Logical Structures in Natural Language: First order Logic (FoL)

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Contents

1. How far can we go with PL?

1. Casper is bigger than John
2. John is bigger than Peter
3. Therefore, Casper is bigger than Peter.

Questions:

How would you formalize this inference in PL?

What do you need to express that cannot be expressed in PL?

Answer:

You need to express: "relations" (is bigger than) and "entities" (Casper, John, Peter)

1.1. Inference

1. Bigger(casper,john)
2. Bigger(john,peter)
3. Therefore, Bigger(casper,peter)

Question: Do you still miss something?

1.2. Exercise: Predicates and entities

Formalize:

- John is bigger than Peter or Peter is bigger than John
- If Raffaella is speaking, then Rocco is listening
- If Peter is laughing, than John is not biting him.

Predicates represent sets of objects (entities). For example:

> "is listening" is the set of all those entities that are listening —i.e. all of you!

1.3. Housing lottery problem

Housing lotteries are often used by university housing administrators to determine which students get first choice of dormitory rooms.

Consider the following problem:

1. Bob is ranked immediately ahead of Jim.
2. Jim is ranked immediately ahead of a woman who is a biology major.
3. Lisa is not near to Bob in the ranking.
4. Mary or Lisa is ranked first.

Is it true that Jim is immediately ahead of Lisa and Lisa is the last of the ranking and Mary is the first?

Questions:
How would you formalize the problem in PL?

What do you need to express that you cannot express in PL?

It would be handy to say, e.g. that *there exists* a woman who is a biology major.

1.4. Graph Coloring Problem

Formalization of our general knowledge about graphs and color assignments.

First of all let $1, 2 \ldots n$ be our vertices, and B (blue), R (red), G (green) ... be our colors, and let $B_i, C_{i,j}$ stand for "i is of color B" and "i is connected to j".

- **Symmetry of edges**: $C_{1,2} \leftrightarrow C_{2,1}, \ldots$ (so for the other connected vertices)
- **Coloring of vertices**: $B_1 \vee G_1 \vee R_1 \ldots$ (so for the colors and for the other vertices)
- **Uniqueness of colors per veritex**: $B_1 \leftrightarrow (\neg G_1 \wedge \neg R_1) \ldots$ (so for the other vertices)

Secondly, we have an explicitly given constraint: the coloring function has to be such that no two connected vertices have the same color:

- **Explicit Constraint**: $C_{1,2} \rightarrow (B_1 \rightarrow \neg B_2) \wedge (R_1 \rightarrow \neg R_2) \wedge (G_1 \rightarrow \neg G_2) \ldots$ (so for the other colors and the other vertices)

Question: Which could be a way to express these properties of colors and edges in a "few words"?

We could speak to properties that hold *for all* colors, or all vertices!

1.5. Syllogism

1. Some four-legged creatures are genus
2. All genus are herbivores
3. Therefore, some four-legged creatures are herbivores

Question:

What is the schema behind the reasoning? What you could not express in PL even if we add relations and entity?

1. Some F are G
2. All G are H
3. Therefore, some F are H

We still need *quantifiers* to express "some" and "all".

1.6. Quantifiers

Quantifiers like truth-functional operators ($\rightarrow, \neg, \vee, \wedge$) are logical operators; but instead of indicating relationships among sentences, they express relationships among the *sets* designated by predicates.

For example, statements like

> "All A are B" assert that the set A is a subset of the set B, $A \subseteq B$;

that is

> all the members of A are also members of B.

1.7. Quantifiers

Statements like

> "Some A are B" assert that the set A shares at least one member with the B $A \cap B \neq 0$;

Note, hence "Some A are B" is considered to be different from standard usage:

- at least one member
- it does not presuppose that "not all A are B"

1.8. Variables and Quantifiers

"All A are B" can be read as saying:

For all x, *if* x is A *then* x is B.

i.e. what we said before: all the members of A are also members of B, i.e A is included in B, $A \subseteq B$.

We write this as: $\forall x.A(x) \to B(x)$

"Some A are B" can be read as saying:

For some x, x is A *and* x is B.

i.e. what we said before: there exists at least a members of A that is also a members of B.

We write this as: $\exists x.A(x) \wedge B(x)$

1.9. Summing up: Motivations to move to FOL

- We can already do a lot with propositional logic.
- But it is unpleasant that we cannot access the *structure* of atomic sentences.
- Atomic formulas of propositional logic are *too atomic* – they are just statements which my be true or false but which have no internal structure.
- In *First Order Logic* (FOL) the atomic formulas are interpreted as statements about *relationships between objects*.

