1 Logic Backgroud

1.1 Residuated and Galois connected functions

Consider two posets $\mathcal{A} = (A, \sqsubseteq_A)$ and $\mathcal{B} = (B, \sqsubseteq_B)$, and functions $f : A \to B$, $g : B \to A$. The pair (f, g) is said to be *residuated* iff $\forall a \in A, b \in B$

$$[RES_1] \qquad f(a) \sqsubseteq_B b \quad \text{iff} \quad a \sqsubseteq_A g(b)$$

The pair (f, g) is said to be *Galois connected* iff $\forall a \in A, b \in B$

$$[GC_1] \qquad b \sqsubseteq_B f(a) \quad \text{iff} \quad a \sqsubseteq_A g(b)$$

Remark Let $\mathcal{B}'$ be a poset s.t. $\mathcal{B}' = (B, \sqsubseteq'_B)$ where $x \sqsubseteq'_B y \stackrel{\text{def}}{=} y \sqsubseteq_B x$, and $h : B \to A$. If (f, h) is a residuated pair with respect to $\sqsubseteq_A$ and $\sqsubseteq'_B$, then it's Galois connected with respect to $\sqsubseteq_A$ and $\sqsubseteq_B$.

$$b \sqsubseteq_B f(a) \quad \text{iff} \quad f(a) \sqsubseteq'_B \quad \text{iff} \quad a \sqsubseteq_A h(b)$$

1.2 The base logic $\mathbf{NL}(\Diamond, \cdot^{\mathbf{0}})$

Transitivity/Reflexivity of the derivability relation, plus

$$\begin{array}{llll}
(\text{RES-L}) & A \bullet B \to C & \text{iff} & A \to C/B \\
(\text{RES-R}) & A \bullet B \to C & \text{iff} & B \to A\backslash C \\
\\
(\text{RES-1}) & \Diamond A \to B & \text{iff} & A \to \Box^{\downarrow} B \\
\\
(\text{GAL}) & A \to {}^{\mathbf{0}}B & \text{iff} & B \to A^{\mathbf{0}}
\end{array}$$

Soundness/Completeness

$$A \to B \ \text{ is provable} \quad \text{iff} \quad \forall F, V,\ V(A) \subseteq V(B)$$

See [Areces, Bernardi & Moortgat 2001], also for Gentzen presentation, cut elimination and decidability.

1.3 Interpretation of the constants

$$\begin{array}{lcl}
V(\Diamond A) & = & \{x \mid \exists y (R^2_{\Diamond} xy \ \&\ y \in V(A)\} \\
V(\Box^{\downarrow} A) & = & \{x \mid \forall y (R^2_{\Diamond} yx \ \Rightarrow y \in V(A)\}
\end{array}$$

$$\begin{array}{lcl}
V({}^{\mathbf{0}}A) & = & \{x \mid \forall y (y \in V(A) \ \Rightarrow \neg R^2_0 yx\} \\
V(A^{\mathbf{0}}) & = & \{x \mid \forall y (y \in V(A) \ \Rightarrow \neg R^2_0 xy\}
\end{array}$$

$$\begin{array}{lcl}
V(A \bullet B) & = & \{z \mid \exists x \exists y [R^3 zxy \ \&\ x \in V(A) \ \&\ y \in V(B)]\} \\
V(C/B) & = & \{x \mid \forall y \forall z [(R^3 zxy \ \&\ y \in V(B)) \Rightarrow z \in V(C)]\} \\
V(A \backslash C) & = & \{y \mid \forall x \forall z [(R^3 zxy \ \&\ x \in V(A)) \Rightarrow z \in V(C)]\}
\end{array}$$

1.4 Other derived properties

Closure Let $(\cdot)^*$ be ${}^{\mathbf{0}}(\cdot^{\mathbf{0}})$, $({}^{\mathbf{0}}\cdot)^{\mathbf{0}}$, $\Box^{\downarrow}\Diamond(\cdot)$, $X/(\cdot\backslash X)$, $(X/\cdot)\backslash X$. $\forall A \in$ TYPE

$$A \to A^*, \quad A^* \to B^* \ \text{ if } \ A \to B, \quad A^{**} \to A^*$$

Triple Let (f_1, f_2) be either the residuated or the Galois pair, $f_1 f_2 f_1 A \leftrightarrow f_1 A$, and similarly $f_2 f_1 f_2 A \leftrightarrow f_2 A$. For example,

