Logical Structures in Natural Language: First order Logic (FoL): Entailment

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1. Satisfiability and Validity in FOL

Similarly as in propositional logic, a formula ϕ can be **satisfiable**, **unsatisfiable**, **falsifiable** or **valid**—the definition is in terms of the pair (I, α).

A formula ϕ is

- **satisfiable**, if there is some (I, α) that satisfies ϕ,
- **unsatisfiable**, if ϕ is not satisfiable,
- **falsifiable**, if there is some (I, α) that does not satisfy ϕ,
- **valid** (i.e., a **tautology**), if every (I, α) satisfies ϕ.

2. Tableaux

$A \wedge B$ A B	$A \vee B$ $A \qquad B$	$A \rightarrow B$ $\neg A \qquad B$
$A \leftrightarrow B$ $A \wedge B \qquad \neg A \wedge \neg B$	$\neg\neg A$ A	$\neg(A \wedge B)$ $\neg A \qquad \neg B$
$\neg(A \vee B)$ $\neg A$ $\neg B$	$\neg(A \rightarrow B)$ A $\neg B$	$\neg(A \leftrightarrow B)$ $A \wedge \neg B \qquad \neg A \wedge B$

(10) $\forall x.A(x)$ $A(t)$ where t is a term	(11) $\exists x.A(x)$ $A(t)$ where t is a term which has *not* been used in the derivation so far.
(12) $\neg\forall x(A(x))$ $\exists x(\neg A(x))$	(13) $\neg\exists x(A(x))$ $\forall x(\neg A(x))$

3. Weaker Results

- If ψ is *valid*, can we conclude it is satifiable, falsifiable or unsatisfiable?

  We can conclude ψ is *satisfiable*:

  $$\text{IF } \forall I, I \models \psi \text{ THEN } \exists I, I \models \psi$$

  Satisfiability is a weaker property then validity.

- If ψ is *unsatisfiable*, can we conclude it is satifiable, falsifiable or valid?

  We can conclude ψ is *falsifiable*:

  $$\text{IF } \forall I, I \not\models \psi \text{ THEN } \exists I, I \not\models \psi$$

  Falsiability is a weaker property then unsatisifiability.

3.1. Consequences

You are asked to prove whether ψ is valid by means of tableaux.

- If all branches of your tableaux are open, what do you conclude?

  ψ is satisfiable. Are you sure you cannot give a stronger answer, i.e. are you sure ψ is not valid? In order to check whether ψ is valid you have to look at $\neg\psi$. If $\neg\psi$ is unsatisfiable then ψ is also valid.

- If all branches close: ψ is unsatisfiable. Can you make a stronger claim? No this is already a strong result, there is no need to look at $\neg\psi$.

4. Formalization

Consider the following situations:

(a) In the Alpha town there were a barber man and a man that the barber shaved. However, any barber man of Alpha shaved *all and only* the men of Alpha who did not shave themselves.

Formalize the problem, and show that the resulting set of formulae is unsatisfiable.

(b) In the Alpha town there were a barber man and a man that the barber shaved. However, any barber man of Alpha shaved *all* the men of Alpha who did not shave themselves.

Formalize the problem, and show that the resulting set of formulae is satisfiable.

(a)

$$\{\exists x(B(x) \wedge \exists y S(x,y)), \forall x(B(x) \rightarrow \forall y(\neg S(y,y,) \leftrightarrow S(x,y)))\}$$

(b)

$$\{\exists x(B(x) \wedge \exists y S(x,y)), \forall x(B(x) \rightarrow \forall y(\neg S(y,y,) \rightarrow S(x,y)))\}$$

5. Entailment

Entailment is defined similarly as in propositional logic.

The formula ϕ is logically implied by a formula ψ, if ϕ is true in all the interpretations in which ψ is true

$$\psi \models \phi \quad \text{iff} \quad I \models \phi \text{ for all models } I \text{ of } \psi$$

6. Exercise: Entailment among given FoL formulae

Prove whether the following entailments are valid and give a counter-example (a domain and an interpretation) if they are not.

- $\forall x(F(x) \rightarrow G(x)), \neg\exists x(G(x)) \models \neg F(a)$
- $\neg\exists x(F(x) \wedge G(x)) \models \neg F(a)$
- $\forall x(F(x)) \rightarrow \forall x(G(x)), \neg\exists x G(x) \models \exists x \neg F(x)$

a) Check by means of tableaux method whether the argument below is valid.

$$\forall y(\texttt{Suspect}(y)), \exists x(\texttt{Murder}(x)) \models \neg\exists x(\forall y(\texttt{Suspect}(x) \rightarrow \texttt{Murder}(y))).$$

b) Build a counterexample if the argumentation is not valid.

7. Animal problem

Consider the following problem

1. The only animals in this house are cats.
2. Every animal that loves to look at the moon is suitable for a pet.
3. When I detest an animal, I avoid it.
4. All animals that don't prowl at night are carnivorous.
5. No cat fails to kill mice.
6. No animals ever take to me, except the ones in this house.
7. Kangaroos are not suitable for pets.
8. All animals that are carnivorous kill mice.
9. I detest animals that do not take to me.
10. Animals that prowl at night always love to look at the moon.

Is it true that I always avoid kangaroo?

(a) Represent the facts above as FOL sentences and formalize the problem. Remember to give the keys of your formalization and add the extra information you might need in order to answer the question. (b) Give the proof of your answer by means of tableaux.

7.1. Solution: Animal Problem

$\texttt{Animal}(x)$:	"x is an animal"
$\texttt{Cat}(x)$:	"x is a cat"
$\texttt{Inhouse}(x)$:	"x is in this house"
$\texttt{Detest}(x,y)$:	"x detests y"
$\texttt{Avoid}(x,y)$:	"x avoids y"
$\texttt{Take}(x,y)$:	"x takes to y"
$\texttt{Kan}(x)$:	"x is a kangaroo"

$$\forall x.(\texttt{Animal}(x) \wedge \texttt{Inhouse}(x)) \rightarrow \texttt{Cat}(x) \qquad [1.]$$

$$\forall x.\texttt{Detest}(r,x) \rightarrow \texttt{Avoid}(r,x) \qquad [3.]$$

$$\forall x.(\texttt{Animal}(x) \wedge \neg\texttt{Inhouse}(x)) \rightarrow \neg\texttt{Take}(x,r) \qquad [6.]$$

$$\forall x.(\texttt{Animal}(x) \wedge \neg\texttt{Take}(x,r)) \rightarrow \texttt{Detest}(r,x) \qquad [9.]$$

$$\forall x.\texttt{Kan}(x) \rightarrow \texttt{Animal}(x) \qquad \text{[extra]}$$

$$\forall x.\texttt{Cat}(x) \leftrightarrow \neg\texttt{Kan}(x) \qquad \text{[extra]}$$

$$\neg(\forall x.\texttt{Kan}(x) \rightarrow \texttt{Avoid}(r,x)) \qquad \text{[negation of Con.]}$$