2. Syntax of FoL

Formulas: $\phi, \psi \rightarrow P(t_1, \ldots, t_n)$ atomic formulas

Formulas:	$\phi, \psi \rightarrow P(t_1, \ldots, t_n)$	atomic formulas
	$\mid\ \bot$	false
	$\mid\ \top$	true
	$\mid\ \neg\phi$	negation
	$\mid\ \phi \wedge \psi$	conjunction
	$\mid\ \phi \vee \psi$	disjunction
	$\mid\ \phi \rightarrow \psi$	implication
	$\mid\ \phi \leftrightarrow \psi$	equivalence
	$\mid\ \forall x.\ \phi$	*universal quantification*
	$\mid\ \exists x.\ \phi$	*existential quantification*

E.g. Everyone in England is smart: $\forall x.\ In(x, england) \rightarrow Smart(x)$

Someone in France is smart: $\exists x.\ In(x, france) \wedge Smart(x)$

3. Domain and Interpretation

- Socrates, Plato, Aristotle are philosophers
- Mozart and Beethoven are musicians
- All of them are human beings
- Socrates knows Plato.
- Mozart knows Beethoven.

Which do you thing is the Domain of discourse?

What's the meaning of "knows", and of "musician", "philosopher" and "human beings"?

Are the statements below true or false in the above situation?

1. $\forall x.HumanBeings(x)$?
2. $\exists x.HumanBeings(x) \wedge Musicians(x)$?
3. $\forall x.Female(x) \rightarrow Musicians(x)$?

3.1. Semantics of FOL: intuition

- Just like in propositional logic, a (complex) FOL formula may be true (or false) with respect to a given interpretation.
- An interpretation specifies referents for
  constant symbols $\rightarrow$ **objects**
  predicate symbols $\rightarrow$ **relations**
- An atomic sentence $P(t_1, \ldots, t_n)$ is true in a given interpretation
  iff the *objects* referred to by $t_1, \ldots, t_n$
  are in the *relation* referred to by the predicate P.
- An interpretation in which a formula is true is called a *model* for the formula.

4. Sub-formulas, free and bound

Sub-formula is a string inside a wff such that it is also a wff. For instance, in the wff: $\forall x(P(x) \wedge Q(y))$ the sub-formulas are:

- $\forall x(P(x) \wedge Q(y))$
- $P(x) \wedge Q(y)$
- $P(x)$
- $Q(y)$

Quantifier It is always before a sub-formula (its scope) and it is associated to a variable (the variable appearing just after the quantifier and of which the quantifier express its quantification.)

Variable A variable inside a formula is said to be *free* if it does not occur in any sub-formula preceded by a quantifier associated to such variable. Otherwise, it is said to be *bound*. In the example above x is bound whereas y is free. A formula is said to be *closed* if it does not contain free variable, it is said to be *open* if it contains free variable.

5. Exercises on FoL syntax

How do you represent in FOL the following sentences?

1. All lectures teach Logic.

   $\forall x.\mathtt{Lectures}(x) \rightarrow \mathtt{Teach}(x, \mathtt{logic})$

2. Not all lecturers teach Logic.

   $\neg(\forall x.\mathtt{Lectures}(x) \rightarrow \mathtt{Teach}(x, \mathtt{logic}))$

3. Some lectures teach logic

   $\exists x.\mathtt{Lectures}(x) \wedge \mathtt{Teach}(x, \mathtt{logic})$

4. Some lectures do not teach Logic

   $\exists x.\mathtt{Lectures}(x) \wedge \neg\mathtt{Teach}(x, \mathtt{logic})$

Question: Are any of these sentences/formulas equivalent?

Yes: 2 and 4!

Question: What do you conclude? How can you generalize this claim?

$\neg\forall x.A = \exists x \neg A$

6. Tableaux

$A \wedge B$ A B	$A \vee B$ $A \qquad\qquad B$	$A \to B$ $\neg A \qquad\qquad B$
$A \leftrightarrow B$ $A \wedge B \qquad\qquad \neg A \wedge \neg B$	$\neg\neg A$ A	$\neg(A \wedge B)$ $\neg A \qquad\qquad \neg B$
$\neg(A \vee B)$ $\neg A$ $\neg B$	$\neg(A \to B)$ A $\neg B$	$\neg(A \leftrightarrow B)$ $A \wedge \neg B \qquad\qquad \neg A \wedge B$

(10) $\forall x.A(x)$ $A(t)$ where t is a term	(11) $\exists x.A(x)$ $A(t)$ where t is a term which has *not* been used in the derivation so far.
(12) $\neg\forall x(A(x))$ $\exists x(\neg A(x))$	(13) $\neg\exists x(A(x))$ $\forall x(\neg A(x))$

7. Heuristics

Prove $\exists x.(\exists y.(R(x,y))) \rightarrow R(a,a)$ is valid.

