Hendriks 93].

□ 2. Residuation In Algebra

Let $\mathcal{A}_i = (A_i, \sqsubseteq_{A_i})$ be partially ordered sets. A pair of functions (f, g) such that $f : A_1 \to A_2$ and $g : A_2 \to A_1$ forms a *residuated pair* if

$$[RES_1] \quad \forall x \in A_1, y \in A_2 \begin{pmatrix} f(x) \sqsubseteq_{A_2} y \\ \text{iff} \\ x \sqsubseteq_{A_1} g(y) \end{pmatrix}$$

Similarly, a triple of functions (f, g, h) such that $f : A_1 \times A_2 \to A_3$, $g : A_1 \times A_3 \to A_2$, $h : A_3 \times A_2 \to A_1$ forms a *residuated triple* if

$$[RES_2] \quad \forall x \in A_1, y \in A_2, z \in A_3 \begin{pmatrix} x \sqsubseteq_{A_1} h(z, y) \\ \text{iff} \\ f(x, y) \sqsubseteq_{A_3} z \\ \text{iff} \\ y \sqsubseteq_{A_2} g(x, z) \end{pmatrix}$$

□ 3. Residuated Triple In Proof-Theory

$$[RES_2] \quad \forall A, B, C \in \mathsf{TYPE} \begin{pmatrix} A \vdash C/B \\ \text{iff} \\ A \bullet B \vdash C \\ \text{iff} \\ A \vdash B\backslash C \end{pmatrix}$$

Logic Rules NL	
$\dfrac{\Gamma \vdash B \quad \Delta[A] \vdash C}{\Delta[(\Gamma \circ B\backslash A)] \vdash C}\ [\backslash L]$	$\dfrac{(B \circ \Gamma) \vdash A}{\Gamma \vdash B\backslash A}\ [\backslash R]$
$\dfrac{\Gamma[(A \circ B)] \vdash C}{\Gamma[(A \bullet B)] \vdash C}\ [\bullet L]$	$\dfrac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma \circ \Delta \vdash A \bullet B}\ [\bullet R]$
$\dfrac{\Gamma \vdash B \quad \Delta[A] \vdash C}{\Delta[(A/B \circ \Gamma)] \vdash C}\ [/L]$	$\dfrac{(\Gamma \circ B) \vdash A}{\Gamma \vdash A/B}\ [/R]$

□ 4. Residuation and Monotonicity In Algebra

Saying that (f, g) is a residuated pair is equivalent to the conditions *i*) and *ii*),

i. f and g are $[\uparrow]$-functions.

$$\text{ii. } \forall y \in A_2, x \in A_1 \begin{pmatrix} f(g(y)) \sqsubseteq_{A_2} y \\ \text{and} \\ x \sqsubseteq_{A_1} g(f(x)) \end{pmatrix}$$

Similarly, saying that (f, g, h) is a residuated triple is equivalent to requiring

i. f is a $[\uparrow, \uparrow]$-function, g is an $[\downarrow, \uparrow]$-function and h is an $[\uparrow, \downarrow]$-function.

$$\text{ii. } \forall x \in A_1, y \in A_2, z \in A_3 \begin{pmatrix} f(x, g(x, z)) \sqsubseteq_{A_3} z \\ \text{and} \\ y \sqsubseteq_{A_2} g(x, f(x, y)) \\ \text{and} \\ f(h(z, y), y) \sqsubseteq_{A_3} z \\ \text{and} \\ x \sqsubseteq_{A_1} h(f(x, y), y) \end{pmatrix}$$

□ 5. Residuated Triple and Monotonicity In Proof Theory

i. $\bullet$ is a $[\uparrow,\uparrow]$-function, $\backslash$ is $[\downarrow,\uparrow]$-function, and $/$ is $[\uparrow,\downarrow]$-function. Viz. $\forall A, B \in \mathsf{TYPE}$, s.t. $A \vdash B$,

$$A \bullet C \vdash B \bullet C \quad \text{and} \quad C \bullet A \vdash C \bullet B$$

$$A/C \vdash B/C \quad \text{and} \quad C/B \vdash C/A$$

$$C\backslash A \vdash C\backslash B \quad \text{and} \quad B\backslash C \vdash A\backslash C$$

ii.

$$\forall A, B, C \in \mathsf{TYPE} \left(\begin{array}{c} A \bullet A\backslash C \vdash C \\ \text{and} \\ B \vdash A\backslash(A \bullet B) \\ \text{and} \\ C/B \bullet B \vdash C \\ \text{and} \\ A \vdash (A \bullet B)/B \end{array} \right)$$

□ 6. Residuated Pair In Proof Theory

$$[RES_1] \quad \forall A, B \in \mathsf{TYPE} \left(\begin{array}{c} \Diamond A \vdash B \\ \text{iff} \\ A \vdash \Box^{\downarrow} B \end{array} \right)$$

