Exercises on Tableaux for FoL

University of Trento

19 March 2014

1 Model

(a) Let $\mathcal{L} = \langle P, D, G \rangle$ be a FOL language with P e D unary symbols and G a binary symbol. Given the model with the following domain $D = \{0, 1, 2\}$ and interpretation:

$$
\begin{aligned}
\mathcal{I}(P) &= \{0, 2\}, \\
\mathcal{I}(D) &= \{1\}, \\
\mathcal{I}(G) &= \{(0, 1), (0, 2), (1, 2)\},
\end{aligned}
$$

verify whether in $\mathcal{M}$ the following formulas are true and explain your answer

(1) $\forall x (\exists y ((P(x) \lor D(x)) \to G(x, y)))$
(2) $\exists x (\forall y (\neg G(x, y)))$

(b) Let $\mathcal{L} = \langle P, S \rangle$ be a FOL language with S and P binary and unary symbols, respectively. Given the model with the following domain $D = \{0, 1\}$, and interpretation:

$$
\begin{aligned}
\mathcal{I}(P) &= \{1\}, \\
\mathcal{I}(S) &= \{(0, 1), (1, 1)\},
\end{aligned}
$$

verify whether in $\mathcal{M}$ the following formulas are true and explain the answer.

$$
\forall x (\exists y (S(x, y) \to P(y)))
$$
$$
\exists x (\forall y (S(x, y) \land P(y)))
$$

2 Tautology

Check whether the following formula is a tautology:

1. $\forall x (F(x) \to F(a))$

2. $\neg (\exists x F x \land \forall x \neg F x)$

3 Valid Entailment

Check whether the following entailments are valid:

1. $\forall x \forall y (Fxy \to \neg Fyx) \models \neg \exists Fxx$

2. $\exists x \forall y Lxy \models \forall x \exists y Lyx$

3. $\exists x \exists y Lxy \models \exists x Lxx$

4. $\forall x \exists y Lxy \models Laa$

4 Solutions

4.1 Tautology

1.

$$\begin{array}{lll}
1 & \neg(\forall x F(x) \rightarrow F(a)) & \\
2 & \exists x \neg(F(x) \rightarrow F(a)) & \text{from 1} \\
3 & \neg(F(b) \rightarrow F(a)) & \text{from 2} \\
4 & F(b) & \text{from 3} \\
5 & \neg F(a) & \text{from 3}
\end{array}$$

2.

$$\begin{array}{lll}
1. & \exists x Fx \wedge \forall x \neg Fx & \\
2. & \exists x Fx & \text{from 1} \\
3. & \forall x \neg Fx & \text{from 1} \\
4. & Fa & \text{from 2} \\
5. & \neg Fa & \text{from 3}
\end{array}$$

4.2 Valid Entailment

1.

$$\begin{array}{lll}
1. & & \text{from} \\
2. & & \text{from} \\
3. & & \text{from} \\
4. & & \text{from}
\end{array}$$

2.

$$\begin{array}{lll}
1. & \exists x \forall y Lxy & \text{Prem} \\
2. & \neg(\forall x \exists y Lyx) & \text{from negation of Conc} \\
3. & \forall y Lay & \text{from 1} \\
4. & \exists x \neg \exists y Lyx & \text{from 2} \\
5. & \neg y Lyb & \text{from 4} \\
6. & \forall y \neg Lyb & \text{from 5} \\
7. & Lab & \text{from} \\
8. & \neg Lab & \text{from} \\
& X & \text{Contradiction } Lab \text{ and } \neg Lab
\end{array}$$

The entailment is valid.

3.

1.	$\exists x \exists y Lxy$	Prem
2.	$\neg(\exists x Lxx)$	negation of Conc
3.	$\exists y Lay$	from Prem
4.	Lab	from 3
5.	$\forall x \neg Lxx$	from 2
6.	$\neg Laa$	from 5
7.	$\neg Lbb$	from 5

The entailment is not valid. The couter-example is given by the model with $D = \{a, b\}$ and $\mathcal{I}(L) = \{(a, b)\}$

4.

1.	$\forall x \exists y Lxy$	Prem
2.	$\neg Laa$	negation of Conc
3.	$\exists y Lay$	from 1
4.	Lab	from

The entailment is not valid. The counter-example is given by the model with $D = \{a, b\}$ and $\mathcal{I}(L) = \{a, b\}$