Logical Structures in Natural Language: Propositional Logic (Tableaux)

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Contents

1. Reminder: Language of Propositional Logic

Alphabet The alphabet of PL consists of:

- A countable set of propositional symbols: $p, q, r, \ldots$
- The logical connectives : $\neg$ (NOT), $\wedge$ (AND), $\vee$ (OR), $\rightarrow$ (implication), $\leftrightarrow$ (double implication).
- Parenthesis: (,) (they are used to disambiguate the language)

Well formed formulas (wff) They are defined recursively

1. a propositional symbol is a wff:
2. if A is a wff then also $\neg A$ is a wff
3. if A and B are wff then also $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$ and $(A \rightarrow B)$ are wff
4. nothing else is a wff.

2. Reminder: Model

A model consists of two pieces of information:

- which collection of atomic propositions we are talking about (*domain*, D),
- and for each formula which is the appropriate *semantic value*, this is done by means of a function called *interpretation function* ($\mathcal{I}$).

Thus a model $\mathcal{M}$ is a pair: $(D, \mathcal{I})$.

3. Reminder: Interpretation Function

The interpretation function, denoted by $\mathcal{I}$, can assign true (T) or false (F) to the atomic formulas; for the complex formula they obey the following conditions. Given the formulas P, Q of L:

a. $\mathcal{I}(\neg P) = T$ iff $\mathcal{I}(P) = F$

b. $\mathcal{I}(P \wedge Q) = T$ iff $\mathcal{I}(P) = T$ e $\mathcal{I}(Q) = T$

c. $\mathcal{I}(P \vee Q) = F$ iff $\mathcal{I}(P) = F$ e $\mathcal{I}(Q) = F$

d. $\mathcal{I}(P \rightarrow Q) = F$ iff $\mathcal{I}(P) = T$ e $\mathcal{I}(Q) = F$

e. $\mathcal{I}(P \leftrightarrow Q) = F$ iff $\mathcal{I}(P) = \mathcal{I}(Q)$

4. Reminder: Truth Tables

	ϕ	$\neg\phi$
$\mathcal{I}_1$	T	F
$\mathcal{I}_2$	F	T
		(1)

	ϕ	ψ	$\phi \wedge \psi$
$\mathcal{I}_1$	T	T	T
$\mathcal{I}_2$	T	F	F
$\mathcal{I}_3$	F	T	F
$\mathcal{I}_4$	F	F	F
			(1)

	ϕ	ψ	$\phi \vee \psi$
$\mathcal{I}_1$	T	T	T
$\mathcal{I}_2$	T	F	T
$\mathcal{I}_3$	F	T	T
$\mathcal{I}_4$	F	F	F
			(1)

	ϕ	ψ	$\phi \rightarrow \psi$
$\mathcal{I}_1$	T	T	T
$\mathcal{I}_2$	T	F	F
$\mathcal{I}_3$	F	T	T
$\mathcal{I}_4$	F	F	T
			(1)

5. Reminder: Tautologies and Contradictions

Build the truth table of $p \wedge \neg p$.

It's a *contradiction*: always false.

Build the truth table of $(p \rightarrow q) \vee (q \rightarrow p)$.

It's a *tautology*: always true.

A formula P is:

- *satisfiabiliy* if there is at least an interpretation $\mathcal{I}$ such that $\mathcal{I}(P) = True$

6. Done and to be done

Last week we have done exercises on Truth tables.

Check the ones done at home (pl2.pdf on the website)

Today we look at Tableaux calculus.

7. Tableaux Calculus

- The Tableaux Calculus is a decision procedure solving the problem of *satisfiability*.
- If a formula is satisfiable, the procedure will constructively exhibit an interpretation in which the formula is true.

$A \wedge B$ A B	$A \vee B$ $A \qquad\qquad B$	$A \to B$ $\neg A \qquad\qquad B$
$A \leftrightarrow B$ $A \wedge B \qquad\qquad \neg A \wedge \neg B$	$\neg\neg A$ A	$\neg(A \wedge B)$ $\neg A \qquad\qquad \neg B$
$\neg(A \vee B)$ $\neg A$ $\neg B$	$\neg(A \to B)$ A $\neg B$	$\neg(A \leftrightarrow B)$ $A \wedge \neg B \qquad\qquad \neg A \wedge B$

8. Heuristics

Apply non-branching rules before branching rules.

Efficiency: order of rule applications.

9. Tableaux: Tautology

You are asked to prove whether ψ is a *tautology* by means of tableaux.

- If all branches of your tableaux are open, you can conclude ψ is satisfiable.

  In order to check whether ψ is a tautology you have to look at $\neg\psi$.

  If $\neg\psi$ is unsatisfiable then ψ is also a tautology.

- If all branches close: ψ is unsatisfiable.

  This is already a strong result, there is no need to look at $\neg\psi$.

10. Tableaux: Heuristics

Apply non-branching rules before branching rules.

11. Tableaux: Exercises on satisfiability

Prove by means of tableaux whether the formulas below are *satisfiable*.

- $A \wedge (B \wedge \neg A)$
- $(A \rightarrow B) \rightarrow \neg B$
- $A \rightarrow (B \rightarrow A)$ [(to be done at home)]
- $(B \rightarrow A) \rightarrow A$ [(to be done at home)]
- $A \rightarrow (B \vee \neg C)$ [(to be done at home)]

11.1. Tableaux: Exercises on tautologies

Prove by means of tableaux whether the formulas below are *tautologies*.