Logic Rules: Unary Operators

$$\frac{\Gamma[\langle A \rangle] \vdash C}{\Gamma[\Diamond A] \vdash C}\ [\Diamond \mathrm{L}] \qquad \frac{\Gamma \vdash A}{\langle \Gamma \rangle \vdash \Diamond A}\ [\Diamond \mathrm{R}]$$

$$\frac{\Delta[A] \vdash C}{\Delta[\langle \Box^{\downarrow} A \rangle] \vdash C}\ [\Box^{\downarrow} \mathrm{L}] \qquad \frac{\langle \Gamma \rangle \vdash A}{\Gamma \vdash \Box^{\downarrow} A}\ [\Box^{\downarrow} \mathrm{R}]$$

□ 7. Residuated Pair and Monotonicity In Proof Theory

i. $\Diamond, \Box^{\downarrow}$ are $[\uparrow]$-funcitons. Viz. $\forall A, B$ s.t. $A \vdash B$,

$$\Diamond A \vdash \Diamond B \quad \text{and} \quad \Box^{\downarrow} A \vdash \Box^{\downarrow} B$$

ii.

$$\forall A \in \mathsf{TYPE} \left(\begin{array}{c} \Diamond\Box^{\downarrow} A \vdash A \\ \text{and} \\ A \vdash \Box^{\downarrow}\Diamond A \end{array} \right)$$

For example,

$$\dfrac{\dfrac{A \vdash A}{\langle \Box^{\downarrow} A \rangle \vdash A}\,[\Box^{\downarrow}L]}{\Diamond\Box^{\downarrow} A \vdash A}\,[\Diamond L] \qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots \\ \Diamond\Box^{\downarrow} A \vdash A\end{array}}{\Diamond\Diamond\Box^{\downarrow} A \vdash \Diamond A}\,[\text{Mon}]}{\langle \Box^{\downarrow}\Diamond\Diamond\Box^{\downarrow} \rangle \vdash \Diamond A}\,[\Box^{\downarrow}L]}{\Box^{\downarrow}\Diamond\Diamond\Box^{\downarrow} A \vdash \Box^{\downarrow}\Diamond A}\,[\Box^{\downarrow}R]$$

□ 8. Binary and Unary residuated operators

$$
\begin{array}{lcl}
A/\underline{B} & \rightsquigarrow & \Box^{\downarrow}A \\
A\bullet\underline{B} & \rightsquigarrow & \Diamond A \\
A\circ\underline{B} & \rightsquigarrow & \langle A\rangle
\end{array}
$$

Logic Rules: Binary Operartors

$$\frac{\Gamma[(A\circ B)]\vdash C}{\Gamma[(A\bullet B)]\vdash C}\ [\bullet L] \qquad \frac{\Gamma\vdash A \quad \Delta\vdash B}{\Gamma\circ\Delta\vdash A\bullet B}\ [\bullet R]$$

$$\frac{\Gamma\vdash B \quad \Delta[A]\vdash C}{\Delta[(A/B\circ\Gamma)]\vdash C}\ [/L] \qquad \frac{(\Gamma\circ B)\vdash A}{\Gamma\vdash A/B}\ [/R]$$

Logic Rules: Unary Operators

$$\frac{\Gamma[\langle A\rangle]\vdash C}{\Gamma[\Diamond A]\vdash C}\ [\Diamond L] \qquad \frac{\Gamma\vdash A}{\langle\Gamma\rangle\vdash\Diamond A}\ [\Diamond R]$$

$$\frac{\Delta[A]\vdash C}{\Delta[\langle\Box^{\downarrow}A\rangle]\vdash C}\ [\Box^{\downarrow}L] \qquad \frac{\langle\Gamma\rangle\vdash A}{\Gamma\vdash\Box^{\downarrow}A}\ [\Box^{\downarrow}R]$$

□ 9. Linguistic Application. Scope Phenomena with GQs

In [Hendriks 93] GQ scope phenomena are accounted for by means of (i) Value Raising (VR), (ii) Argument Lowering (AL), and (iii) Argument Raising (AR).

VR, AL are theorems of NL since / (resp. \) is $[\uparrow, \downarrow]$ (resp. $[\downarrow, \uparrow]$)-function. Wheras AR is not, viz. $C/A \not\vdash C/B$.

In [Moortgat 91] NL is extended with the rule of use below.

$$\frac{\Delta[A] \vdash B \quad \Gamma[C] \vdash D}{\Gamma[\Delta[q(A, B, C)]] \vdash D}\ (qL)$$

A GQ is assigned the type $q(np, s, s)$: It acts locally like a noun phrase but takes scope semantically at a higher sentential level.

From such a system all logically possible GQs scope combinations are derivable: NL + qL = AL + VR + AR.

□ 10. Linguistic Application. GQ

[Beghelli and Stowell 1997] QGs show *different* scope possibilities, for example, with respect to negation:

1.)	John didn't pass *an exam.*	$\exists\neg$	$\neg\exists$
2.)	John didn't pass *any exam*		$\neg\exists$
3.)	John didn't pass *some exam*	$\exists\neg$	
4.)	**Any student* left.		

