Logical Structures in Natural Language: Propositional Logic II (Truth Tables and Reasoning

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Contents

1. What we have said last time

- Logic
  - Language: syntax, semantics.
  - Reasoning
- Semantics
  - Meaning of a sentence = Truth value
  - Compositional meaning: truth-functional connectives
  - Interpretation Function: $\texttt{FORM} \rightarrow \{true, false\}$
- Reasoning: $Premises \models \alpha$ iff $W(Premises) \subseteq W(\alpha)$

Today we look more into Propositional Logic (PL)

2. Remind: Propositional Logic: Basic Ideas

Statements:

The elementary building blocks of propositional logic are *atomic statements* that cannot be decomposed any further: *propositions*.

E.g.,

- "The box is red"
- "The proof of the pudding is in the eating"
- "It is raining"

and logical connectives "and", "or", "not", by which we can build **propositional formulas**.

3. Remind: Language of Propositional Logic

Alphabet The alphabet of PL consists of:

- A countable set of propositional symbols: $p, q, r, \ldots$
- The logical connectives : $\neg$ (NOT), $\wedge$ (AND), $\vee$ (OR), $\rightarrow$ (implication), $\leftrightarrow$ (double implication).
- Parenthesis: (,) (they are used to disambiguate the language)

Well formed formulas (wff) They are defined recursively

1. a propositional symbol is a wff:
2. if A is a wff then also $\neg A$ is a wff
3. if A and B are wff then also $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$ and $(A \rightarrow B)$ are wff
4. nothing else is a wff.

4. Reminder: From English to Propositional Logic

Eg. If you don't sleep then you will be tired.

Keys: p = you sleep, q= you will be tired. Formula: $\neg p \rightarrow q$.

Exercise I:

1. If it rains while the sun shines, a rainbow will appear
2. Charles comes if Elsa does and the other way around
3. If I have lost if I cannot make a move, then I have lost.

1. $(rain \wedge sun) \rightarrow rainbow$
2. $elsa \leftrightarrow charles$
3. $(\neg move \rightarrow lost) \rightarrow lost$

Use: http://www.earlham.edu/~peters/courses/log/transtip.htm

5. Reminder: Semantics: Intuition

- Atomic propositions can be *true* `T` or *false* `F`.
- The truth value of formulas is determined by the truth values of the atoms (*truth value assignment* or *interpretation*).

Example: $(a \vee b) \wedge c$: If a and b are false and c is true, then the formula is not true.

6. Reminder: Interpretation Function

The interpretation function, denoted by $\mathcal{I}$, can assign true (T) or false (F) to the atomic formulas; for the complex formula they obey the following conditions. Given the formulas P, Q of L:

a. $\mathcal{I}(\neg P) = T$ iff $\mathcal{I}(P) = F$

b. $\mathcal{I}(P \wedge Q) = T$ iff $\mathcal{I}(P) = T$ e $\mathcal{I}(Q) = T$

c. $\mathcal{I}(P \vee Q) = F$ iff $\mathcal{I}(P) = F$ e $\mathcal{I}(Q) = F$

d. $\mathcal{I}(P \to Q) = F$ iff $\mathcal{I}(P) = T$ e $\mathcal{I}(Q) = F$

e. $\mathcal{I}(P \leftrightarrow Q) = F$ iff $\mathcal{I}(P) = \mathcal{I}(Q)$

7. Reminder: Truth Tables

	ϕ	$\neg\phi$
$\mathcal{I}_1$	T	F
$\mathcal{I}_2$	F	T
		(1)

	ϕ	ψ	$\phi \wedge \psi$
$\mathcal{I}_1$	T	T	T
$\mathcal{I}_2$	T	F	F
$\mathcal{I}_3$	F	T	F
$\mathcal{I}_4$	F	F	F
			(1)

	ϕ	ψ	$\phi \vee \psi$
$\mathcal{I}_1$	T	T	T
$\mathcal{I}_2$	T	F	T
$\mathcal{I}_3$	F	T	T
$\mathcal{I}_4$	F	F	F
			(1)

	ϕ	ψ	$\phi \rightarrow \psi$
$\mathcal{I}_1$	T	T	T
$\mathcal{I}_2$	T	F	F
$\mathcal{I}_3$	F	T	T
$\mathcal{I}_4$	F	F	T
			(1)

8. Reminder: Model

A model consists of two pieces of information:

- which collection of atomic propositions we are talking about (*domain*, D),
- and for each formula which is the appropriate *semantic value*, this is done by means of a function called *interpretation function* ($\mathcal{I}$).

Thus a model $\mathcal{M}$ is a pair: $(D, \mathcal{I})$.