- $(B \to A) \to A$
- $(A \to B) \to \neg B$ [(to be done at home)]
- $(\neg A \to B) \land (\neg A \lor B)$ [(to be done at home)]

11.2. Tableaux: Exercises on sets of formulas

Determine by means of tableaux whether the following sets of logical forms are satisfiable; in other words, you are asked to check whether there is at least an interpretation in which all the formulas in the set are true.

$$\{\neg B \to B, \neg(A \to B), \neg A \vee \neg B\} \text{ and } \{\neg A \vee B, \neg(B \wedge \neg C), C \to D, \neg(\neg A \vee D)\}$$

First set done in class, second to be done at home.

12. Recall: Reasoning

$$\{P_1, \ldots, P_n\} \models C$$

a *valid* deductive argumentation is such that its conclusion cannot be false when the premises are true.

In other words, there is no interpretation for which the conclusion is false and the premises are true.

$W(Premise)$, the set of interpretations for which the premises are all true, and $W(C)$ the set of interpretations for which the conclusion is true:

$$W(Premises) \subseteq W(C)$$

The argumentation is said to be valid iff α is true for all the interpretations for which all the premises are true.

13. Recall: Example of argumentations

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Today is Monday or today is Thursday        P v Q
Today is not Monday                         not P
=================                           =====
Today is Thursday                            Q
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$$\{P \vee Q, \neg P\} \models Q$$

	P	Q	$P \vee Q$	$\neg P$	Q
$\mathcal{I}_1$	T	T	T	F	T
$\mathcal{I}_2$	T	F	T	F	F
$\mathcal{I}_3$	F	T	T	T	T
$\mathcal{I}_4$	F	F	F	T	F

$$\{\mathcal{I}_3\} \subseteq \{\mathcal{I}_1, \mathcal{I}_3\}$$

14. Entailment: Refutation Method

Let's think more about entailment:

to prove

$$\{P_1, \ldots P_n\} \models C \quad W(P_1, \ldots P_n) \subseteq W(C) \text{ which means}$$

$$\forall \mathcal{I}.\text{If } \mathcal{I}(P_1) = T, \ldots, \mathcal{I}(P_n) = T \text{ then } \mathcal{I}(C) = T$$

we could try to prove

$$\{P_1, \ldots P_n\} \not\models C \quad W(P_1, \ldots P_n) \not\subseteq W(C) \text{ which means}$$

$$\exists \mathcal{I} \text{ s.t. } \mathcal{I}(P_1) = T, \ldots, \mathcal{I}(P_n) = T \text{ and } \mathcal{I}(C) = F, \mathcal{I}(\neg C) = T$$

14.1. Entailment

$$\{P_1, \ldots P_n\} \models C$$

1. P_1 $[\mathcal{I}(P_1) = T]$
2. $\ldots$ $[\ldots]$
3. P_n $[\mathcal{I}(P_n) = T]$
4. $\neg C$ $[\mathcal{I}(C) = F]$

If all branches close (there is a contradiction in all branches), then $\models$ is valid.

If there is at least one branch that does not close, then that branch gives an interpretation that falsifies the entailment.

14.2. Tableaux: Exercises on entailment

Check by means of tableaux if the following arguments are valid.

Let's do the first two in class, the others as home work to be checked next time.

- $\{P \wedge \neg P\} \models \neg P \rightarrow P$
- $\{P \rightarrow Q\} \models \neg(Q \rightarrow P)$
- $\{P \rightarrow Q, Q \rightarrow R\} \models P \rightarrow R$
- $\{P, P \rightarrow Q, Q \rightarrow R, R \rightarrow (\neg S \wedge T)\} \models \neg S$ [(to be done at home)]
- $\{P \rightarrow Q, Q \rightarrow R, P\} \models R$ [(to be done at home)]

15. Natural Deduction (just to give an idea!)

The natural deduction system is a proof system.

It consists of inference rules: elimination and introduction rules of each logical connective.

$$\frac{P \wedge Q}{Q} \wedge E^r \qquad \frac{P \quad Q}{P \wedge Q} \wedge I$$

$$\frac{P \to Q \quad P}{Q} \to E \qquad \frac{\begin{array}{c}[P] \\ \vdots \\ Q\end{array}}{P \to Q} \to I$$

Similarly, for the other connectives.

16. Exercises: Truth Tables vs. Natural Deduction

Prove the following derivation by using (a) truth tables (b) Natural Deduction.

1. $P, P \to Q, Q \to R, R \to (\neg S \wedge T) \vdash \neg S$
2. $P \to Q, Q \to R \vdash P \to R$
3. $P \to Q, Q \vdash P$

17. Decidibility and Complexity

Decidable The problem of establishing whether a formula of propositional language is satisfiable is a decidable problem: it can be solved by considering all interpretation functions possible combinations and computing the truth value of the complex formula by applying the interpretation function definition.

Complexity The problem belongs to a class of problems known as NP-complete, non deterministic problems solvable in polynomial time, in other words it is possible to find an algorithm that solves it "quickly".

18. Next time

On Wednesday, bring the solutions to the exercises.

We will start introducing First order Logic.