In [Bernardi and Moot 00] the derivability relation $\Diamond\Box^{\downarrow}s \longrightarrow s \longrightarrow \Box^{\downarrow}\Diamond s$ is used to account for these different scopal possibilities of GQs.

Lexicon	
study	$\in (np\backslash\Diamond\Box^{\downarrow}s)/np$
didn't	$\in (s/(np\backslash s)\backslash s)\backslash(np\backslash s)$
a N	$\in q(np, s, s)$
some N	$\in q(np, \Box^{\downarrow}\Diamond s, \Box^{\downarrow}\Diamond s)$
any N	$\in q(np, \Diamond\Box^{\downarrow}s, \Diamond\Box^{\downarrow}s)$

□ 11. Scope Phenomena (GQs)

$$
\dfrac{
\dfrac{
\dfrac{np \vdash np \qquad \dfrac{np \vdash np \qquad \boxed{\Diamond\Box^{\downarrow}s \vdash s_1}}{np \circ np\backslash\Diamond\Box^{\downarrow}s \vdash s_1}\,(\backslash L)}{np \circ ((np\backslash\Diamond\Box^{\downarrow}s)/np \circ np) \vdash s_1}\,(/L) \qquad \boxed{s_2 \vdash s}
}{np \circ ((np\backslash\Diamond\Box^{\downarrow}s)/np \circ \boxed{q(np, s_1, s_2)}) \vdash s}\,(qL)
}{(np\backslash\Diamond\Box^{\downarrow}s)/np \circ q(np, s_1, s_2) \vdash np\backslash s}\,(\backslash R)
\qquad
\dfrac{np \vdash s/(np\backslash s) \qquad s \vdash \Box^{\downarrow}\Diamond s}{np \circ (s/(np\backslash s))\backslash s \vdash \Box^{\downarrow}\Diamond s}\,(\backslash L)
$$

$$
\dfrac{\cdots}{\underbrace{np}_{\text{subject}} \circ (\underbrace{((s/(np\backslash s))\backslash s)/(np\backslash s)}_{\text{negation}} \circ (\underbrace{(np\backslash\Diamond\Box^{\downarrow}s)/np}_{\text{trans. verb}} \circ \underbrace{q(np, s_1, s_2)}_{\text{gen. quantifier}})) \vdash \Box^{\downarrow}\Diamond s}\,(/L)
$$

Wide Scope Negation (¬GQ).

$\Box$ 12. Scope Phenomena (GQs)

$$
\cfrac{
\cfrac{
np \vdash np \qquad
\cfrac{
\cfrac{\Diamond\Box^{\downarrow}s \vdash s \quad np \vdash np}{np\backslash\Diamond\Box^{\downarrow}s \vdash np\backslash s}\;(\backslash R - L)
\qquad
\cfrac{np \vdash s/(np\backslash s) \quad \boxed{s \vdash s_1}}{np \circ (s/(np\backslash s))\backslash s \vdash s_1}\;(\backslash L)
}{np \circ (((s/(np\backslash s))\backslash s)/(np\backslash s) \circ np\backslash\Diamond\Box^{\downarrow}s) \vdash s_1}\;(/L)
}{np \circ (((s/(np\backslash s))\backslash s)/(np\backslash s) \circ ((np\backslash\Diamond\Box^{\downarrow}s)/np \circ np)) \vdash s_1}\;(/L)
\qquad \boxed{s_2 \vdash \Box^{\downarrow}\Diamond s}
}{
\underbrace{np}_{\text{subject}} \circ (\underbrace{((s/(np\backslash s))\backslash s)/(np\backslash s)}_{\text{negation}} \circ (\underbrace{(np\backslash\Diamond\Box^{\downarrow}s)/np}_{\text{trans. verb}} \circ \underbrace{\boxed{q(np, s_1, s_2)}}_{\text{gen. quantifier}})) \vdash \Box^{\downarrow}\Diamond s
}\;(qL)
$$

Narrow Scope Negation (GQ¬).

$\Box$ 13. Scope Phenomena. Polarity Items: Problem

The string *Any student left* is proved to be of type $\Box^{\downarrow}\Diamond s$.

$$\dfrac{\dfrac{np \vdash np \quad \Diamond\Box^{\downarrow}s \vdash \Diamond\Box^{\downarrow}s}{np \circ np\backslash\Diamond\Box^{\downarrow}s \vdash \Diamond\Box^{\downarrow}s}\ (\backslash L) \qquad \boxed{\Diamond\Box^{\downarrow}s \vdash \Box^{\downarrow}\Diamond s}}{\underbrace{q(np, \Diamond\Box^{\downarrow}s, \Diamond\Box^{\downarrow}s)}_{\text{Any student}} \circ \underbrace{np\backslash\Diamond\Box^{\downarrow}s}_{\text{left}} \vdash \Box^{\downarrow}\Diamond s}\ (qL)$$

The linear pattern given by residuation,

$$\Diamond\Box^{\downarrow}s \longrightarrow s \longrightarrow \Box^{\downarrow}\Diamond s$$

is not enough to deal with Polarity Phenomena.

