About the notion of natural logic: historical and theoretical remarks

Paolo Feguglia,
L'Aquila University (Italy)

Raffaella Bernardi
Utrecht University (NL) and Chieti University (Italy)

Abstract

The paper is meant as a historical and theoretical analysis of some notions we claim to characterize a "natural logic". As an example we focus our attention on the syllogistic theory. After giving a brief introduction to the classical syllogism, we examine the techniques of the syllogistic 'demostratio' with respect to the analytic and synthetic methods and we characterize them by means of elimination and introduction rules. In the second part of the paper we propose a comparison with Gentzen's Natural Deduction.

1 Introduction: Syllogisms

The syllogistic 'demostratio' has been traditionally considered to be the unique logical instrument for formalizing natural reasoning inferences. In the philosophical discussions of the XVI century a prominent role was played by the analysis of mathematical proofs. In particular the attention was focused on the *quaestio de certitudine mathematicarum* (see Piccolomini, Catena, Clavio [2] and Barozzi [1]) and the methodological symmetry between analysis and synthesis. The *resolutio* and the *compositio*, their synonyms, were described by Dupleix [3], as:

> Analytique [...] est un mot Grec derivé d'*Analysis*, c'est à dire *Resolution*: qui n'est autre chose qu'un regrés ou retour d'une chose en ses principes [...] tellement que c'est le contraire de la composition [...].

The increased interest for the analysis was motivated by two events: the discussions on Euclide, from the one hand, and on Pappo and Diofanto, from the other. These two fact pushed the mathematicians to focus their attention on the method they were using. From this methodological studies the analysis acquired a new role, and was considered to be as important as the synthesis.

Following this idea, in the paper we define the syllogism in terms of mathematical rules able to expres both the synthesis and the analysis.

As it is well-known the classical syllogisms are based on four types of categorical propositions: universally affirmative (A), universally negative (E), particularly affirmative (I), particularly negative (O). The problem of how to interpret the categorical propositions correctly has been deeply discussed in Lukasiewicz [8]. We will represent these four propositions as:

A	$F_1(p,q)$:	Every p is q
E	$F_2(p,q)$:	No p is q
I	$F_3(p,q)$:	Some p are q
O	$F_4(p,q)$:	Some p are not q

Summing up we say that: $F_i(x,y)$ stands for a categorical proposition where x and y are the subject and the predicate, respectively.

A *syllogism* is an inference schema composed by two premises which share a term, known as medium term, and which carry the other two terms which will be repeated in the conclusion. Accordingly to the different positions these three terms occupy in the schema, different figures are obtained. As an example we discuss the first figure.

Let z be the medium term, x and y the terms in the major and minor premises, respectively,

1^{st} Figure

$$\begin{array}{cc} z & x \\ y & z \\ \hline y & x \end{array}$$

Different modes are obtained combining the four possible categorical propositions. We summarize them in the table below:

	mode_1	mode_2	mode_3	mode_4	mode_5	mode_6
figure_1	AA $\vdash$ A	EA $\vdash$ E	AI$\vdash$ I	EI$\vdash$ O	AA$\vdash$I	EA$\vdash$ O

where the XY $\vdash$ Z denotes the inference of the conclusion Z from the major and minor premises X and Y, respectively.

In order to interpret the classical syllogisms as introduction and elimination of the medium term, we represent the above inferences by means of a more general schema which has been orginally proposed in Freguglia [5] where an algebra of categories was considered as a model for the syllogisms.

Definition 1. Let $F_i(\mu, p)$, $F_j(s, \mu)$ and $F_k(s, p)$ be categorical propositions, we define a syllogism as the following transformation the validity of which is independent from the semantic truth value of the categorical propositions it consists of. Let $i, j, k \in \{1, 2, 3, 4\}$

$$\Sigma[F_i(\mu, p), F_j(s, \mu)] = F_k(s, p) \tag{1}$$

We call (1) σ-transformation, $F_i(\mu, p)$ and $F_j(s, \mu)$ the major and minor premises, respectively, and $F_k(s, p)$ the conclusion. This schema satisfies the modes of the first figure, similar schemata can be however given for the other figures simply changing the subject/predicate positions. As an example we give the inference in Darii first in the traditional way, then with the format proposed in def. 1:

Example 1: DARII

$$\frac{\text{Every roman is stubborn} \quad \text{Some italians are romans}}{\text{Some italians are stubborn}} \; DARII$$

$$\Sigma[\text{Every(roman,stubborn), Some(italians,romans)}] = \text{Some(italians,stubborn)}$$

