Background on Vectors

Raffaella Bernardi

University of Trento

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Background: Vector and Matrix

Vector Space

A vector space is a mathematical structure formed by a collection of vectors: objects that may be added together and multiplied ("scaled") by numbers, called scalars in this context.

Vector an n-dimensional vector is represented by a column:

$$\begin{bmatrix} v_1 \\ \dots \\ v_n \end{bmatrix}$$

or for short as $\vec{v} = (v_1, \dots v_n)$.

Background: Vectors

Sum and difference

$$\vec{v} + \vec{w} = (v_1 + w_1, \ldots, v_n + w_n)$$

$$\vec{v} - \vec{w} = \vec{v} + (-\vec{w}) = (v_1 - w_1, \ldots, v_n - w_n)$$

Background: Vectors

Dot product or inner product

The inner product of two vectors is:

$$\vec{v} \cdot \vec{w} = (v_1 w_1 + \ldots + v_n w_n) = \sum_{i=1}^{n} v_i w_i$$

		dog	bark	apple	home	run	cat	school	jump
v	=	(3,	3,	3,	3,	0,	0,	0,	0)
w	=	(1,	1,	1,	1,	0,	0,	0,	0)

$$\vec{v} \cdot \vec{w} == (3 \times 1) + (3 \times 1) + (3 \times 1) + (3 \times 1) = 12$$

Background: Vector

Length

Length $||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{\sum_{i=1}^{n} v_i^2}$

For instance, given the vector:

$$\vec{d} = (4, 4, 4, 4)$$

its length is computed as:

$$|\vec{d}| = \sqrt{\sum_{i=1}^{n} d_i^2} = \sqrt{16 + 16 + 16 + 16} = \sqrt{64} = 8$$

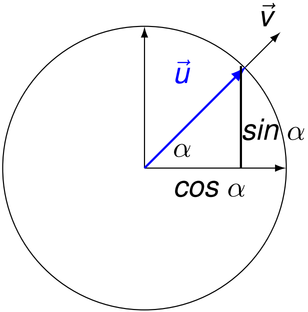
Background: Vector

Unit vector

Unit vector is a vector whose length equals one.

$$\vec{u} = \frac{\vec{v}}{||\vec{v}||}$$

is a unit vector in the same direction as $\vec{v}$. (normalized vector)



$\vec{u} = \frac{\vec{v}}{||\vec{v}||} = (cos\ \alpha, sin\ \alpha)$

Background: Vector

How close to each other are two vectors?

We want to know how close to each others are two vectors.
The angle δ they form gives us the answer.
The smaller the angle δ, the closer are the two vectors.
How do we compute this angle?

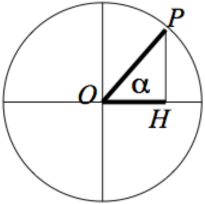
1. we compute the unit vectors (=have the same length, 1)
2. we compute the inner product of their unit vectors. In this way, we obtain the cosine of δ.
3. The bigger is *cos* δ, the smaller the angle δ, the closer the two vectors are.

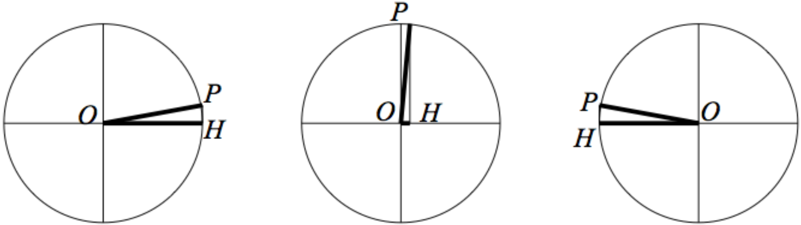
Background: Vector

Cosine and angle

$$\text{Cos } \alpha = \frac{OH}{OP}$$

With $OP = 1$, $\text{Cos } \alpha = OH$





$\cos(0°) = 1$ $\cos(90°) = 0$ $\cos(180°) = -1$

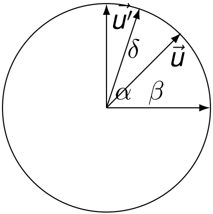
The angle measure *increases*, and the cosine measure *decreases*.

Background: Vector

Cosine formula

Given δ the angle formed by the two unit vectors $\vec{u}$ and $\vec{u'}$, s.t.
$\vec{u} = (cos\ \beta, sin\ \beta)$ and $\vec{u'} = (cos\ \alpha, sin\ \alpha)$

$$\vec{u} \cdot \vec{u'} = (cos\ \beta)(cos\ \alpha) + (sin\ \beta)(sin\ \alpha) = cos(\beta - \alpha) = cos\ \delta$$



Given two arbitrary vectors x and y:

$$cos(\vec{x}, \vec{y}) = cos\ \delta = \frac{\vec{x}}{||\vec{x}||} \cdot \frac{\vec{y}}{||\vec{y}||} = \frac{\sum_{i=1}^{n} x_i \times y_i}{\sqrt{\sum_{i=1}^{n} x_i^2} \times \sqrt{\sum_{i=1}^{n} y_i^2}}$$

Background: Vector

Example

$$cos(\vec{x}, \vec{y}) = \frac{\sum_{i=1}^{n} x_i \times y_i}{\sqrt{\sum_{i=1}^{n} x_i^2} \times \sqrt{\sum_{i=1}^{n} y_i^2}}$$

		dog	bark	apple	home	run	cat	school	jump
v	=	(3,	3,	3,	3,	0,	0,	0,	0)
w	=	(1,	1,	1,	1,	0,	0,	0,	0)
z	=	(0,	0,	0,	0,	1,	1,	1,	1)

$$cos(\vec{v}, \vec{w}) = \frac{\vec{w} \cdot \vec{w}}{||\vec{v}|| \times ||\vec{w}||} = \frac{(3 \times 1) + (3 \times 1) + (3 \times 1) + (3 \times 1)}{6 \times 2} = \frac{12}{12} = 1$$

$$cos(\vec{v}, \vec{z}) = \frac{\vec{v} \cdot \vec{z}}{||\vec{v}|| \times ||\vec{z}||} = \frac{0}{6 \times 2} = \frac{0}{12} = 0$$

v and *w* are more similar/are closer.

Background: Vector

Matrix and vector

A matrix is represented by [nr-rows x nr-columns]. Eg. for a 2 x 3 matrix, the notation is:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

a_{ij} i stands for the row nr, and j stands for the column nr.

Background: Vector

Function application=Matrix multiplication

The multiplication of two matrices is obtained by

Rows of the 1st matrix x columns of the 2nd.

A matrix with m-columns can be multiplied only by a matrix of m-rows:

[n x m] x [m x k] = [n x k].

$$A\,\vec{x} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (1,0)\cdot(x_1,x_2) \\ (-1,1)\cdot(x_1,x_2) \end{bmatrix} = \begin{bmatrix} 1(x_1)+0(x_2) \\ -1(x_1)+1(x_2) \end{bmatrix} =$$

$$= \begin{bmatrix} x_1 \\ x_2 - x_1 \end{bmatrix} = \vec{b}$$

A is a “difference matrix”: the output vector $\vec{b}$ contains differences of the input vector $\vec{x}$ on which “the matrix has acted.”