□ 14. Adverb Ordering

Cinque's Hierarchy (1999)

Cinque's Hierarchy (1999)

[$_{mood_{speech-act}}$ frankly [$_{mood_{evaluative}}$ fortunately [$_{mood_{evidential}}$ allegedly [$_{mod_{episthemic}}$ probably [$_{T_{past}}$ once [$_{T_{future}}$ then [$_{mod_{irrealis}}$ perhaps [$_{mod_{necessity}}$ necessarily [$_{mod_{possibility}}$ possibly [$_{asp_{habitual}}$ usually [$_{asp_{repetetive}}$ again [$_{asp_{freq(I)}}$ often [$_{mod_{volitional}}$ intentionally [$_{asp_{celerative(I)}}$ quickly [$_{T_{anterior}}$ already [$_{asp_{terminaitive}}$ no longer [$_{asp_{continuative}}$ still [$_{asp_{perfect(?)}}$ always [$_{asp_{retrospective}}$ just [$_{asp_{proximative}}$ soon [$_{asp_{durative}}$ briefly [$_{asp_{generic/progressive}}$ characteristically(?) [$_{asp_{prospective}}$ almost [$_{asp_{sg.completive(I)}}$ completely [$_{asp_{pl.completive}}$ tutto [$_{voice}$ well [$_{asp_{celerative(II)}}$ fast/early [$_{asp_{repetetive(II)}}$ again [$_{asp_{freq(II)}}$ often [$_{asp_{sg.completive(II)}}$ completely]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]

Some Support

(1) a. *John **almost probably** forgot about it.

b. *John **completely allegedly** forgot about it.

c. *John **completely briefly** forgot about it.

Some counterexamples

(2) a. I **still usually** treat with Amox, with an occasional dose of Rocephin if the child looks pretty sick.

b. It's **always allegedly** to protect the wolves.

c. Clinton announces the US will **no longer intentionally** degrade GPS signals.

□ 15. Ernst's "Fact-Event Object Calculus"

1. Simplifies Cinque's order
2. Attempts to provide a semantic motivation for it

events ⟶ propositions ⟶ facts ⟶ EXPRESS(facts)

(3) a. John **paradoxically didn't** forget about it.

b. *John **didn't paradoxically** forget about it.

(4) a. John **didn't completely** forget about it.

b. *John **completely didn't** forget about it.

(5) a. John **paradoxically completely** forgot about it.

b. *John **completely paradoxically** forgot about it.

paradoxically $\prec$ didn't $\prec$ completely

16. Ernst in **NL**

paradoxically	$= [_{FACT}$ paradoxical $[_{FACT}\ ]]$	$\rightsquigarrow F/F$
didn't	$= [_{PROPOSITION}$ not $[_{PROPOSITION}\ ]]$	$\rightsquigarrow P/P$
completely	$= [_{EVENT}$ complete $[_{EVENT}\ ]]$	$\rightsquigarrow E/E$

Let $E, F, P \in TYPE$, s. t. $E \vdash P \vdash F$

$$\dfrac{\mathit{paradoxically} \vdash F/F \qquad \dfrac{F \vdash F \qquad \dfrac{\dfrac{\mathit{didn't} \vdash P/P \qquad \dfrac{P \vdash P \qquad \begin{array}{c}\Delta \vdash E\\ \vdots\\ \Delta \vdash P\end{array}}{(P/P \circ \Delta) \vdash P}\,/L}{(\mathit{didn't} \circ \Delta) \vdash P}\,\mathit{Cut}}{(\mathit{didn't} \circ \Delta) \vdash F}\,(\mathit{by}\, P \Rightarrow F)}{(F/F \circ (\mathit{didn't} \circ \Delta)) \vdash F}\,/L}{(\mathit{paradoxically} \circ (\mathit{didn't} \circ \Delta)) \vdash F}\,\mathit{Cut}$$

$$\dfrac{\mathit{didn't} \vdash P/P \qquad \dfrac{P \vdash P \qquad \dfrac{\dfrac{\mathit{completely} \vdash E/E \qquad \dfrac{E \vdash E \qquad \Delta \vdash E}{(E/E \circ \Delta) \vdash E}\,/L}{(\mathit{completely} \circ \Delta) \vdash E}\,\mathit{Cut}}{(\mathit{completely} \circ \Delta) \vdash P}\,(\mathit{by}\, E \Rightarrow P)}{(P/P \circ (\mathit{completely} \circ \Delta)) \vdash P}\,/L}{(\mathit{didn't} \circ (\mathit{completely} \circ \Delta)) \vdash P}\,\mathit{Cut}$$