2 Syllogisms by means of Introduction and Elimination rules

As the reader might have noticed the schema given in def. 1 simply consists in the elimination of the medium term. We can replace the above definition with a more elegant and simple one:

Definition 2 Let a and b be the major and minor premises, respectively, c the conclusion and μ the medium term shared by a and b, the syllogism is defined by the following operation:

$$E\mu[a,b] = c \tag{2}$$

Example 2

a:= Every roman is stubborn,
b:= Some italians are romans,
c:= Some italians are stubborn,

$$E\text{‘roman’[a,b]} = \text{c}$$

From an epistemological perspective we can look at the elimination rule as the *synthetic* method (the *demostratio propter quid* or deduction), see Szabo[10]. Therefore we can now define the *analysis* (i.e. the *demostratio quic* or logical induction) in terms of transformations. The analytic method starts from the results and goes back to the assumptions (see Hitikka-Remes[7]). Applying this process to the syllogism means to start from the conclusion and reconstruct the premises it has been derived from. In order to establish all the possible syllogisms with the given proposition as the conclusion, we need to use two transformations: one which gives the set of possible major premises and one which gives the set of possible minor premises.

Definition 3 Let A and B be sets of terms, $F_k(p,s)$ the given categorical proposition, we define the transformations Γ_M and Γ_m which return the major and minor premises, respectively. Let $k,i,j \in \{1,2,3,4\}$

$$\Gamma_M[F_k(s,p)] = F_{in}(\mu,p) \tag{3}$$

$$\Gamma_m[F_k(s,p)] = F_{jt}(s,\mu) \tag{4}$$

where n,t indicates the number of $\mu \in A$ and of the $\mu \in B$, respectively. We call (3) and (4) γ-transformations.

Theorem 1 Given the two sets of categorical propositions $F_{in}(\mu,p)$ and $F_{jt}(s,\mu)$ obtained by the transformations (3) and (4), $\exists\mu \in (A \cap B)$ iff a specific transformation of the type given in (1) is found as well.

From this theorem it follows that in the same way we have formalized the Σ transformation by means of the elimination rule, we can now replace the two schemata (3) and (4) with an operation, namely the inverse of the elimination rule, viz. the introduction one.

Definition 4 Let c be the given conclusion, and a, b any of the possible categorical propositions which can be the major and minor premises, the analysis is formalized by the introduction operation:

$$\mathrm{I}\mu[\mathrm{c}] = [\mathrm{a,b}] \tag{5}$$

Example 3

Let A and B be sets of terms, M and m sets of categorical propositions obtained by means of (3) and (4), respectively, and F_k(s,p) the given conclusion:

A = {girl, roman, cat}

B = {abruzzese, roman, tuscan, girl}

F_k(s,p) = Some(italians, stubborn)

M = { M_1: 'Every roman is stubborn', M_2: 'Every girl is stubborn', M_3: 'Every cat is stubborn'}

m = { m_1:'Some italians are roman', m_2: 'Some italians are girls', m_3: 'Some italians are tuscan', m_4: 'Some italians are abruzzesi'}

applying the theorem 1 we can give an example of the transformation (1) in the following way:

1. Σ[Every(roman,stubborn),Some(italians,romans)]= Some(italians,stubborn)

2. Σ[Every(girl,stubborn),Some(italians,girls)] = Some(italians,stubborn)

Starting from the conclusion we have reconstructed the reasoning (i.e the inferences) which had brought to it.

3 Dialectic argumentation

Having both the analysis and the synthesis we can give the rules which formalize the dialection argumentations as the combinations of "analysis" and "synthesis". Human beings' argumentations are in fact made of these two different moments, one in which we ask how our claim has been deduced, and one in which we look at the consequences of our argumentation.

In order to formalize this natural way of reasoning, we use the rules below. For both the Introduction and Elimination operator, we give two types of rules to which we refer as *proper* vs. *conventional*. The former are the rules we have obtained from the transformations discussed above – 1a, 2a below; whereas the later are new ones which are introduced by convention – 1b, 2b.

Having these second rules we can proceed in our dialectic combination asking the same kind of question twice – i.e. combining the E (resp. I) operator with itself. As a constrain to correctly build the dialectic combination we require our "reasoning" to start always from either 1a or 2a; whereas 1b and 2b can be applied only as a second step. As we will see with an example, although formally all combinations of the operators E and I are possible we obtain a result only when we fix the medium term μ, i.e. when the rules combined are operating on the same term.