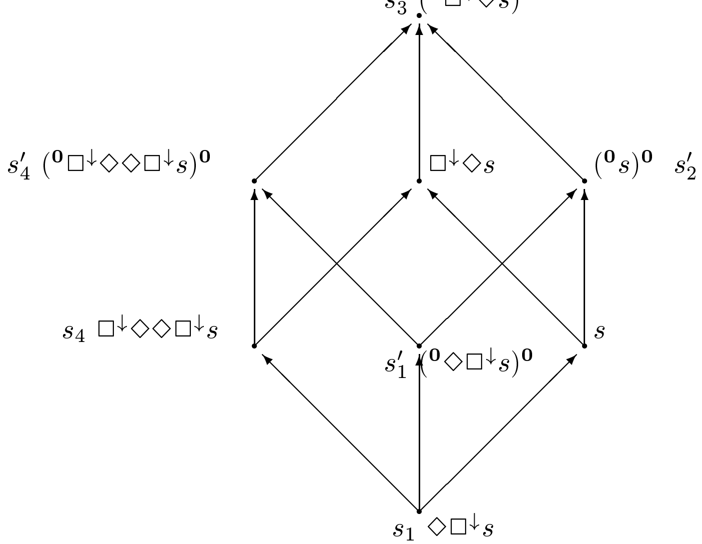
$$\Diamond\Box^{\downarrow}\Diamond A \leftrightarrow \Diamond A \ \text{ and } \ \Box^{\downarrow}\Diamond\Box^{\downarrow} A \leftrightarrow \Box^{\downarrow} A.$$

2 Memo. From the slides

2.1 Some useful derived properties

Compositions	$\Diamond\Box^{\downarrow}A \to A$		$A \to \Box^{\downarrow}\Diamond A$
	$A \to {}^{\mathbf{0}}(A^{\mathbf{0}})$		$A \to ({}^{\mathbf{0}}A)^{\mathbf{0}}$
(Iso/Anti)tonicity	$B \to C$	implies	$B/A \to C/A$ $A/C \to A/B$
			$A\backslash B \to A\backslash C$ $C\backslash A \to B\backslash A$

2.2 The cube



3 Linguistic Applications

3.1 Greek

Greek: Lexicon

PPI: $q(np, s_4, s_4)$, *kapjos*
API: $q(np, s'_1, s'_1)$, *kanenan*
modal: $(((s'_4/np)\backslash s'_4)\backslash s_1)/(np\backslash s'_4)$, *bori*
cond.: $(s_1/s'_1)/s'_3$, *an*

NPI: $np\backslash s'_2$, *ipe leksi*
FCI: $q(np, s'_4, s'_4)$, *opudhipote*
neg.: $(np\backslash s_1)/(np\backslash s'_2)$, *dhen*

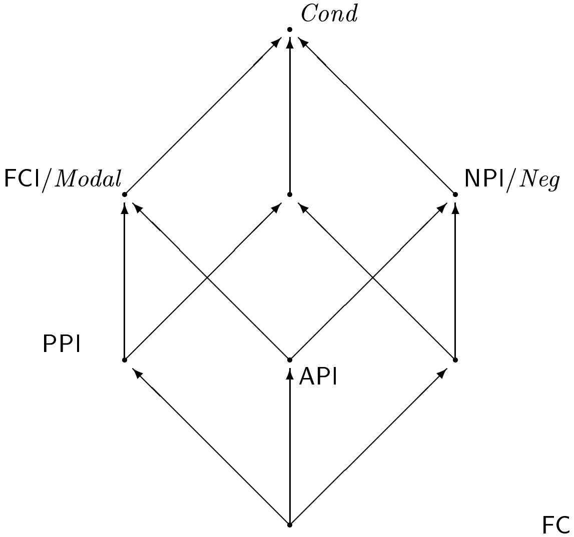
Italian: Lexicon

PPI: $q(np, s_4, s_4)$, *qualcuno*
API: $(np\backslash s_1)\backslash(np\backslash s'_1)$, *mai*
modal: $(((s''_4/np)\backslash s''_4)\backslash s_1)/(np\backslash s''_4)$, *puó*
cond: $(s_1/s'_1)/s'_4$, *se*

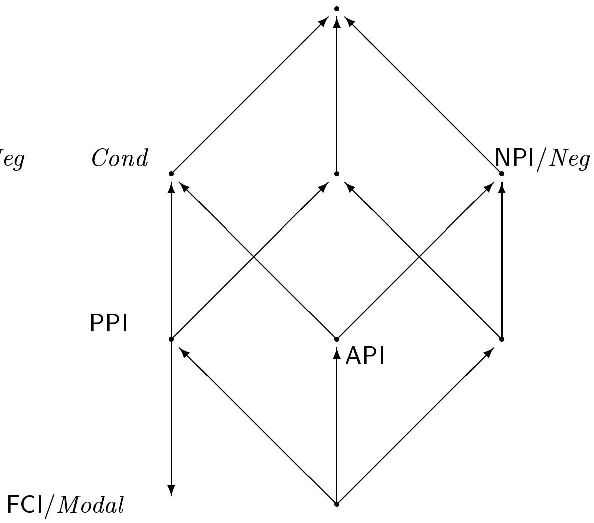
NPI: $q(np, s'_2, s'_2)$, *nessuno*
FCI: $q(np, s''_4, s''_4)$, *chiunque*
neg.: $(np\backslash s_1)/(np\backslash s'_2)$, *non*

$$q(A,B,C) \cong (A \to B) \to C$$

3.2 Comparison. Greek and Italian



Greek



Italian