$\Box$ 17. Ernst in $\mathsf{NL}(\Diamond)$

Since / is a $[\uparrow, \downarrow]$-function, we have that

$E/E \vdash P/E \vdash F/E$	VR
$P/P \vdash F/P$	VR
$P/P \vdash P/E$	AL
$F/F \vdash F/P \vdash F/E$	AL

The data are correctly predicted, for instance, $(completely \circ paradoxically \circ \Delta) \not\vdash F$

The residuated unary connectives of $\mathsf{NL}(\Diamond)$ are the types we need, since:

$$\boxed{\Diamond\Box^{\downarrow}s \;\longrightarrow s \;\longrightarrow \Box^{\downarrow}\Diamond s}$$

$$E : \Diamond\Box^{\downarrow}s \quad \longrightarrow P : s \quad \longrightarrow F : \Box^{\downarrow}\Diamond s$$

□ 18. Problems: “facts” and “propositions”

Claim: “paradoxically P” is not “factive”.

(6) a. The current King of the USA is in the corridor.

b. There is a current King of the USA in the corridor.

c. Paradoxically, there is a current King of the USA in the corridor.

d. John said that Bill paradoxically went home.

e. It is paradoxical that Bill went home.

□ 19. Counterexamples to linearity (I)

possibly not always

(7) a. Once I figured out that the Branch Davidians **didn't possibly** kill themselves, though, I looked at the Commandments and wondered how ...

b. Once I figured out that the Branch Davidians **possibly didn't** kill themselves, though, I looked at the Commandments and wondered how ...

(8) a.

Jens	kom	**muligens**	**ikke**	tidsnok.
J	came	possibly	not	on.time

b.

*Jens	kom	**ikke**	**muligens**	tidsnok.
J	came	not	possibly	on.time

$\Downarrow$

Norwegian: $\boxed{\text{muligens} \prec \text{ikke}}$

□ 20. Counterexamples to linearity (II)

possibly not always

(9) a. John **didn't always** show up.

b. John **always didn't** show up.

(10) a. Jens kom **ikke alltid** tidsnok.
J came not always on.time

b. *Jens kom **alltid ikke** tidsnok.
J came always not on.time

$\Downarrow$

Norwegian: $\boxed{\text{ikke} \prec \text{alltid}}$

□ 21. Counterexamples to linearity (II)

Expectation: $\boxed{\text{muligens} \prec \text{alltid}}$

(11) a. Jens kommer **muligens alltid** for sent.
J comes possibly always too late.

b. Jens kommer alltid muligens for sent
J comes **always possibly** too late
"It is always possible that he comes too late."

(12) a. This is a fun, free game where you're **always possibly** a click away from winning $1000!

b. I farm pigs, cattle and arable and the protest was **always possibly** going to get to this stage.

alltid muligens isn't (necessarily) a constituent

(13) Derfor kommer **alltid** Jens **muligens** for sent.
therefore comes always J possibly too late

ikke muligens is possible if *alltid* intervenes

(14) Jens kommer **ikke alltid muligens** for sent
J comes not always possibly too late

$\Box$ 22. Deriving the facts with $\Box^{\downarrow}, \Diamond$

The derivability square for $\Diamond$ and $\Box^{\downarrow}$

$$\begin{tikzcd} & s_3 : \Box^{\downarrow}\Diamond\Diamond\Box^{\downarrow} s \arrow[dr] & \\ s_1 : \Diamond\Box^{\downarrow} s \arrow[ur] \arrow[dr] & & s_4 : \Box^{\downarrow}\Diamond s \\ & s_2 : s \arrow[ur] & \end{tikzcd}$$

Lexicon	
Δ	$\in s_1$
muligens	$\in s_2/s_4$
ikke	$\in s_3/s_3$
alltid	$\in s_1/s_2$

Given that / is a $[\uparrow, \downarrow]$-function, we have that e.g.

$s_3/s_3 \vdash s_4/s_1$	AL and VR
$s_4/s_2 \vdash s_4/s_1$	AL
$s_2/s_1 \vdash s_4/s_1$	VR

$$
\dfrac{muligens \vdash s_2/s_4 \qquad \dfrac{s_2 \vdash s_2 \qquad \dfrac{\dfrac{ikke \vdash s_3/s_3 \qquad \dfrac{s_3 \vdash s_3 \qquad \begin{array}{c}\Delta \vdash s_1\\ \vdots\\ \Delta \vdash s_3\end{array}}{(s_3/s_3 \circ \Delta) \vdash s_3}\;/L}{(ikke \circ \Delta) \vdash s_3}\;Cut}{(ikke \circ \Delta) \vdash s_4}\;(by\; s_3 \Rightarrow s_4)}{(s_2/s_4 \circ (ikke \circ \Delta)) \vdash s_2}\;/L}{(muligens \circ (ikke \circ \Delta)) \vdash s_2}\;Cut
$$