For reasons which will be clear soon, we include the identity operation $\mathcal{I}$ as well. For this operator as well we give two schemata considering the $\mathcal{I}$ as a function which takes one or two arguments. However in this case the different behavior of the two rules does not hold: our reasoning can start with both of them, since the two rules are both "natural".

[1a.]	E[a,b] = [c]	[1b.]	E[c] = [c]
[2a.]	I[c] = [a,b]	[2b.]	I[a,b] = [a,b]
[3a.]	$\mathcal{I}$[c] =[c]	[3b.]	$\mathcal{I}$[a,b] = [a,b]

Using the above rules we can combine the operations $\mathcal{I}$, E, I.

Examples 4 of dialectic combination.

(i) I(EI)[a,b] = EI[a,b] = I[c] = [a,b]

(ii) (IE)I[a,b] = IE[a,b] = I[c] =[a,b]

where in (i) the rules are applied from the left to the right, i.e. first (3b), then (1a) finally (2a); whereas in (ii) the rules are applied from the right to the left following the same order. Therefore, the associative law holds:

(6) IEI[a,b] = [a,b]

which says: "Given two categorical propositions a and b if I ask first which syllogism can have them as conclusions, then which syllogisms they can be the premises of, and finally once again from which syllogism they can be derived, the same propositions are found, i.e. I obtain the same categorical propositions I started from".

As a linguistic example we can consider the categorical propositions given in Example 3. Let μ_1 and μ_2 be two different terms in $A \cap B$, for example, the terms 'roman' and 'girl', respectively. If we combine the operators as in (i) we obtain the conclusion c, whereas the combination given in (ii) is unable to go back to the original premises.

Example 5

(i) $\mathrm{E}\mu_1\mathrm{I}\mu_1[\mathrm{M}_1,\mathrm{m}_1] = \mathrm{I}[\mathrm{c}] = [\mathrm{M}_1,\mathrm{m}_1]$

(ii) $\mathrm{E}\mu_1\mathrm{I}\mu_2[\mathrm{M}_1,\mathrm{m}_1] = \mathrm{I}[\mathrm{c}] = [\mathrm{M}_2,\ \mathrm{m}_2]$

4 Algebraic Aspects

The situation described in the previous paragraph can be better expressed introducing an explicate operation to compose $\mathcal{I}$, I and E. Simply applying the rules 1-6 we obtain the table below:

$$(7) \qquad \begin{array}{c|ccc} \circ & \mathcal{I} & \mathrm{E} & \mathrm{I} \\ \hline \mathcal{I} & \mathcal{I} & \mathrm{E} & \mathrm{I} \\ \mathrm{E} & \mathrm{E} & \mathrm{E} & \mathcal{I} \\ \mathrm{I} & \mathrm{I} & \mathcal{I} & \mathrm{I} \end{array}$$

In order to verify that this table correctly synthesize the above rules, we can look at the composition $\mathrm{E} \circ \mathrm{I}$. As it results from the example (i) applying first E and then I to (a,b) we have obtained the same result which is given by the application of $\mathcal{I}$ to the same argument, i.e. $\mathrm{E} \circ \mathrm{I} = \mathcal{I} = \mathrm{I} \circ \mathrm{E}$. We can therefore conclude that the following theorem holds:

Theorem 2 The combination of the σ and γ-transformations, viz. (1) and (3), (4), follows the structure of a finite commutative and idempotent group.

The commutative property (as well as the associative one) is due to the constrain we have required when speaking of the dialectic combination of the operators, viz. our argumentation can never start applying the "conventional" rules (1b and 2b).

The representation of "cognitive" processes by means of operations, and even more the interpretation of the possible combinations of these operations via an algebraic structure recall Piaget's theory, see Piaget [9]. Our proposal of considering the dialectic combination as a group based on $\mathcal{I}$, I, E is, in fact, similar to the idea of having the group of INRC as the algebraic structure behind our mental operations. Having found this algebraic structure as a model for the syllogistic theory is of particular interest because it sheds light on a class of logics which we can define as natural logic, i.e. a logic in which it is possible to *introduce* and *eliminate* logical elements. Intuitively this definition brings into the mind besides the syllogistic theory also the natural deduction system proposed by Gentzen [6]. However, despite the similarity between the meta-operators which characterize the two systems, some important differences occur. We are going to discuss this point in the next section.