9. A formula: Tautology, Contradiction, Satisfiable, Falsifiable

Recall, a formula P is:

- *tautology* if for all the interpretations $\mathcal{I}$, $\mathcal{I}(P) = True$ (it's always true)
- *contradiction* if for all the interpretations $\mathcal{I}$, $\mathcal{I}(P) = False$ (is always false)

A formula P is:

- *satisfiabiliy* if there is at least an interpretation $\mathcal{I}$ such that $\mathcal{I}(P) = True$
- *falsifiable* if there is at least an interpretation $\mathcal{I}$ such that $\mathcal{I}(P) = False$

A formula that is false in all interpretation is also called *unsatisfiable*.

9.1. Example

	P	$\neg P$	$\neg P \vee P$
$\mathcal{I}_1$	T	F	T
$\mathcal{I}_2$	F	T	T

$\neg P \vee P$ is a tautology.

10. Tautologies and Contradictions

Build the truth table of $p \wedge \neg p$.

It's a *contradiction*: always false.

Build the truth table of $(p \rightarrow q) \vee (q \rightarrow p)$.

It's a *tautology*: always true.

11. Reminder: exercises

Build the truth tables for the following formulas and decide whether they are satisfiable, or a tautology or a contradiction.

- $(\neg A \to B) \wedge (\neg A \vee B)$
- $P \to (Q \vee \neg R)$

12. An argumentation: Validity

$$\{P_1, \ldots, P_n\} \models C$$

a *valid* deductive argumentation is such that its conclusion cannot be false when the premises are true.

In other words, there is no interpretation for which the conclusion is false and the premises are true.

$W(Premise)$, the set of interpretations for which the premises are all true, and $W(C)$ the set of interpretations for which the conclusion is true:

$$W(Premises) \subseteq W(C)$$

13. Example of argumentations

```
Today is Monday or today is Thursday         P v Q
Today is not Monday                          not P
=================                            =====
Today is Thursday                             Q

If today is Thursday, then today I've a lecture   Q --> R
Today is Thursday                                 Q
===============                                   =======
Today I've a lecture                              R
```

$$P \vee Q, \neg P \models Q \qquad Q \to R, Q \models R$$

Try to build truth tables to verify: $P \vee Q, \neg P \models Q$

	P	Q	$P \vee Q$	$\neg P$	Q
$\mathcal{I}_1$	T	T	T	F	T
$\mathcal{I}_2$	T	F	T	F	F
$\mathcal{I}_3$	F	T	T	T	T
$\mathcal{I}_4$	F	F	F	T	F

$$W(Premesse) \subseteq W(Q)$$

$$\{\mathcal{I}_3\} \subseteq \{\mathcal{I}_1, \mathcal{I}_3\}$$

13.1. Example

	P	Q	$P \vee Q$	$\neg P$	Q
$\mathcal{I}_1$	T	T	T	F	T
$\mathcal{I}_2$	T	F	T	F	F
$\mathcal{I}_3$	F	T	**T**	**T**	T
$\mathcal{I}_4$	F	F	F	T	F

$$W(Premesse) \subseteq W(Q)$$

$$\{\mathcal{I}_3\} \subseteq \{\mathcal{I}_1, \mathcal{I}_3\}$$

13.2. Exercises

Check whether the following arguments are valid:

> If the temperature and air pressure remained constant, there was no rain. The temperature did remain constant. Therefore, if there was rain then the air pressure did not remain constant.

> If Paul lives in Dublin, he lives in Ireland. Paul lives in Ireland. Therefore Paul lives in Dublin.

(i) Give the keys of your formalization using `PL`; (ii) represent the argument formally, and (iii) Apply the truth table method to prove or disprove the validity of the argument.

14. Counter-example

Counterexample an interpretation in which the reasoning does not hold. In other words, an interpretation such that the premises are true and the conclusion is false.

Exercise: together Take the previous exercise and build a counter-example if the argumentation is not valid

> If the temperature and air pressure remained constant, there was no rain. The temperature did remain constant. Therefore, if there was rain then the air pressure did not remain constant.

> If Paul lives in Dublin, he lives in Ireland. Paul lives in Ireland. Therefore Paul lives in Dublin.

Exercises: alone See printed paper (pl3)

15. Reminder: exercises

Build the truth tables for the following entailments and decide whether they are valid

1. $P \vee Q \models Q$
2. $P \rightarrow Q, Q \rightarrow R \models P \rightarrow R$
3. $P \rightarrow Q, Q \models P$
4. $P \rightarrow Q \models \neg(Q \rightarrow P)$

16. Summary of key points.

- Tomorrow bring the solutions for the exercises.
- Today key concepts
  - Syntax of PL: atomic vs. complex formulas
  - Semantics of PL: truth tables
  - Formalization of simple arguments
  - Interpretation function
  - Domain
  - Model
  - Entailment
  - Satisfiability