1.	$\neg(\exists x.(\exists y.R(x,y))) \rightarrow R(a,a))$	
2.	$\exists x.(\exists y.(R(x,y)))$	Rule (8) applied to 1.
3.	$\neg R(a,a)$	Rule (8) applied to 1.
4.	$\exists y.R(a,y)$	Rule (11) applied to 2.
5.	$R(a,a)$	Rule (11) applied to 4.

Can we conclude that the $\exists x.(\exists y.(R(x,y))) \rightarrow R(a,a)$ is valid?

Let us interpret the above theorem as following:

- Let the natural numbers be our domain of interpretation.
- Let $R(x,y)$ stand for $x < y$

Then, $\exists x.(\exists y.(R(x,y)))$ is satisfiable. E.g. $2 < 4$.

From this it follows that $R(a,a)$ should be true as well, but it is not.

Hence, $\exists x.(\exists y.(R(x,y))) \rightarrow R(a,a)$ is not satisfiable in this interpretation, much less valid.

On line 5. Rule (11) has violeted the constraint: the term a was already used. Similarly, a could have not been used in line 4. either. Hence we don't get a contraddiction and the tableau is not closed.

Note : The same term can be used many times for universal instantiation.

$$\downarrow$$

Heuristic : When developing a semantic tableau in FOL use the existential instantiation rule (Rule 11) before the universal instantiation rule (Rule 10).

7.1. Heuristic (II)

1.	$\neg(\forall x.(A(x) \wedge B(x)) \rightarrow \forall x.A(x))$	
2.	$\forall x.(A(x) \wedge B(x))$	Rule (8) applied to 1.
3.	$\neg(\forall x.A(x))$	Rule (8) applied to 1.
4.	$\exists x.\neg A(x)$	Rule (12) applied to 3.
5.	$\neg A(a)$	Rule (11) applied to 4.
6.	$A(a) \wedge B(a)$	Rule (10) applied to 2.
7.	$A(a)$	Rule (1) applied to 6.
8.	$B(a)$	Rule (1) applied to 6.
	Closed	

Note: if we had used Rule 10 before we would have not been able to apply Rule 11.

8. Exercises

Prove each of the following using semantic tableaux:

1. $\forall x.(A(x) \rightarrow (B(x) \rightarrow A(x)))$
2. $\exists x.A(x) \rightarrow A(a)$
3. $(\exists x.A(x) \wedge \exists x.B(x)) \rightarrow \exists x.(A(x) \wedge B(x))$

$\forall x.(A(x) \to (B(x) \to A(x)))$

1.	$\neg(\forall x.(A(x) \to (B(x) \to A(x))))$	
2.	$\exists x.(\neg(A(x) \to (B(x) \to A(x))))$	Rule (12) applied to 1.
3.	$\neg(A(a) \to (B(a) \to A(a)))$	Rule (8) applied to 2.
4.	$A(a)$	Rule (12) applied to 3.
5.	$\neg(B(a) \to A(a))$	Rule (8) applied to 3.
6.	$B(a)$	Rule (8) applied to 5.
7.	$\neg A(a)$	Rule (8) applied to 5.
	Closed	

2. $\exists x.(A(x)) \to A(a)$

1.	$\neg(\exists x.(A(x)) \to A(a))$	
2.	$\exists x.A(x)$	Rule (8) to line 1
3.	$\neg A(a)$	Rule (8) to line 1
4.	$A(b)$	Rule (11) to line 2

We cannot close the branch.

3. $(\exists x A(x) \wedge \exists x B(x)) \rightarrow \exists x (A(x) \wedge B(x))$

1.	$\neg((\exists x A(x) \wedge \exists x B(x)) \rightarrow \exists x (A(x) \wedge B(x)))$	
2.	$\exists x A(x) \wedge \exists x B(x)$	Rule (8) to line 1
3.	$\neg(\exists x (A(x) \wedge B(x)))$	Rule (8) to line 1
4.	$\forall x \neg(A(x) \wedge B(x))$	Rule (13) to line 3
5.	$\exists x A(x)$	Rule (1) to line 2
6.	$\exists x B(x)$	Rule (1) to line 2
7.	$A(a)$	Rule (11) to line 5
8.	$B(b)$	Rule (11) to line 6
9.	$\neg(A(a) \wedge B(a))$	Rule (10) to line 4
10.	$\neg A(a)$ | $\neg B(a)$	Rule (6) to line 9

We cannot close the right branch.

We could have used b in line 9 to give $\neg(A(b) \wedge B(b))$, but then the left branch could have not been closed.

9. Next steps

- Wednesday 12th of April FoL and tableaux (ex. on reasoning) [shall we start at 12:30? or 12:45?]
- Thursday 13rd of April FoL and tableaux (formalization)
- Wednesday 19th Recap (what you tell me we should practice more)
- Thu. 20th of April sample Exam
- Wed. discussion sample exam