5 Gentzen's calculus and the syllogistic theory: A possible comparison

As we have mentioned if we look at the syllogisms as applications of introduction and elimination rules, it seems natural to assimilate the obtained formalization to Genzten's Natural Deduction. However, the similarity disappear as soon as we consider the composition of the operators. Before comparing the two systems, we briefly present Gentzen's calculus.

Natural Deduction is a proof-system introduced by Gentzen in order to reach a calculus closer to natural reasoning than Hilbert's assiomatic system. With this intend he has characterized a logic by means of introduction and elimination rules for each logical connective the language is built from. Simplifying:

$$\mathrm{I} * [\ldots, \mathrm{r}, \mathrm{p}, \mathrm{q}] = \mathrm{r} * \mathrm{s} \qquad \mathrm{E} * [\ \ldots, \mathrm{p} * \mathrm{q}] = \mathrm{r}$$

where $*$ is any connective which is introduced by means of I, and eliminated by means of E. In the introduction rule the conclusion may contain a subformula which does not occur in the premises[1]; in the elimination rule, instead, the conclusion is always a subformula of one of the premises.

[1]This is the case of the introduction rule for the $\rightarrow$ and $\vee$

Already at this point we can consider an important difference between our formalization of the syllogistic theory and this calculus. In the latter the rules eliminate or introduce connectives, whereas in the former they eliminate or introduce terms. As an example we give the introduction and elimination rules for the conjunction $\wedge$ and the conditional $\rightarrow$. Let $a_1, a_2 \in P$ – where P is a set of propositions –, and $i \in \{1, 2\}$, the logical rules for $\wedge$ and $\rightarrow$ are:

$$\frac{a_1 \quad a_2}{a_1 \wedge a_2} \wedge I \qquad\qquad \frac{a_1 \wedge a_2}{a_i} \wedge E$$

$$\frac{\begin{matrix} a_1 \\ \vdots \\ a_2 \end{matrix}}{a_1 \rightarrow a_2} \rightarrow I \qquad\qquad \frac{a_1 \rightarrow a_2 \quad a_1}{a_2} \rightarrow E$$

To be noticed is the particular behavior of the introduction rule of the $\rightarrow$. The two arguments a_1 a_2 are linked to each other: the latter is a consequence of the former through a derivation.

If we compare these inferences with the rules (2) and (5) we easily see the similarity and the differences between the two systems. In particular, we notice that in both the two systems the 'synthesis' is formalized in a similar way by means of the elimination rules; whereas in the natural deduction one the 'analysis' is missing. This difference can be better understood looking at the introduction rules which are 'deductive' as well as the elimination one: from a set of premises they give back a conclusion. To facilitate the comparison we give the above rules in the format used in the previous section.

$$\begin{array}{ll} I \wedge [a_1, a_2] = a_1 \wedge a_2 & E \wedge [a_1 \wedge a_2] = a_i \\ I \rightarrow [a_1, a_2] = a_1 \rightarrow a_2 & E \rightarrow [a_1 \rightarrow a_2, a_1] = a_2 \end{array}$$

Due to the differences between the introduction rules in the systems we are considering, the dialectic combination differs as well. Therefore, the algebraic structure represented in (7) cannot be used as a model of Gentzen's calculus.

Moreover, as we have said before, in the previous section it has been possible to build the finite commutative and idempotent group thanks to the combination of the standard rules with the "conventional" one. Therefore, this combination cannot be done in Gentzen system since there is no way to return the premises and because only the proper rules are given.

We now give some examples to show how the lack of the analysis moment makes the system unable to formalize natural reasoning at least in the way we have said before.

Examples 6

$$\mathrm{EI}[\mathrm{a}\!\rightarrow \mathrm{b},\, \mathrm{a}] = \mathrm{I}[\mathrm{b}] = \,?$$

Let a:= 'It rains', b:= 'I cannot come', the above schema becomes:

$$\cfrac{\textbf{If}\text{ it rains, }\textbf{then}\text{ I cannot come} \quad \text{It rains}}{\cfrac{\text{I cannot come}}{?} \rightarrow I} \rightarrow E$$

The question marker means that we are unable to reconstruct the reasoning which had lead to the conclusion 'I cannot come'.

6 Conclusions

In the paper we have proposed a formalization of the syllogistic theory which is able to account for both the dialectic moments: the synthesis and the analysis. The motivation behind this proposal are of two natures: a historical one which has brought us to consider the *resolutio* involved in natural reasoning as important as the *compositio*; and a physiological one – which can be traced back to Piaget's theory – which has made us thinking of the finite group as a basic and 'natural' structure in human beings' mental processes.

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