$$
\dfrac{ikke \vdash s_3/s_3 \qquad \dfrac{s_3 \vdash s_3 \qquad \dfrac{\dfrac{alltid \vdash s_1/s_2 \qquad \dfrac{s_1 \vdash s_1 \qquad \begin{array}{c}\Delta \vdash s_1\\ \vdots\\ \Delta \vdash s_2\end{array}}{(s_1/s_2 \circ \Delta) \vdash s_1}\;/L}{(alltid \circ \Delta) \vdash s_1}\;Cut}{(alltid \circ \Delta) \vdash s_3}\;(by\; s_1 \Rightarrow s_3)}{(s_3/s_3 \circ (alltid \circ \Delta)) \vdash s_3}\;/L}{(ikke \circ (alltid \circ \Delta)) \vdash s_3}\;Cut
$$

$$
\dfrac{muligens \vdash s_2/s_4 \qquad \dfrac{s_2 \vdash s_2 \qquad \dfrac{\dfrac{alltid \vdash s_1/s_2 \qquad \dfrac{s_1 \vdash s_1 \qquad \begin{array}{c}\Delta \vdash s_1 \\ \vdots \\ \Delta \vdash s_2\end{array}}{(s_1/s_2 \circ \Delta) \vdash s_1}\;/L}{(alltid \circ \Delta) \vdash s_2}\;Cut}{(alltid \circ \Delta) \vdash s_4}\;(by\, s_2 \Rightarrow s_4)}{(s_2/s_4 \circ (alltid \circ \Delta)) \vdash s_2}\;/L}{(muligens \circ (alltid \circ \Delta)) \vdash s_2}\;Cut
$$

$$
\dfrac{ikke \vdash s_3/s_3 \qquad \dfrac{s_3 \vdash s_3 \qquad \dfrac{\dfrac{alltid \vdash s_1/s_2 \qquad \dfrac{s_1 \vdash s_1 \qquad \dfrac{muligens \vdash s_2/s_4 \qquad \dfrac{s_2 \vdash s_2 \qquad \begin{array}{c}\Delta \vdash s_1 \\ \vdots \\ \Delta \vdash s_4\end{array}}{(s_2/s_4 \circ \Delta) \vdash s_2}\;/L}{(muligens \circ \Delta) \vdash s_2}\;Cut}{(s_1/s_2 \circ (muligens \circ \Delta)) \vdash s_1}\;/L}{(alltid \circ (muligens \circ \Delta)) \vdash s_1}\;Cut}{(alltid \circ (muligens \circ \Delta)) \vdash s_3}\;(by\, s_1 \Rightarrow s_3}{(s_3/s_3 \circ (alltid \circ (muligens \circ \Delta))) \vdash s_3}\;/L}{(ikke \circ (alltid \circ (muligens \circ \Delta))) \vdash s_3}\;Cut
$$

□ 24. Minimal Pattern

The minimal pattern we would need is the following:

$$\begin{tikzcd} & s'_3 \\ s'_1 \arrow[ur] \arrow[dr] & \\ & s'_2 \end{tikzcd}$$

Lexicon	
Δ	$\in s'_1$
muligens	$\in s'_3/s'_2$
ikke	$\in s'_2/s'_1$
alltid	$\in s'_1/s'_3$

This would make it easier to find semantic motivation, since it does not exhibit the unpleasant collapse into the same type (s_4/s_1) we saw above.

□ 25. Discourse anaphora and adverbs:

(15) a. This house **possibly** has a bathtub. #**It** is green.

b. This house **doesn't** have a bathtub. # **It** is green.

c. New houses **always** have a bathtub. **It** is green.

Speculation: Dynamic closure (cf. $\downarrow\uparrow\varphi$ in DMG) is associated with s_2'?

$\square$ 26. The other algebraic law in NL

We have seen that $\square^{\downarrow}B$ is obtained from $A\backslash B$ fixing the negative argument.

What do we obtain if we fix the positive one?

In NL the relations below hold. Note that they are based on a fixed element in the positive position of $/$ and $\backslash$.

$$\forall A \left(\begin{array}{c} A \vdash B/(A\backslash B) \\ \text{and} \\ A \vdash (B/A)\backslash B \end{array} \right)$$

This is the lifting theorem which has found interesting linguistic pheonomena.

In algebraic term these operators $(\backslash B, B/)$ are called Galois connected operators.

□ 27. Galois Operators In Algebra

Let $\mathcal{A}_i = (A_i, \sqsubseteq_{A_i})$ be a partially ordered set. Consider a pair of functions $f : A_1 \to A_2$ and $g : A_2 \to A_1$. The pair (f, g) is called a *Galois connection* if

$$[\mathrm{GC}] \qquad \forall x \in A_1, y \in A_2 \begin{pmatrix} y \sqsubseteq_{A_2} f(x) \\ \text{iff} \\ x \sqsubseteq_{A_1} g(y) \end{pmatrix}$$

The equivalent formulation of this property in terms of the monotonicity behavior and a composition rule is given by

i. f and g are $[\downarrow]$-functions;

ii. $\forall x \begin{pmatrix} x \sqsubseteq f(g(x)) \\ \text{and} \\ x \sqsubseteq g(f(x)) \end{pmatrix}$

□ 27. Galois operators in Proof Theory

(Areces-Bernardi)

Logical Rules	
$\dfrac{\Gamma \vdash \Delta}{{}^0(\Delta) \vdash \flat\Gamma}\ [{}^0(\cdot)\mathrm{L}]$	$\dfrac{\Delta \vdash \sharp\Gamma}{\Gamma \vdash {}^0(\Delta)}\ [{}^0(\cdot)\mathrm{R}]$
$\dfrac{\Gamma \vdash \Delta}{(\Delta)^0 \vdash \sharp\Gamma}\ [(\cdot)^0\mathrm{L}]$	$\dfrac{\Delta \vdash \flat\Gamma}{\Gamma \vdash (\Delta)^0}\ [(\cdot)^0\mathrm{R}]$
$\dfrac{\Gamma \vdash \Theta \quad \Theta \vdash \Delta}{\Gamma \vdash \Delta}\ [\mathrm{Cut}]$	

The proofs below show that the $(\cdot)^0$ and ${}^0(\cdot)$ operators are indeed Galois connected,

$$\dfrac{\dfrac{A \vdash (B)^0 \quad \dfrac{B \vdash B}{(B)^0 \vdash \sharp B}\ [(\cdot)^0\mathrm{L}]}{A \vdash \sharp B}\ [\mathrm{Cut}]}{B \vdash {}^0(A)}\ [{}^0(\cdot)\mathrm{R}] \qquad \dfrac{\dfrac{A \vdash {}^0(B) \quad \dfrac{B \vdash B}{{}^0(B) \vdash \flat B}\ [{}^0(\cdot)\mathrm{L}]}{A \vdash \flat B}\ [\mathrm{Cut}]}{B \vdash (A)^0}\ [(\cdot)^0\mathrm{R}]$$

Moreover, the operators satisfy the appropriate Galois composition laws.

$$\dfrac{\dfrac{A \vdash A}{(A)^0 \vdash \sharp A}\ [(\cdot)^0\mathrm{L}]}{A \vdash {}^0((A)^0)}\ [{}^0(\cdot)\mathrm{R}] \qquad \dfrac{\dfrac{A \vdash A}{{}^0(A) \vdash \flat A}\ [{}^0(\cdot)\mathrm{L}]}{A \vdash ({}^0(A))^0}\ [(\cdot)^0\mathrm{R}]$$

From these, the fact that the operators are [$\downarrow$]-functions follows immediately.

$\square$ 28. Galois: Linguistic Application

The addition of Galois operators to the pure logic of residuation $\mathsf{NL}(\Diamond)$ generates new derivability patterns.

The linear logic relation which characterizes the residuated pair $\Diamond, \square^{\downarrow}$ is extended as shown below:

$$
\begin{array}{ccccc}
 & & {}^{0}(s^{0}) & & \\
 & & \uparrow & & \\
\Diamond\square^{\downarrow}s & \longrightarrow & s & \longrightarrow & \square^{\downarrow}\Diamond s \\
 & & \downarrow & & \\
 & & ({}^{0}s)^{0} & &
\end{array}
$$

The pattern required by the adverb order analysis is given by the Galois operators:

$$s_1 = s,\ s_2 = {}^{0}(s^{0}) \ \text{ and } \ s_3 = ({}^{0}s)^{0}$$

$\square$ 29. Galois: Linguistic Application
Polarity Items

1. **Any student* left.
2. John didn't see *any student.*
3. *Some student* left.
4. John didn't see *some student.*

Lexicon	
didn't	$\in ((s/(np\backslash s))\backslash s)/(np\backslash({}^0 s)^0)$
any N	$\in q(np, ({}^0 s)^0, ({}^0 s)^0)$
some N	$\in q(np, \square^{\downarrow}\diamond s, \square^{\downarrow}\diamond s)$

$$\dfrac{\dfrac{np \vdash np \quad s \vdash ({}^0 s)^0}{np \circ np\backslash s \vdash ({}^0 s)^0}\,(\backslash L) \qquad \boxed{({}^0 s)^0 \not\vdash \square^{\downarrow}\diamond s}}{\underbrace{q(np, ({}^0 s)^0, ({}^0 s)^0)}_{\text{Any student}} \circ \underbrace{np\backslash s}_{\text{left}} \vdash \square^{\downarrow}\diamond s}\,(qL)$$

$\square$ 30. Polarity Items (I)

$$
\dfrac{
\dfrac{
\dfrac{
np \vdash np \qquad
\dfrac{np \vdash np \qquad \boxed{s \vdash s_1}}{np \circ np\backslash s \vdash s_1}\,(\backslash L)
}{np \circ ((np\backslash s)/np \circ np) \vdash s_1}\,(/L)
\qquad \boxed{s_2 \vdash ({}^0 s)^0}
}{np \circ ((np\backslash s)/np \circ \boxed{q(np, s_1, s_2)}) \vdash ({}^0 s)^0}\,(qL)
}{(np\backslash s)/np \circ q(np, s_1, s_2) \vdash np\backslash({}^0 s)^0}\,(\backslash R)
\qquad
\dfrac{np \vdash s/(np\backslash s) \qquad s \vdash \square^{\downarrow}\diamond s}{np \circ (s/(np\backslash s))\backslash s \vdash \square^{\downarrow}\diamond s}\,(\backslash L)
}{\underbrace{np}_{\text{subject}} \circ (\underbrace{((s/(np\backslash s))\backslash s)/(np\backslash({}^0 s)^0)}_{\text{negation}} \circ (\underbrace{(np\backslash s)/np}_{\text{trans. verb}} \circ \underbrace{q(np, s_1, s_2)}_{\text{gen. quantifier}})) \vdash \square^{\downarrow}\diamond s}\,(/L)
$$

Wide Scope Negation ($\neg$GQ).

□ 31. Polarity Items (II)

$$
\dfrac{
  \dfrac{
    np \vdash np
    \qquad
    \dfrac{
      \dfrac{s \vdash ({}^0 s)^0 \quad np \vdash np}{np\backslash s \vdash np\backslash({}^0 s)^0}\;(\backslash R - L)
      \qquad
      \dfrac{np \vdash s/(np\backslash s) \quad \boxed{s \vdash s_1}}{np \circ (s/(np\backslash s))\backslash s \vdash s_1}\;(\backslash L)
    }{np \circ (((s/(np\backslash s))\backslash s)/(np\backslash({}^0 s)^0) \circ np\backslash s) \vdash s_1}\;(/L)
  }{np \circ (((s/(np\backslash s))\backslash s)/(np\backslash({}^0 s)^0) \circ ((np\backslash s)/np \circ np)) \vdash s_1}\;(/L)
  \qquad
  \boxed{s_2 \vdash \Box^{\downarrow}\Diamond s}
}{
  \underbrace{np}_{\text{subject}} \circ (\underbrace{((s/(np\backslash s))\backslash s)/(np\backslash({}^0 s)^0)}_{\text{negation}} \circ (\underbrace{(np\backslash s)/np}_{\text{trans. verb}} \circ \underbrace{\boxed{q(np, s_1, s_2)}}_{\text{gen. quantifier}})) \vdash \Box^{\downarrow}\Diamond s
}\;(qL)
$$

Narrow Scope Negation (GQ¬).

□ 32. Future Research Connections with DMG

From the linguistic analysis of adverb order and scope pheomena, we have seen that there seems to be a correspondence between type assignment based on $\Box^{\downarrow}$, $\Diamond$ (resp. ${}^{0}\cdot$, $\cdot^{0}$) and linguistic strings involving dynamic (resp. static) "meaning".

Is there a logic connection?

Making explicit the connection between $\Box^{\downarrow}$ and $\Diamond$ with the / and $\bullet$ also at the level of lambda terms, we have that

$$\Box^{\downarrow}A : \lambda p.(Bp)_a \quad \text{and} \quad \Diamond A : (a, p)$$

In our approach sentences are of type $\Box^{\downarrow}\Diamond s$, and therefore their lambda term representation is:

$$\lambda p.(\phi, p)$$

which closely recall $\uparrow \phi =_{def} \lambda p.(\phi \wedge^{\vee} p)$, viz. the "dynamic" meaning assigned to sentence in DMG.

□ 33. Future Research Argument Raising by means of Galois

In $\mathsf{NL}(\Diamond)$ AR is not derivable. In Hendriks'thesis it is shown that it can be used to account for scope phenomena of GQs.

In the extended system $\mathsf{NL}(\Diamond, \cdot^0)$, the downward montonic unary operators, give the possibility of reversing the type-order. In particular,
$\forall A, B$ s.t. $A \vdash B$

$$C/A^0 \vdash C/B^0$$

In other words we can control the application of AR.

$$\dfrac{\dfrac{A \vdash B \quad C \vdash C}{C/B \circ A \vdash C}\,[/L]}{C/B \vdash C/A}\,[/R] \qquad \dfrac{\dfrac{\dfrac{A \vdash B}{B^0 \vdash A^0}\,[\mathrm{Mon}] \quad C \vdash C}{C/A^0 \circ B^0 \vdash C}\,[/L]}{C/A^0 \vdash C/B^0}\,[/R]$$

□ 34. Conclusion

We have shown that looking at the unary operators of $\mathsf{NL}(\diamond)$ as unary couterpart of its binary operators

1. helps understand their logical behavior and shed lights on their linguistic applications;
2. helps classifing linguistic phenomena in new ways;
3. opens the way to new unary operators, the Galois connected operators, which introduce no-linear order in the type hierarchy and the possibility of reversing it;
4. provides some insight on possible connection between the $\Box^{\downarrow}, \diamond$ and the $\uparrow, \downarrow$ opening the way to a better understanding of their (linguistic) meaning.

□ 35. References

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